Basic Calculus for Engineers, Scientists and Economists
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Lecture – 19
Functions of two or more variables

Welcome once again. We start our 4th week course, already we done three weeks for this Calculus course hope you are enjoying it. Of course, in 20 minutes we are just trying to give the main glimpse of the ideas. However, today we want to mention that we are going to make a paradigm shift today, from Functions of one real variable to Function of two real variables.

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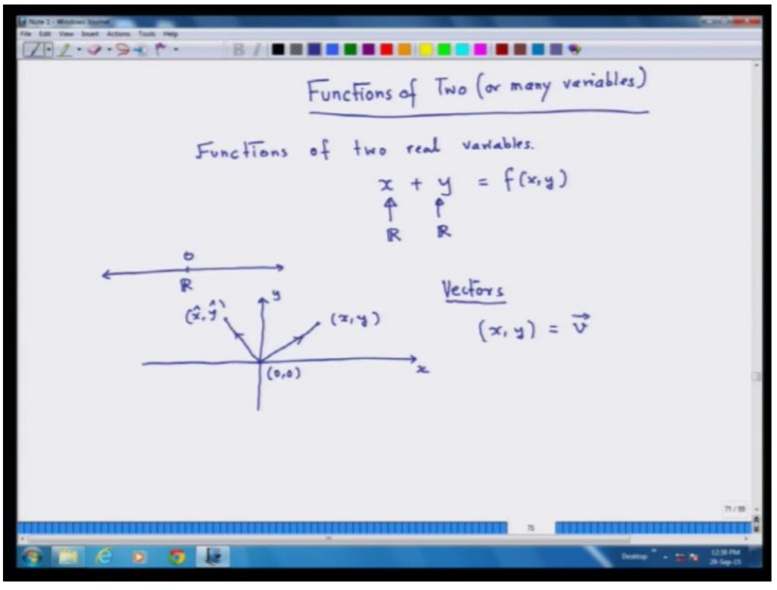

Many variables are just an extension so we do not have to bother about it. We talk about essentially Functions of two real variables. You might be wondering that this is something very special or very artificial thing and the function of one real variable was an natural thing, but I would like to convenience you that, the function of two variable is quite a natural thing because for example, when you add two quantities which you have studied in our child hood, when you add two quantities x and some y you are essentially performing our function of two variable operation, where x is some number and y is some number when you are adding it and let us call that addition F of x y.

And hence, this is an example our function of two variables, so this is a real number coming from R and this is another real number coming from the real line that is all. This addition is something which you learnt at childhood. So, when you start your childhood you essentially start with function of two variables and not with function of one variable your childhood in school mathematics essentially talks about that in that disguise which you of course do not know.

See in function of two variables my domain as suddenly shifted from the real line R which was my earlier domain is denoted by the number line to now the coordination plane with the x coordinates and y coordinates. You have to understand if I take an x y point in the coordination plane then of course from 0 again join the point with the origin 00. If you look at this line not only as a line, but it as a particular direction corresponding to this given x y because if I change my x y to say at some other point I can take x hat and y hat, then the line that joins this point x hat y hat 200 as a different direction.

Such quantities which have both magnitude and direction are called Vectors, and you must have studied some of them in physics. So, that is why function of two real variables are sometimes called Vector functions, real functions of vector variables this x and y together represent some vector V. Now, after giving a few examples of functions of two variables we would now also trying to talk about what is the meaning of limit and continuity when you are talking functions of two variables.

When you are learning calculus the paradigm shift is from the function of one variable to the function of two variables, lot of stories which occur in the function of one variable does not occur in the function of two variables. Once you know about two variables you can just keep one adding more variables to get the same story what you just have to write more thing do more algebra. But otherwise, the real conceptual clock lies in the change from the real line to the complex plane.

The biggest interesting point in the from the real line the complex plane is a loss of well ordering in the sense that if me give me to real numbers I can always say whether this now given at number a or b I can always say whether a is bigger than b or b is bigger than a or they are equal, such a thing cannot be said when you are in the plane. So, let us take some examples.

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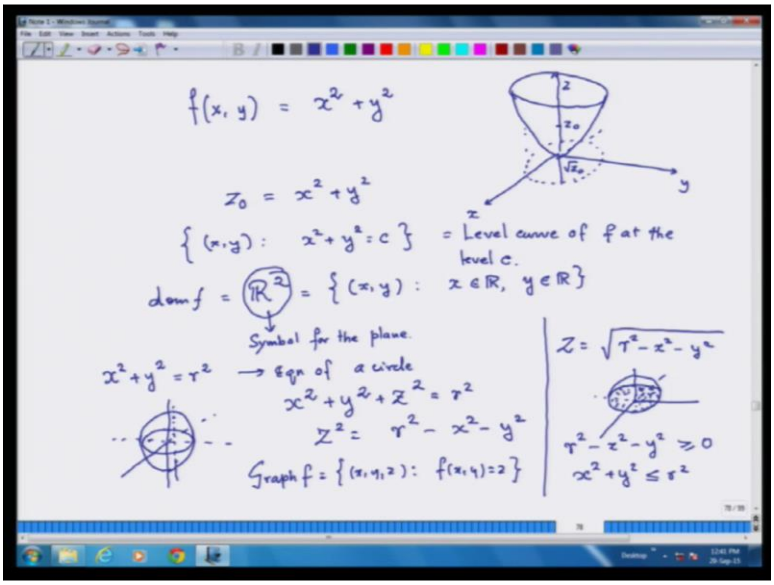

The example is - for example, I think there are some jumbling of papers is rest we corrected at the end, but just follow the talk. For examples, let I do generalise my idea of a parabola and then what you get is called a Paraboloid at a 00 it takes a value 00. Of course there is nothing it is like a whole inside, it is like some sort of a bowl. An important role is played by something called Level curves. For example, it fixes a value of z naught.

Basically, I want those x square and y square which gives me the value z naught. So, what will be those x's and y's. Those x's and ys would be those x and ys on the plane for which x square plus y square is equal to z naught, or this is a circle with radius so this has to be a circle with radius root of z naught. These curves, that is for examples set a all x and y because that x square plus y square equal to sum c this is called the Level curve. Level curve of f at the level c, they are quite useful in many many contexts.

Now you might ask me what is the domain in range of a function of two variables. Like we are task spoken about domain and range of function of one variable - the domain and range of function of two variables, of course has to be done according to the structure of the function. Here for example, it does not matter whatever x y I put in I can actually compute x square plus y square, it does not matter what is x and what is y and the values that I will get are only positive quantity.

In this case we will write that domain of f for this particular function is whole of the plane R 2. The plane R 2, the standard Cartesian plane is the collection of these coordinates x y, where x is in the real line and y is in the real line and both are real numbers. This is nothing but my standard plane which you are did it with it, its symbol is R 2 that is essentially telling that Cartesian product of R with itself. There might be some people who do not know what Cartesian product forget. So, plane is a symbol for the plane. Once this is done let us take another example.

For example, we have known about a circle. So, how do you write? This is the equation of a circle of radius r. What about a sphere, how do we write the equation of a sphere of radius r. So, sphere of radius r. Can I represent it by the function of two variables? See you know what as sphere is at the end of the day; this is our standard object that you are seen. For examples, a cricket ball or football ball, these are spherical objects. Even the earth is sometimes thought to be spherical and most studied you considered it to spherical, but though it is really not with.

So, how do I write? You see I can always write z square as r square minus, x square minus, y square, and if you take the root that will be plus and minus. So corresponding to two values of x y and one given value of x y their two values of z. Technical it is really not a function, because give me an x y I am corresponding to and I draw up perpendicular through it it will cut the sphere at two points. If you take any x y and try to draw perpendicular through it, it will cut two points. This is very simple thing to understand. This is really not a function, but we can make it into a function by if I just write z equal to, just take the positive root and write.

Basically, I am in this function actually the graph of this function. Of course, you can ask me what is the graph of a function of two variables which is two symbol that I have not even mentioned it, that is collection of all x y z because you will now been three dimension. If, want to talk about graph of f, is the collection of all x y z, such that f of x y is equal to z. Here, the graph of this function would essentially be the hemisphere, upper hemisphere, and if you take minus sign it will be the lower hemisphere. We are in the northern hemisphere.

Now here if you look at this thing very carefully, does it mean that this function is defined on this function, if I put any x and y it will define the answer is no. Here, the question comes and what is exactly the domain. Where as in this particular function you should always have r square minus x square minus y square must be greater than equal to 0, or x square plus y square must be less than equal to r square.

For this function to be defined, that this function z to be a numerical real value one must have always that the functions of that this x and y must be on and inside a circle of radius r. So, if this is my circle of radius r, so I can only considered x and y from this region. Any x and y which I take outside this region will immediately make this a negative number and we are really not talking about complex number here because we want the real quantity because we want to represent it in the graph, you want to represent put the point on the real line on the z axis.

Here, as you observe the three axis which we always draw x, y and this is the z axis. So, there are 3 axis. It is not so easy to visualise things in three-dimension because in three dimensions you are essentially looking at things standing in your own dimensional. We are in the three dimensional space where we have length breadth and height. But it is always to view things in two dimensional that is why functions of real variables are so easy to study, because we are viewing the functions in two dimensions.

Essentially because, we are in three dimensions but when once it comes for three dimension things become slightly difficult, so that is why I do not know solid geometry course are almost abolish from the school level but it is a very very good thing that if somebody as an understanding of solid geometry he as the much more better understanding of functions of two variables.

For examples here, the domain is of course the circular zone and for these particular functions z is equal to root of r square minus x square minus y square, and it gives me real numbers. Its co domain is of course, you can say whole are all 2, whole of r, but its range of course is not the whole of r, its range is of course not the whole of positive numbers is range where is from 0 to r.

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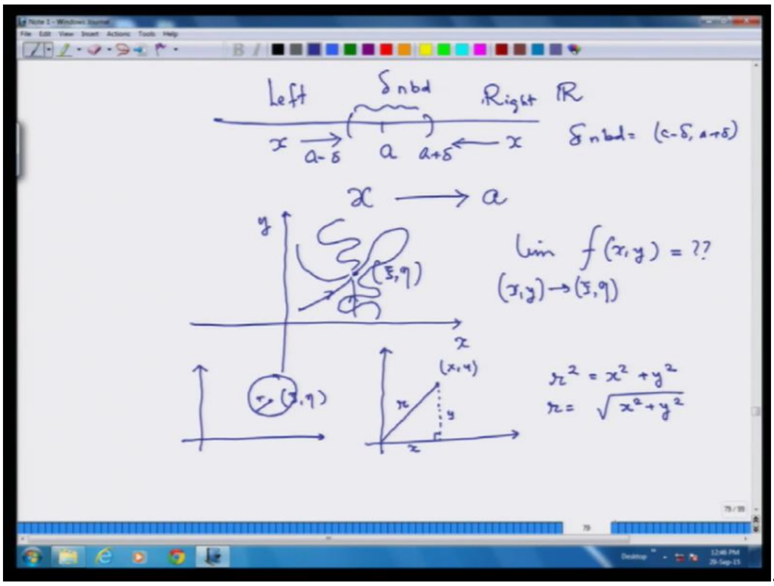

Now, we are going to end today’s talk by talking about what is the meaning of limit. How do you talk about a limit in R2? We will compute them tomorrow limit continuity those issues will come tomorrow. But, it is very essentially that you understand what is a meaning of taking a limit. If you look at a point a in the real line, is a real line and I am talking about x approaching a, so what does this mean? It means, here I have only two degrees of freedom that is x can come either from this side or x can came either from the other side. Means x can come either from the right side which I can right as right and or it can come from the left side which I write as left. So, any of the two ways there is no other way. But, this is exactly the paradigm shift I am talking about.

When you come to a function of two variables and you talk about finding a limit of x and y when x and y are going to some psi and eta, and you ask the question what is the meaning of this then the this whole thing immediately takes you out of different kind of conceptual regime. It might be in the schema of your things initially, but it is a very very important thing that you have to get used to and you have to understand this in clearly.

This is your point psi eta, and I want to say that x y is approaching psi eta. What is the meaning of the term x y approaching psi eta? That is the crucial thing. So, x y can come to it like this from a straight line, or you can approach x y, approach psi eta along a curve and approach psi eta along this curve, and approach psi eta in vary complex ways.

There are infinite ways and this is uncountable infinite ways just like the uncountable infinite of the real line its uncountable infinite ways that even approach psi and eta. When I am talking about existence of a limit here, I am saying that whenever I compute the limit I should compute through all possible paths. And you know that practically when you thing about theoretically the difficult thing, how can I take a limit or all possible path because I cannot even enumerate all possible path there is no question where enumeration, it is uncountable infinite. So, what I am supposed to do?

This definition is not very clear if you just write that limit of f of x y is equal to l. Once you want to do that then you will see the power of epsilon and delta concept. So that is why now first I have to define what is a meaning of a neighbourhood of a point, just like we have defined a delta neighbourhood of the point a. For example, a minus delta and a plus delta, so it is a open interval around a on the real line and we call this open interval delta neighbourhood, n b d short for neighbourhood. For a real line delta neighbourhood is equal to a minus delta to a plus delta. So, what do I mean by neighbourhood here? That is a very very important concept.

For us, for all our studies neighbourhood who do mean the following; that, if I had taken a point psi eta, a neighbourhood of a given radius r is a circular disk of radius r around the point psi eta. But I am taking everything inside the circular disc but not considering its boundary. To understand what is this we have to know how to compute the distance between a coordinate x y and the origin and this is a fundamental importance.

How do I consider the distance? You can drop here a perpendicular, so this would be x and this would be y. The distance between them by the Pythagorean Theorem this is say my distance r, then r square is x square plus y square, distance is r can be written as root over x square plus y square.

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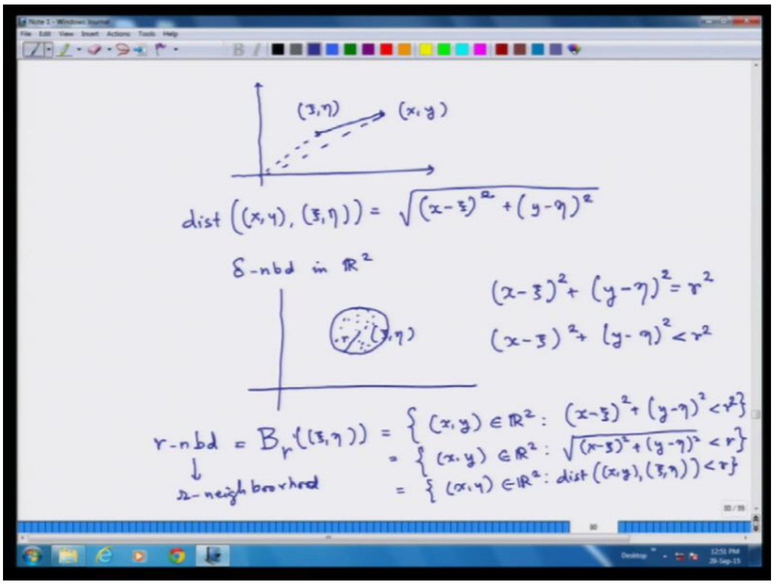

Now, what about the distance between two more points in the plane that is more important? Suppose, this is x and y and this is psi and eta. The distance between x and y, I am writing simply dist is our short form of distance between x and y comma psi and eta is equal to root over x minus psi so you are subtracting the x coordinates from 1 and then y minus eta whole square. Now how do I now define our neighbourhood again? So what is my delta neighbourhood in R2? See, if you take the point psi eta, essentially you have drawn a circle of radius r.

What is the circle of radius r? We have everything here, but not anything on the boundary; a circle of radius r which centre at psi eta is given by these definition, this equation rather which you know from your high school. But I do not want things on the boundary; I want things which are strictly lesser than that. Basically, I want to collect all x and y such that these hold. The delta neighbourhood is usually given by this symbol B delta at psi eta delta neighbourhood.

It collects all x and y from the plane which satisfies this formula, which satisfies x minus psi square plus y minus eta square is strictly less than r, which is same as x y element R2 root over x minus psi whole square plus y minus eta whole square strictly less than r which is same as x y is in R2, such that the distance between the points x y and psi eta must be less than r. Sorry, this is not delta; delta neighbourhood this called Odd neighbourhood, because my radius is this delta actually talks about the radius so I have taken here r so its r neighbour. Let me just write the r neighbourhood.

See it consist of all the x y which are not on the boundary of the circle these, but they are inside the boundary. So, it has all points for which they are not equal but strictly less. So I collect all those points and then that points are called the r neighbourhood, sorry I just mention the delta neighbourhood because of talking about the real line just few minutes ago. So that is same as this we would take the root, so if you to take the root this will become r not r square.

Then this is same as telling from these definitions that the distance between these two points are strictly less than r. So, consider all x y not to such the distance between x y and psi eta must be strictly less than r, so that is called a Neighbourhood. Now once we have these definitions we can then speak about limits and continuity which we will start in the next class, and we will do it through the epsilon delta definition.

Thank you very much.