Basic Calculus for Engineers, Scientists and Economists
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Lecture - 21
Differentiation Functions of Variable – 1

Welcome once again and we are going to start about start talking about partial derivatives. It is about the derivatives, some kind of derivative of functions of 2 variables.

Now as we go on from here the next last part of this course, we expect a slightly more maturity from you and slightly more amount of concentration because here the concept starts becoming different and may be slightly difficult for immediate comprehension. You should not ever read science, by on thinking that science is all about absolute truths science is largely not about absolute truths, it is just creating some perception about the physical reality in which we are in and mathematics is a tool which aids to that perfection.

So, mathematics you cannot say it is really a part of science, but mathematics is language mixed with logic which aids in developing that perception of reality, which we are trying to do through science is very important to have this view in the beginning. Now, when you have a function of 2 variables, Z is function of x and y.

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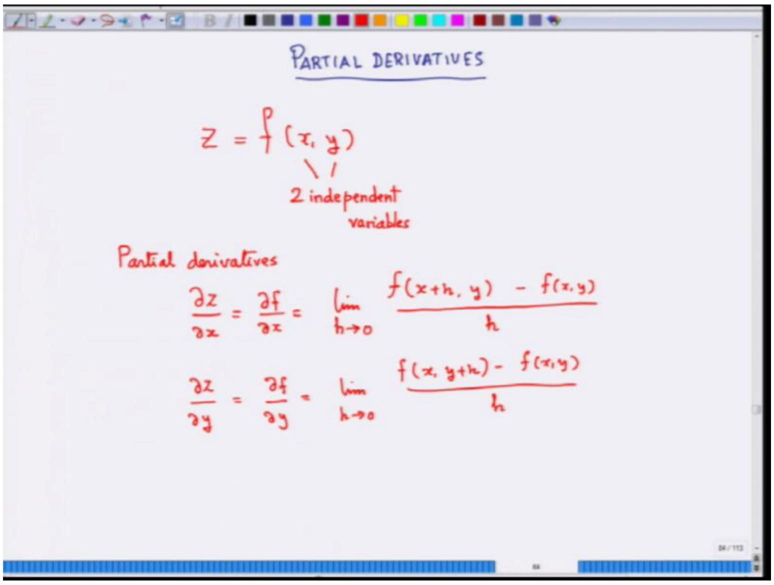

So, here you have 2 independent variables. Now when we at function of one real variable we had talked we had spoken about derivative with respect to that variable. So, how can I speak of derivative with respect to these 2 variables this is something which is not. So, easily understood there are couple of ways of looking at it I will not get into the deeper way of doing things, but I want to tell you that if I fix y.

For example, in this function then I have z to be a function of x only, and if I fix up the x put some values x is equal to 3 then, I have z to be a function of y only. If I take these 2 aspects of this function then I can obviously, use my standard idea of derivative and do it and that leads to the notion of partial derivatives.

So, when I am talking about partial derivative at x which I symbolize as del z del x which is also written as del f del x. How do I define these I define it is as same as the one I have known already, but what I do is I hold y constant I think that this is the constant a number as why. So I do not have to bother about it is differentiation. It is differentiation of that with respect to x would be 0 of course. I only increase or I only make an increment in the x variable and keep the y variable fixed and these gives me what is called my partial derivative with respect to x.

Similarly, you can talk about a partial derivative with respect to why and of course, you can have a function of more variables and you can go on, but we are just going to concentrate on the same placed ideas and then when you do this you hold the variable x to be constant and make an increment only in the variable y. So, f of x given x x y plus h minus f of x y by h remember this is nothing, but taking the ordinary derivative these 2 limits, because these limits are limits not a functions of 2 variables, but functions of one variable because y is constant here.

Now, if I want to compute the function of 2 variables, Let us take an example and let us compute it.

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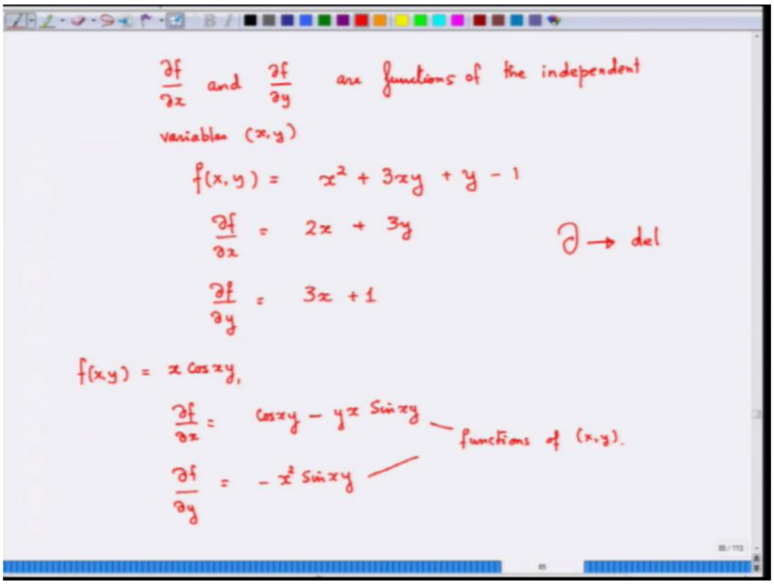

So, an example which will show you some important thing that del f del x and del f del y they must be feeling that del f del x is just the function of x and del f and del y is the function of y, but no both are functions of x and y. You are holding y fixed for that moment, but actually you can give whatever values of y you want. When you have computed the value of del f del x, you can keep on giving values whatever value of y you essentially want. The partial derivative here this actually if you go by strict things then these actually means the partial derivative of f with respect to x computed at the point x y.

So, what partial because you are holding one of the things fixed. These; this and this are functions of x y functions of the independent variables x y. Now take for example, which I take it out straight from the book Thomas and Finney, sorry book Thomas as calculus. So, unfortunate Thomas and Finney both are dead. If you look at this function my first job is to take del f del x. So, what do I mean by del f del x here I am now looking at y as if y as fixed.

Now I am differentiating with respect to x why you can think whatever 2, 3 or whatever in your mind. I take delivery of x square which is 2 of x now y is constant. So, 3 in to y is the coefficient of x. It is just 3 y and the derivative of x is one this is the constant. The derivative with respect is 2 x 0 and minus one is a; obviously, constant derivative is 0. It is 2 x plus 3 y and it you see it is again a function of independent variables x and y.

Now, if I compute del f del y then you would immediately realize that this would become 0 because you because this is now x is been held as constant and then it will become 3 x plus the derivative of y here one and this is anyway constant. It is 3 x plus 1, but you observe here that the del f del y is here in these case is the function of x not even a function of y. So, you have to be slightly cautious about this for example, if you want to find the slightly more complicated function again I am taking it from the book which is the standard text for this course.

I am sure many of you can actually buy it not all, but many surely it is around 400 rupees I guess now when I bought it, but I guess it can and should be kept as a treasure. So, if I want to finds a del f del x here how do I do it? Note, here this is a very complex situation because here you have x both here and in the function cos x y. It is you can you have to treat is as a product of 2 functions why I am keeping it fixed I you can put cos 2 x it does not matter and then you replace wherever the 2 comes as y.

So, basically it will become x the derivative of x into cos x y minus x into the derivative of cos x y which is sin x y and then you have to take the derivative as the chain rule this is the function, so a 2 of x. So, you have to take the derivative of that. So, y would come here up again. So, here it is you see a function of x and y again I take del of del y.

Now, this x is constant it will remain. It is minus x sin x y again derivative of this with x y with respect to y will again bring me an x. It will become minus x s square. So, you see in both these cases I have these both of them to be functions of x and y.

Now I come to a crucial issue, is the issue of chain rule because what would happen if you have more than on a if you have the functions x y being related to some other variables say t or some psi eta we will come them, but let me answer you a very important question before I go. You might think that, what happens if you have a function of one variable and if that function is differentiable we have mentioned that such a function is always continuous.

Any function of one variable which is differentiable must be continuous, I would ask you to find the proof from any where you want, but this is very well known fact. What happens if I have a function who is real, but this partial derivatives exists will that mean the function is continuous the answer surprisingly is not let me give an example.

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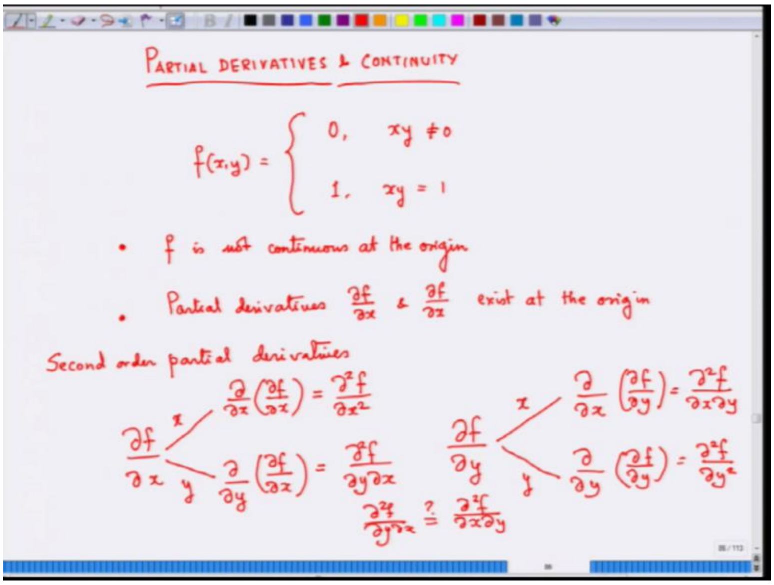

This is this is what I am going to say partial derivatives and continuity that is or feature and then we will move to the issue deducted with the computing of derivatives that is the chain rule. Partial derivatives and it is associated equations called partial derivative equations rules physics and engineering and chemistry whatever.

Let us get into this very important issue now, take an example is equal to 0, when x y is not equal to 0. When the product x y is 0 it is 1. The important part is that I want to figure it out you to figure it out as an exercise that f is not continuous at the origin. See people who have difficulty with these problems can ask this question in the forum and my (Refer Time: 13:03) will answer you back and the last one which is more interesting that the partial derivatives with respect to x and y exists at 00 and may there they are finite numbers del f del x and del f del y this is called the del of not d d x of f, but del del x of f or del f del x.

These symbol the partial derivatives which we had been using I have not mentioned this is called the del symbol this symbol is called the del symbol which will taken out of the delta symbol it is called the del. Del symbol these exist at the origin this is the key fact.

So, what part what conditions you should have on the partial derivatives. That they are the function is also continuous at that point actually they should be bounded by a number which is independent of x and y. That is the answer, let us forget about that and before going into the chain rule let us make certain additional things.

We will now introduce something of second order derivatives, second order partial derivatives. Now you have del f del x as a function of x y and you have del f del y as a function of x y. Now because del f del x is the function of xy you can then takes it is partial derivative with possible with respect to x and then with respect to y. If you take it is partial derivative with respect to x of del f del x then you write it has del del x of del f del x and this is symbolically written as del 2 f del x 2 similarly, you take the partial derivative with respect to y. This is called the mixed partial derivative when you have a combination of a x and y in the derivative.

This is symbolized as del 2 f del y del x similarly from f. Is the partial derivative with respect to x and partial derivative with respect to y. This is with respect to x and this is with respect to y this is with respect to x and this is with respect to y. So, you have here del del x because of us derivative with respect to x del f del y which we symbolize as and pronounce as del 2 f del x del y and this one is del del y of del f del y, that would this symbolize as del 2 f del y 2.

Now if you observe that these 2 are almost similar looking these 2 are almost similar looking things if these 2 are very similar looking things, how do you I mean do. How do you feel that almost this should be same, but in generally it need not be same you have that del 2 f. The question is where are these 2 same this is a very deep result actually in analysis or calculus.

It says that if all of these variables because, now this is the function of x y. These are all 4 different partial second order partial derivatives that you see or functions of x y and then if each of them are continuous as functions of x y then these 2 mixed derivatives del 2 here del y del x must equal to del 2 here; del x del y. Now let us come to the last part about the talk which is about Chain Rules.

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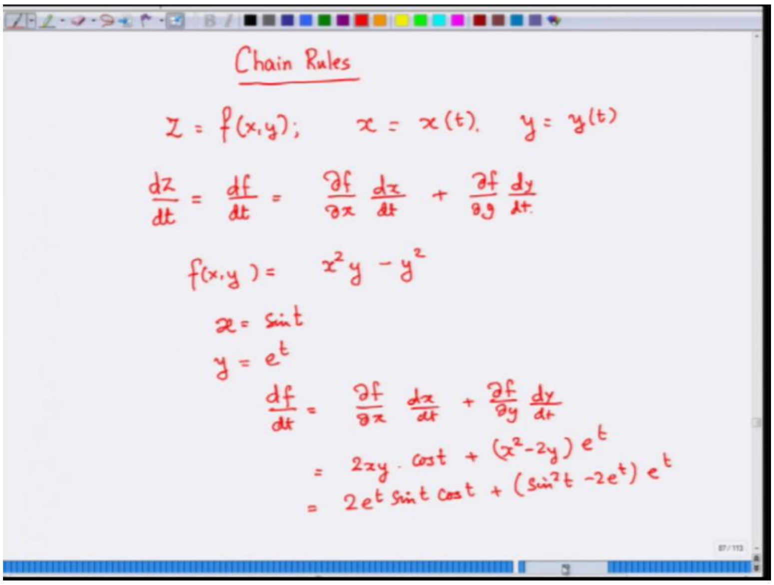

Now we come to the chain rule now suppose you have a function say z is f of x y and x is a function of t and y is also a function of t. So, y is y t I can write it h t g t does not matter just for simplicity I am writing this now ultimately z is the function of t because if I put the actual function of formulation of x with respect to t in place of x and in place of y I put whatever functional formulation y as with respect to t then the function of the f.

So, z now becomes a function of t. So, essentially what is d z d t. I am asking this question. This would be as follows the chain rule is del f del x d x d t. If there was only one variable, then it would have become instead of d f d s d s d t ah

So, because here I have 2 variables I would have add the other part also del f del y del y d t this is this is the first simple chain rule you can have simple examples of this for example, if you have say f x y is equal to x square minus y minus y square then suppose x is given equal to sin t and y is given equal to e to the power t. So, give me a t for example, you know it is the. It is a particle moving on a plain and at every moment time t you are giving the positions of the particle.

It is a problem of mechanics now essentially what you are telling that d x d t and d y d t themselves represents the velocity of the particles right. So, of course, it is not. So, easy to your physical meaning to what we are doing here at this moment, but you can easily calculate out d f d t. I will this is d f d x d x d t plus del f del y d y d t now del f del x you can figure out very simply it is 2 x y into d x d t is cos t plus del f del y here is x square minus 2 y and del f I mean del y d t is e t now you see every there are mixture of functions of x y and t. So, what do you do? It is 2 e 2 the power t sin t cos t plus sin square t minus 2 into the power t e to the power t that is the derivative that is d of t.

Now, we going to go to the second formulation second formulation means, I have z is a function of x y, but then x is again of function of 2 more variables and y is again a function of 2 more variables.

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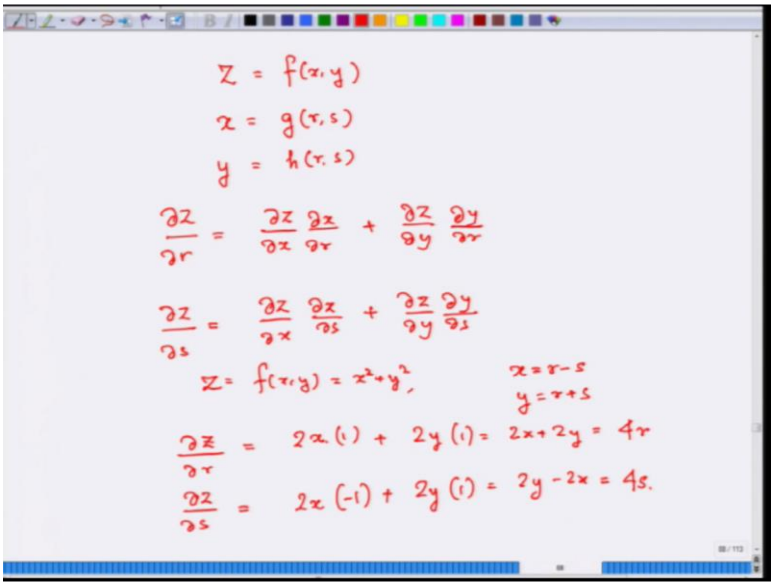

Let me say z is a function of x y and I have x now as a function of 2 variable or an s and y as another function of 2 variables r and s, and now hence the whole is given as z itself is a function of r and s. It is now function of 2 variables knows I cannot talk about del z, del r and I can also talk about del z, del s and now I want to find what are these in using the data that I already have. This will give you del z del x and del x del r. So, you first differentiate with respect to x partial derivative and then differentiate x with respect to r similarly you have to do it for y del z del y del y del r.

Similarly, in this case you have del z del x. So, a del z del x del x del s and then plus del z del y into del y del s that is the chain rule. So, you can take a simple example a z is equal to x square plus y square. So, z is equal to f x y is equal to x square plus y square now I have written that x here is r minus s and y is equal to r plus s then what is del z del r it is del z of del x del z of del x is 2 x into del x del r which is one plus del z of del y which is 2 y del y of del r which is one. It is 2 x plus y 2 x plus 2 y now what is del z of del s del z of del s is you have again you start with del z of del x del z of del x is again 2 of x and del x of del s is minus 1.

Similarly, you have del z of del y which is 2 of y as before and what is del y del s it is plus one. It is 2 y minus 2 x and that is simply the answer of course, you can write x and y in now terms of course, but now I have to express this these are the function of r s del z del r and del z also the function of del z del s also the function of r s. I will put down x as r minus s and y as r plus s and this will become four r and this will become four s that is it. These are simple examples which to illustrate the chain rule.

I hope you have a fairly good working idea about what are these proving etcetera is different thing we are not going to get into that, but let us just get into the just have a look at the working details and with the examples you have to work with it.

Thank you. And we will talk about total derivative the mean value theorem for partial derivatives or mean value theorem for functions of 2 variables in the next class and also about directional derivatives.