Basic Calculus for Engineers, Scientists and Economists

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Lecture – 20
Limits and Continuity of Function of Two Variables

We are back again. We are going to discuss about the Limits and Continuity of Two Variables.

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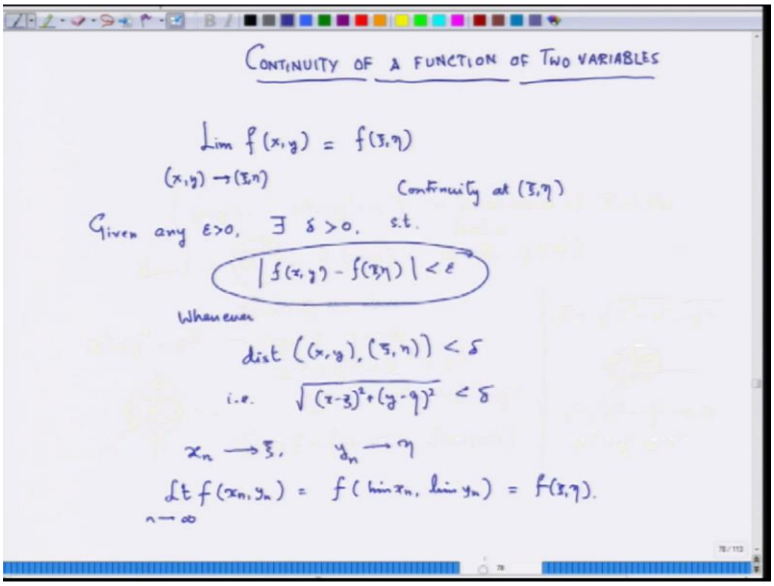

Limit is we are not going to stress too much and finding limits because this whole issue of finding limits will come when you are going to talk about finding to talk about continuity. So, what do you mean by continuity? Again if I go back and try to mimic the definition that I have learnt for real variables, continuity would mean the following, but what does it really mean in terms of the neighbourhood that we have defined in the last lecture. This is crucial.

If I am talking about continuity, my thing would be as follows that given any epsilon, there would exists a delta greater than 0 such that difference between f x, meaning continuity at psi eta. It is basically continuity at psi eta. So, f x y minus f psi eta is less than epsilon, whenever the distance of the point x y from psi eta is strictly less than delta that is root over x minus psi whole square plus y minus eta whole square is strictly less than delta. That is the meaning of continuity.

Of course you can saytherecan be anything in terms of sequence which we have learnt infunctions of one real variable. The answer is that suppose we have a sequence x and converging to x and a converging to psi and a sequence y and converging to eta, then limit f of xn yn as end lines to infinity. So, basically now looking at the sequential limit of the function, this is same as f of limit x n limit y n and that is same as f of psi eta.

Let us go through it again is all as important to go through things which are not so easy to understand. I am trying to now mimic the definition of what I have learnt in functions of one real variable. In that mimicking gives me any epsilon I can show that the distance between the function values is strictly less than epsilon whenever the distance between the two points is strictly less than delta, but these distance not just taking the absolute value because you know distance that we have defined in case of functions of two variables is very different.

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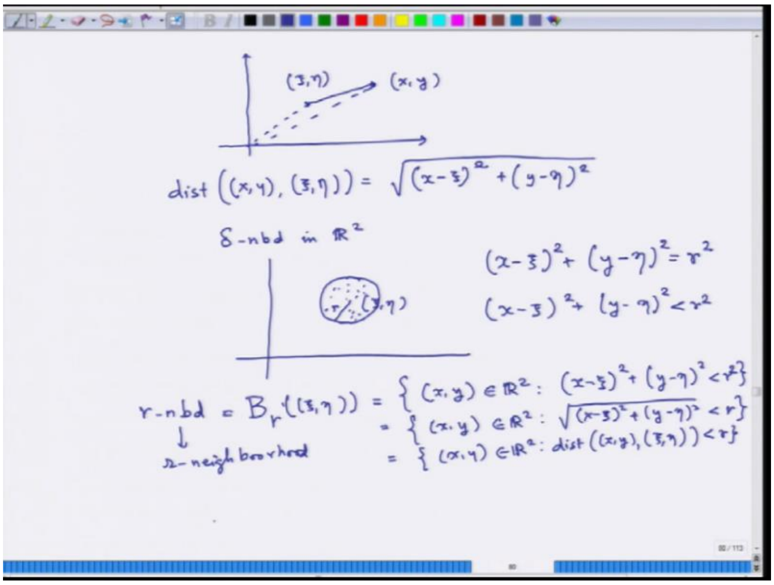

This is exactly the distance. Basically the delta at the r neighbourhood, we are talking about, sorry this should have also been r neighbourhood. I corrected. The r neighbourhood that we are talking about is exactly the distance.

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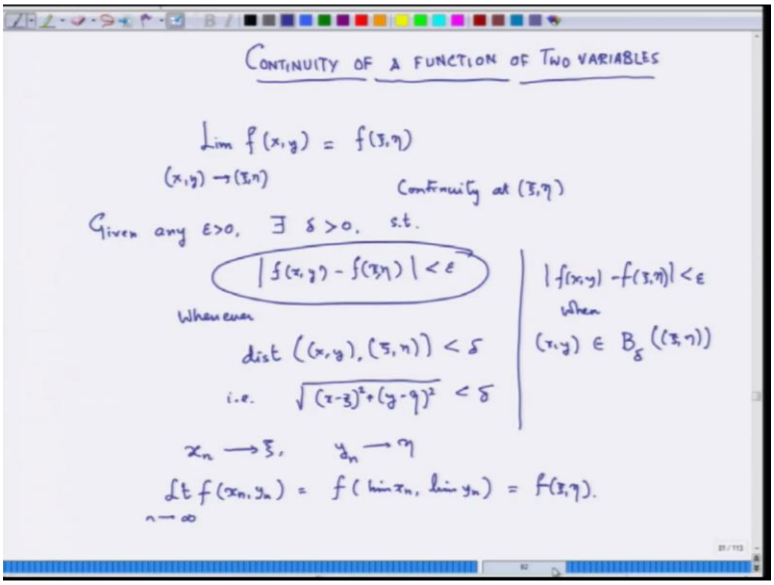

So, whenever the distance is within a given R which means that if I want to write it more precisely, I can write that mod f of x y minus f of psi eta is strictly less than epsilon when x and y is in the ball of radius delta and centred at psi eta is ball, of course means the open value. Just from the definition of the previous day, this is exactly the whole thing. So, once we have done this, it is important to know how to detect continuity of a function of two variables.

Let us know some. Let me tell you one thing. Functions of two variables things are very strange. A function of two variables can be such that if you fix up y, make it a constant, and then it is continuous in x. If you fix up the x that it becomes just a function of y, it is continuous in y, but it may not be continuous at every point as a function of x y.

Finally, we have to understand when you are looking at function of real variables, when you are talking at points of discontinuity; they are only one point of discontinuity here. The function can be discontinuity along a whole curve and things can be quite different and that is a sort of thing that I want to show you right here before we go into any further discussions.

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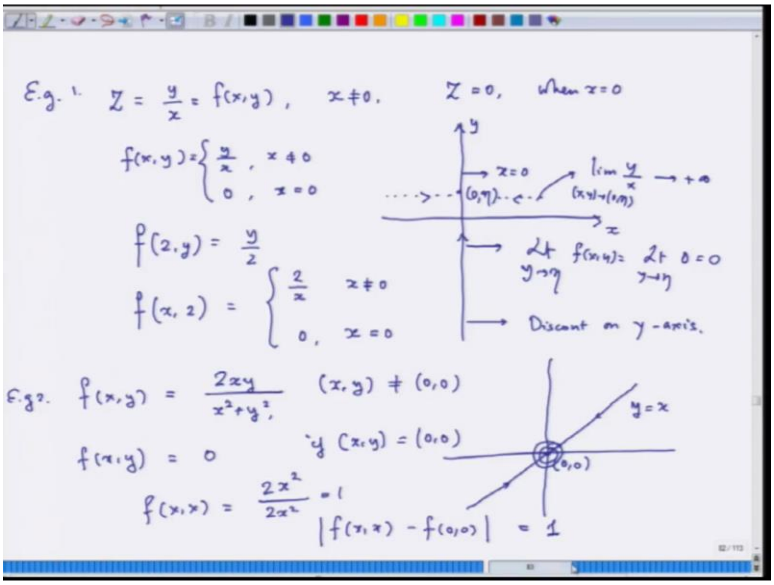

Let us for example talk about the function say z equal to y by x and that is your f of x y. This is an example. Now, this is defined for x naught equal to 0 and we say that z is equal to 0 - when x is equal to 0 or if you do not want to write it in this cryptic fashion, you can write it as f x y as a y by x. When x is not equal to 0, when is equal to 0, when x is equal to 0, do we have continuity. Suppose I fix up the x say I put x equal to 2.

While look at this function f 2 y, what does it mean? Now, it becomes the function of one real variable. This is nothing, but y by 2. Another function of why it is continuous, of course it is continuous. It is obvious just a linear function. So, you see this happens. Now, if I fix my y and vary my x, what would happen? Then, this function is 2 by x, then x is not equal to 0 and equal to 0. When x is equal to 0, this function becomes this continuous as a function of x.

Now, let us look at this function y f x y equal to y equal to x. This is discontinuous only at 0 at the point 0 say when I am fixing up y and allowing x to move. Now, if I look at this function f x y, take all the points x y, all or take the linewhere you have x equal to 0 means basically the x axis is the y axis. I am through this line on y, you have x is equal to 0. So, why this line is same as x? The line x equal to 0, if you are taking x equal to 0, but I will not tell you is that along this line x equal to 0, this function is discontinuous. It does not have when your left continuity and right continuity, it becomes different. So, can you try to prove this fact?

Take any point here on the boundary which is 0 and say eta. Then, what you can do is, you can move from this side say suppose you can move this side which every way you want and you see if you move from this side, you have negative values of y. Now, suppose you are approaching these two parts. In these parts of course now at these points, if you take an x is not 0, right so you take this point, these are the points. So, at this point x is not 0. At these points, x is not 0. It is y y x.

At this point, y is negative and here x is negative. It will give me the positive thing. It will give me the positive thing right now. The limit would be same, but now somewhere I approach from these directions to 0 and you are approaching from these direction. See why approach on these directions? I have x 0. So, my limit is actually 0 0 0 0 0. If there is only 0 on this line, but when I am approaching 0 eta from these directions, the limit would be finally positive. It would actually blow up because you are bringing x towards x towards 0and you are bringing towards eta. These limits would actually blow up.

Then the left limit and when the two limits from two different paths do not. If you come by these particular path, then limit y by xas y x extends to sorry as xy extends to 0 and eta, this will blow up. This will just shoot up while if you are coming on this line here because x is always 0, function value is 0; limit y approaching eta because x is always 0, this function while f of x y is actually 0 because x is 0. Limit y approaching eta 0 is 0. So, when you approach through 1, the function will close up through approach.

Further approach should have taken the approach to y axis, the function value gives us 0 that the limiting value is 0. So, you see the limit does not really exist at 0 eta. The limit does not really existing at 0 eta. It is 2 for any point on this line. So, you cannot talk about continuity along this line any more. These function f is discontinuous was on the x y axis.

Let us look at any other function like this f of x y is much easy to prove the function is not where not continuous was an actually proving continuityand f x y defined as 0. If x and y is equal to 0 is just an origin. So, when y is not equal to 0, it does not matter whether x is 0 or not the function value because even x is 0, the function value is 0; then even when y is 0, the function value is 0. So, when I fix my x and change my y, I get a continuous function and when I fix my y and change my x, I get a continuous function, but the fact is that this function does not have continuity at the point 0 0. It does not have continuity at the point 0 0.

Why you think that this point does not have continuity at the point 0 0? Just look at the line approach. It approaches 0 0 through this line. The line y is equal to x and you put y equal to x here. When you put y equal to x here, so what would happen? So, f of x x would become 2 x square and this will become 2 x square f x x value is 1. So, difference between f x x and f 0 0 is 1. This cannot be made arbitrary small given any epsilon. I cannot make it as small as I like even though I can have points on y and x which are arbitrary close to 0.

You see if I take a point here, it take a small ball around here as small I can make the radius smaller and smaller, but I will still find points on y and x in those intervals points which are lying on the line y equal x, but the function value cannot be made less than1. This is a very crucial example which tells you that you see that when you are fixing x, it is not continuous. It is continuous that over the whole domain does not matter. What is your x and y and whatever x you choose, once you fix y, but it is not true that this will be continuous at 0 0. That is the interesting fact.

So, as the function of y if I fix the x for example, this becomes continuous, but at the point 0 0 when I am looking at the function of the function of two variables, it is not continuous because of this very simple fact that when I am putting y equal to x, it comes out to be this and this is the distance though points on y will bethe line y equal to x is an arbitrary close to 0 0, the difference between the function values cannot be made arbitrarily small which violates delta types definition of continuity. So, you see is also important in this context to learn if this epsilon and delta type definition of continuity.

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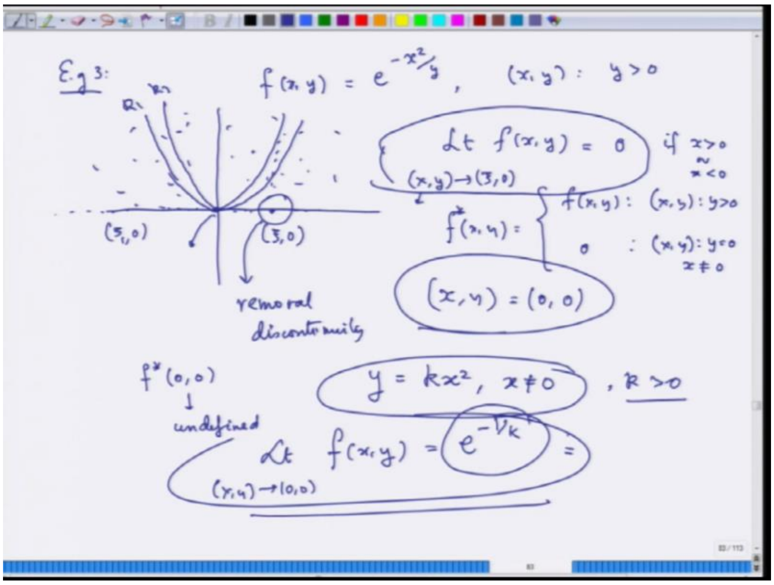

Now, let us look into another example. For example, the last one for this class we shalldefine let us consider this for all x y with y strictly bigger than 0. We can say what would happen if y is negative, then it will become anyway positive and e to the power x is defined when y is strictly less thanyou can do for that. We are just considering this function taking y strictly bigger than 0.

Now, if I take a condition, now let me take a point. I am essentially looking at this function value for all these points except this line. I am not considering y is greater than 0. I am only considering these points. This is what I am making my domain to be I have defined these as my domain or which I have defined this function. This boundary again on this domain, you can define this function on the lower half plane. On the upper half plane, you have this and then what is important now is to observe.

Now, it takes a point here say psi 0. Basically you are now looking at, suppose I look at the limit of f x y as x y goes to psi 0, what would this limit be because when this will go to 0, this whole thing is going towards infinity. It is e. This whole thing goes towards e to the power minus infinity, but when x goes towards minus infinity or t goes to a minus infinity e to the power minus t actually goes towards 0. This limit is actually 0. So, you can redefine the function. So, just like in the case this is called a. This is case of removable discontinuity point.

In this case because I know this is the limit, I can say that f I define f star x y is equal to f of x yfor all x y with y greater than 0 and is equal to 0, for all x y with y is equal to 0. Of course, here also you have to take x naught equal to 0 because here on the line, but at the point 0 0. They of course here x x cannot be 0 because you have 0 0. So, when I am taking the limit here of course taking x to be positive number, psi is a positive number and I am approaching it from all these sides where x is a non-0 number.

Here my psi is positive. I could take it here also say psi 1 0, then I can also approach with x negative from this side. So, x is either negative or positive. It cannot be 0 and then only you have the limit. This limit that I am circling now is true if x continuous to remain non-0. So, basically here is what you are doing. This function now can be redefined for all points on the boundary. You can remove the discontinuity and define it like this. So, only at the point 0 0, this function remains discontinuous because this function is defined at that point.

So, you see even is very interesting that on the whole line which the function does not seem to have a, but we are not defined it on the whole line. We can redefine the function on the whole line, but still they will be this origin point where the function is not defined and this is not continuous, but except the point 0 0. We have now made this new function f start continuous all about. So, you see slightly sticky and when you go through the examples and homework, you be very careful in handling these things. Always observe this.

What happens now? Why do I say that f that f star 0 0 is undefined and this is also discontinuous. Why? It is because suppose I know approach the origin through this parabola, I am approaching the origin through this parabola for positive x. For x now non-0 for x 1 0, I am having y non-0, right and then if I go through this.

Now what is f of x y? In this case when I put y as this it is e to the power minus 1 by k, so this limit for various case is just e to the power minus 1 by k. If I take the limit x and y, then this whole thing is on approaching 0 0upward. If this approach is 0, y is approaching 0 k, I can take positive for the (Refer Time: 21:02). Then for every different k, every different family, every different path that you take, this is say for k1, this is for k2. Every different path that you take, the limit is different. This will be the limit.

So, because the limit is different, there cannot be any limit obtained that f 0 0 and the limit is not 0 0 either. In fact, the limit does not exist as you cannot have a limit. This limit itself does not exist because for every different k, for every different path, you are having the different limit and hence, the function is not discontinuous. It is not continuous at 0 0.

Thank you very much and we end our talk today.