Basic Calculus for Engineers, Scientists and Economists
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Lecture - 18
Riemann Integration-2

Now, we come to the second part of Riemann integrals. Here we are going to ask you some stupid looking questions.

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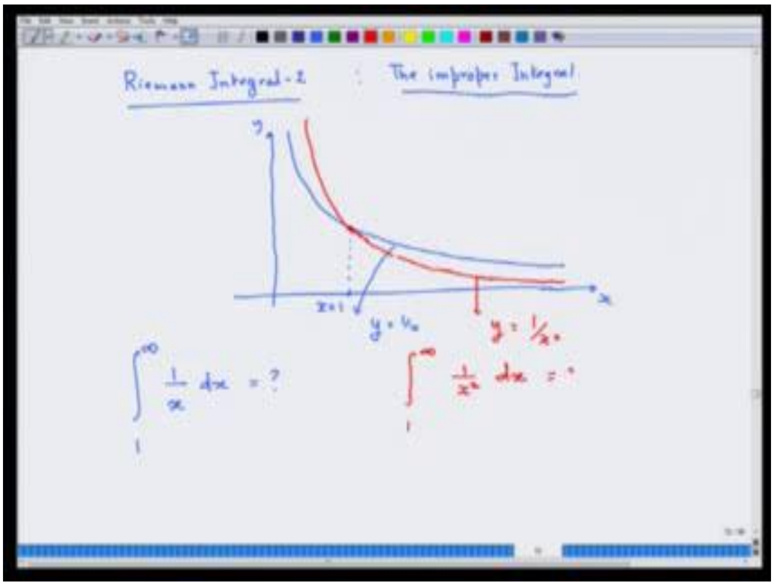

I would like to say take the graph or look at the graph of the function y is equal to 1 by x. And then also look at the graph of the function y equal to 1 by x square, which coincides which is value at 1 and then goes down. If I ask you to tell you that, find the area under the graph of y equal to 1 by x from 1 to infinity that is find the area which on the right hand side of x equal to 1, and lying under the graph of the curve y equal to 1 by x, and find the area of the graph of 1 by y equal to 1 by x square.

Find its area, again on the right hand side of x is equal to 1, and all the area lying under the curve. What do you mean by this? So, essentially I am telling you, can you compute following integrals, do these integrals have any meaning.

Let us look at in more intuitively this whole idea. If you possibly put in two fingers here, and then you want to; if you put a ball or a ball which is rolling into the part under the graph of y equal to 1 by x. So, you see you can keep on ball over say have very small radius, it can keep on rolling for certain time. And still stay under the graph of y equal to 1 by x, but that would not happen with y equal to 1 by x square, it will soon come in hit the wall, it hit the graph.

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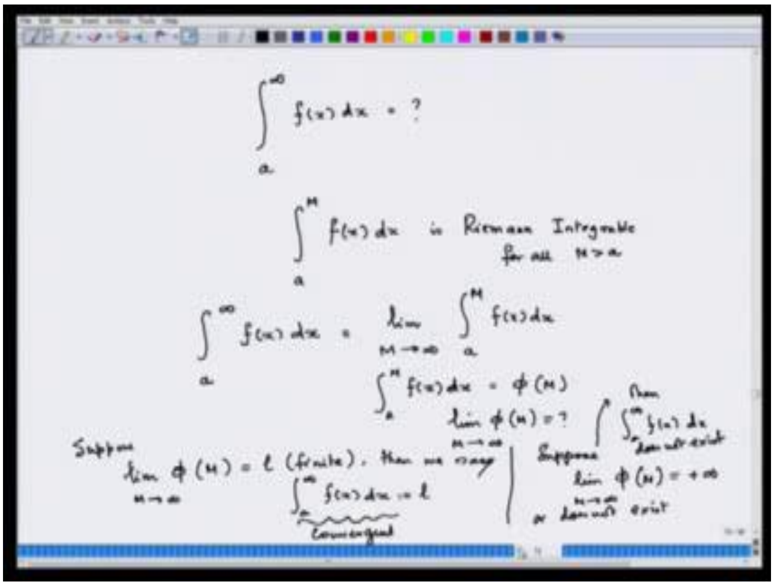

In general, I am asking how you will come back to these two things. In general, I am asking what is the meaning of can I attach any meaning to this. If the function is nice, so over the given interval, so if say from a to any M which is finite, I can say that ok think the function is nice continues or even bounded on that and Riemann integral for every M that you take.

That is suppose I can compute it is this thing f(x), this is Riemann integral or if you what is not very comfortable with this Riemann integral business at this moment just take f to a continuous function between a to M. For all M bigger than N. Suppose, I can compute to this very simply then what should I do, then a sensible idea might be to define this one has limit of M rushing towards infinity, and then calculating basically I am calculating a to M f(x) dx. What I am doing is that for every M, I takes take an M bigger than a, and then take another M bigger than that and keep on increasing the value of M n keep on computing this value of this integral and then take the limit of the values or the sequence that we create rather.

So, of course, you can get a sequence M 1, M 2, M 3, M 4; but in general, you can just write this whole thing would become a function of M, this integral a to M, because M is my unknown number, some number. This is nothing but a function phi of M. And basically I am trying to look at the limit of phi M as M tends to infinity.

Now suppose that limit phi M suppose that limit of phi M is finite, say it is l, it is finite then we say integral a to infinity f(x) dx is equal to l that is the way you define, this is a definition. And then we define the integral from a to infinity as l. Do not get into any philosophical issues here. These are limiting concept how do you say it is exactly the area and all those things. Then we define the integral as this and we say that the area is exactly l, so then we said because we already define the integral as the area. So, we will continue to define for this case also as the area and say that the area is l.

Suppose and then this case, we say that this integral is said to be convergent, when this limit is finite. Now suppose limit of phi of M does not exist, it is a plus infinite or does not exist, we cannot compute it; we do not know limit does not exist. So, you are left limit as M bigger you does not go to M, where does not go to any number rather goes to infinite or does not settle in some number just keeps on oscillating. Then we say then if there is a case then integral a to infinity f(x) dx does not exist, or sometimes it also diverges. Of course, when limit M tends to infinity phi is plus infinity, it diverges or it does not exist.

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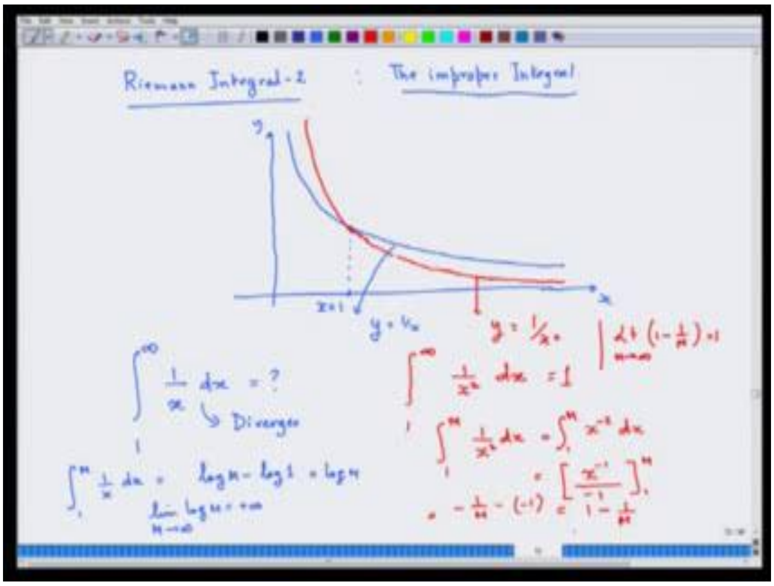

Now, let us go back to the previous thing, let us try to compute this integrals. Let me just now first compute integral 1 to M 1 by x dx, and this because it now it is log mod x, but here the in integral is log mod x 1 to m, but here it is basically log M minus log 1, so it is log m. Limit of log of M, if you remember the graph of the logarithmic function as capital M tends to infinity that will take you to infinity. So, we say that this one, this integral diverges something amazing is going to happen here.

Let us see what happens here. Let me now compute. It is 1 by M x to the power minus 2 dx. It will become x to the power minus 2 plus 1 minus 1 by minus 2 plus 1 minus 1 - 1 to M. This will become minus 1 by M minus minus 1, so that will become 1 minus 1 by M. Now limit of 1 minus 1 by M as M tends to infinity is known to you that is 1 and so we can say that the area under the curve y equal to x square from 1 to infinity is exactly equal to 1 that is slightly amazing. You see that you cannot you can just swim inside 1 by x, but you cannot swim inside 1 by x square, the graph would push you to down very well.

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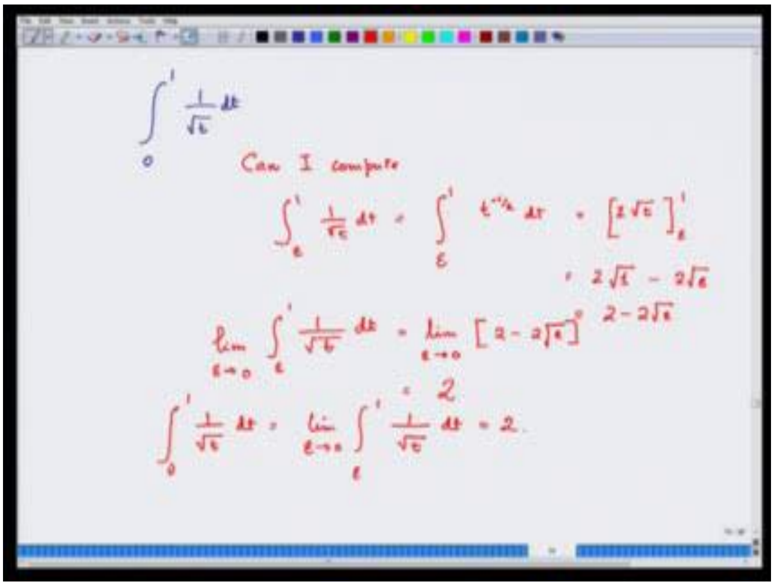

Now, for example, if you look at a similar sort of thing take another example that was our goal that we will look into things through examples and that makes a lot of sense. Now for example, now we look at integral, say this particular integral, what it will give me, what does it give me; it does not give me anything. Strange I would wait, wait, wait is this improper integral, yes, because though it does not have infinity anywhere it has this bad situation as function not been defined at 0.

So, how will you get across this situation, what will you do here. You can say ok, let it not be defined over 0, so question is can I compute 1 to epsilon - some epsilon 1 by root t dt let me check it out So, epsilon to 1 t to the power minus half dt. So, you know what is 1 plus minus half, it is half; and 1 plus it half's. It is root t 2 root t epsilon 2 1. It is 2 of root 1 minus 2 of root epsilon. It is 2 minus 2 root epsilon.

Now, I will make this epsilon smaller and smaller and smaller and smaller, and then I will try to find the limit what happens if when epsilon goes to 0. This is another type of the improper integral. This we into basically then we are computing the limit of and as this that epsilon is going to 0, this is 2 into 2 root epsilon, you immediately see that the answer is 2. Now then we can define that 0 to 1, 1 by root t dt is equal to limit epsilon going to 0 epsilon to 1 1 by root t dt, and this is equal to 2. There are harder examples which we will not get into, but for example, where do such things come up.

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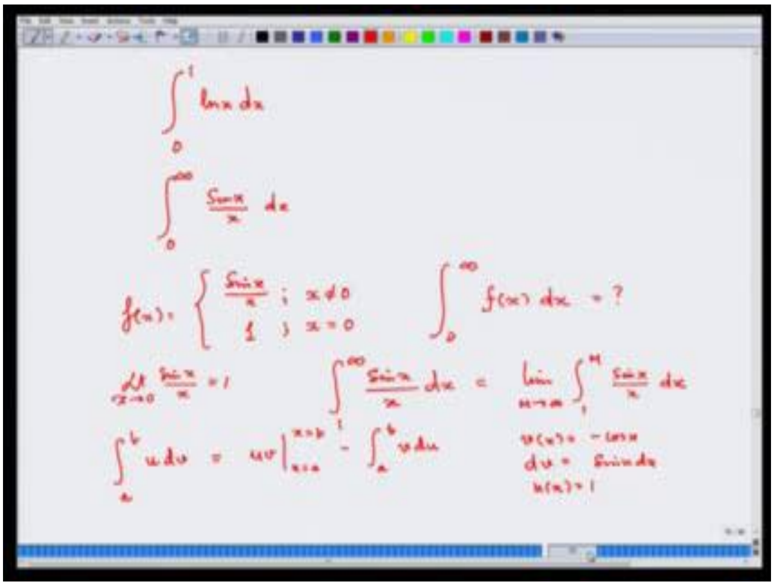

For example, I would give talk about an example this one, 0 to 1 log x dx. This is an improper integral, because log 0 is not defined. So, how do you how would talk about it, there are other improper integrals for example, if I can talk about 0 to 1 or 0 to infinity sin x by x dx. So, you say it also bad at 0 as you know I can manage at 0; I will define the function f (x) is equal to sin x by x when x is not 0. And define it has 1, when x is equal to 0.

Now you see because you know that limit of sin x by x that is x tends to 0 that limit is one then if I define the function f(x) like this then the function actually becomes a continuous function. So, basically I am now looking at the integral of this function, and I am asking whether this as this is convergent. So, how will I go forth looking about this function?

Let me just do a trick. Let me now I know that the function is continuous from 0 to 1. It is a continuous function, it must be bounded between 0 - 1 and hence I can find the area under the curve, because continuous function of the integrable, every continuous function is even integral. Now the fun starts, 0 to 1; even take 0 to k, also any positive k, does not matter, I am just taking 0 to 1. So, essentially I have to see whether I can compute this part f(x) when x is not equal to 0 is actually having this value really can I compute this part. Now what does this sort of writing rings are well in your mind, how to compute it. So, you would say ok, let us do this, great.

But essentially how would you compute the indefinite integral of this that has to be done through integration by parts. So, again if you are taking definite integrals, integration by parts u dv is equal to u v x equal to a to x equal to b it is the way you write the integration by parts a to b v du. Here you can understand, you have sin x into dx. If I take f(x) is equal to minus cos x, if I take my v(x) is equal to minus cos x then dv is equal to sin x dx. So, u x is here equal to 1.

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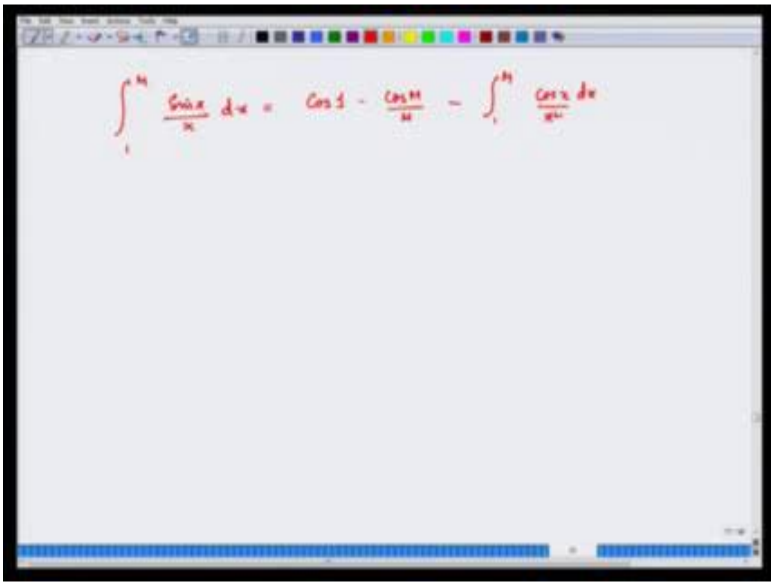

Now if I just use this formula and get the integration done in integral that is 1 to M sin x by x so 1 by x is u, so it is u v, and cos x is minus cos x.

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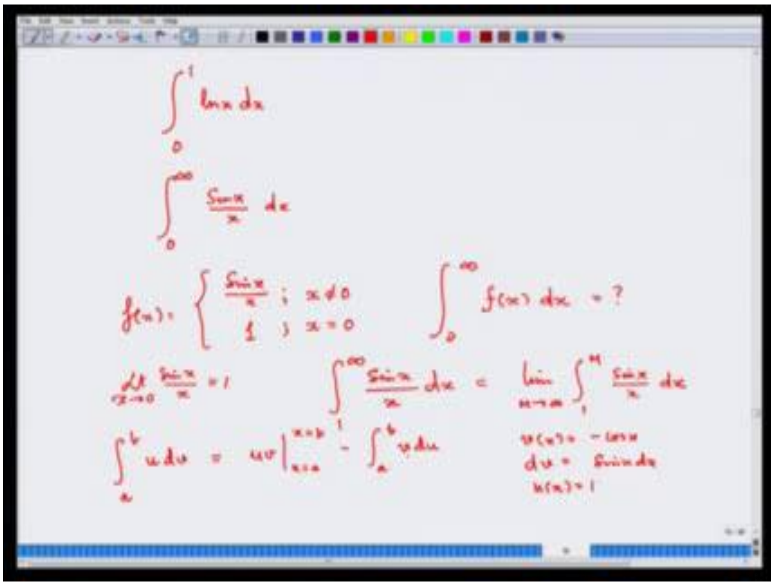

Finally, you will have this formula which I request you to calculate yourself. It is at the end u dv then v du right, so u dv.

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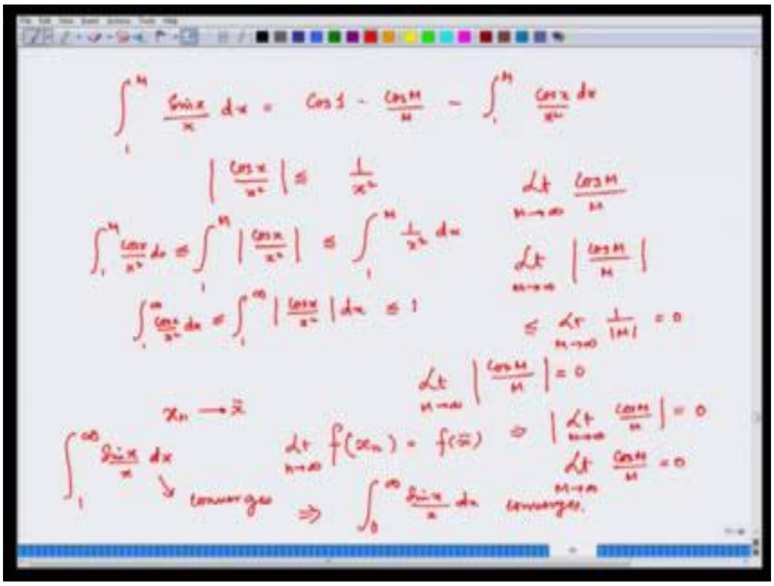

So, your u here is 1 by x right, and v here is minus cos x. So, you immediately see how to calculate it. Once you get there, what do you have? You observe that this function is less than because mod cos x is less than 1, it is less than 1 by x square. Integral 1 to M, so this is you know, if I take the limit as M tends to infinity, I can do something with it. This is a number, so cos M by M, this will be 1 by 0 form and you can take the when you take the limit here; you know that this will always happen by Riemann integration.

Now you know that if I take the limit of this, this limit is 1. So, 1 by 1 to infinity mod cos x square dx is less than equal to 1, and which means again by using a cos x by x square dx is less than equal to 1, because integral of f is less than equal to integral of mod f. I will take I will basically I can write if we want it I can write it like this. So, which means that this integral is convergent and what about this limit, limit cos M by M has M tends to infinity cos M is a bounded function.

Let me compute the limit of M tends to infinity cos M by m. Now, this says same as taking the limit of M tends to infinity 1 by mod M and this 0. Now you know that limit of M tends to infinity cos M by M is 0. Now what is this limit, observe that we if when you have a continuous function, we have learnt when you are talking about continuous function, so if x n is a sequence and you take any sequence x n going to x bar, and the function is continuous at x bar.

Then if you take the limit of x n, you will have f of x bar same as f of x bar, whatever sequence you take going to x bar, this the meaning of continuity. Now which means I can actually take the limit inside, so modulus as function or absolute value function is a continuous function, so I can take the limit inside. It means limit M tends to infinity cos M by M is 0.

Now absolute value of something is 0 that thing must be the distance from 0 is 0. This thing must be 0, wonderful. In the limit this becomes 0. And this also is convergent though I cannot tell you the value. I can say that this function, so this integral one to infinity sin x by x dx converges. And once this converges, I just have to add it with 0 to 1 sin x by dx, which is the proper integral whose area I can find then the whole integral converges which implies 0 to infinity sin x by x dx converges. This was a lit flash on analysis for you, I think we would not get into so much of analysis again; we will do some more simpler things of doing calculating stuff that would be more fun for you.

With this, I end the third week's syllabus or third week's course; do not get about the syllabus, just to enjoy it. Just go though the lectures couple of times, you see you will gradually have a better feel for calculus or the calculus.