Basic Calculus for Engineers, Scientists and Economists
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Lecture – 17
Riemann Integration-1

We are now going to discuss about the Riemann Integral.

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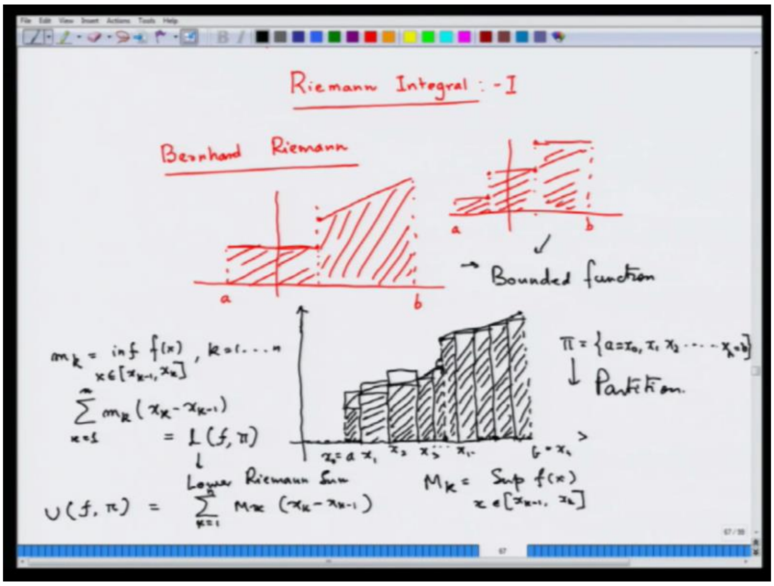

And that is part one. And part two essentially talks about improper integrals but I just called it Riemann integral two because, when you talk about Riemann integral you are not truly in the domain of just what you call calculus its slightly advance calculus or I will say it is called mathematical analysis you are taking a deeper look into the issue of calculus.

My idea is not to give a very deep mathematical idea or very rigorous idea of what is Riemann integral. But I would just like to tell you that this was an idea which was quite important in those times in the 19th Century where you move forward from the idea of just integrating continuous functions to the idea of integrating bounded functions and it was introduced by Bernhard Riemann, one of the greatest mathematicians of all times.

Now, sorry Bernhard h r d, but you do not know I am writing it is women's I think it is a correct spelling. So, what was his idea? His idea was pretty simple, that suppose I just have a look, if you observe a curve like a continuous function you know how to find the area you already have studied in school.

But suppose I have a function like this which is here and then it is discontinuous it is from here it is here, and then I am trying to find the area between this point a and this point b. Do they have any area? Can we define an area under this curve? You might feel ok it is slightly problematic at this end point, but I would say why both are just compute the area of this rectangular zone, and just compute the area of this. Which can also be easily computed or you can also take a rectangular. For example, in electrical engineering what you have is staff functions.

Functions as we just something do we switching circuits it is called (Refer Time: 02:58) site staff function. Do they have an area? Observe that if I want to find the area of my - a to b area I would really can easily define an area under this curve by just computing out the area of these rectangles. See the idea is interesting in the sense, now if a function is a bounded Riemann said that you can extend the idea concept of Newton’s integration from continuous functions to bounded functions, because every bounded function need not be continuous because here we see that be these two functions are bounded functions but discontinuous at several points.

So, how do I look at such a function? Let me make a drawing Riemann’s which I was a following, that let us just take this curve the graph over a bounded function. Let us just for just check we take one discontinuity. The discontinuity in a function does not have any; sorry I will just make it much easier. The function comes up to here, but does not have any value at this point which is c. So, how do I start integrating? What we do is let us forget this c point for the time being. At the c the value is this so there is a discontinuity, but the function is bounded. What we do is, let us divide this whole line a b into some partitions of equal length or may not be equal length but ok it does not matter. This is say x 1, so a - I put as x 0, is x 1, x 2, x 3, and dot dot dot x i and so and so this x n.

What we are doing? We are drawing rectangle, we are just joining and drawing one set of rectangles, just joining up with the upper part of the graph. A partition which is slightly clearly, these is my partition points. So, what Riemann did was a set up the function is bounded so which ever partition I take say I take this partition, the function of the maximum value and a minimum value here minimum value obtained at a, so taking this height as f a draw a rectangle.

The same case you do with the point x 1. If the minimum value is x 1 the function here is increasing, but here are the minimum values also the function is increasing so you take it here. Here also it is increasing you take it up, you take the minimum values, it is here, here, here, here, and here.

Now, you take the minimum value over that interval take it has the height of the rectangle and if the base the as a length of the interval, compute the area of the rectangle. If you some of that area. So what I am doing, I am taking for each interval m k as the infimum, it is obviously achieved because here the function is bounded.

It is the infimum of f x then x is lying between just for the sake of computing on it just for the simplicity is put m k to be between x k minus 1 and x k, you can take rather wise also. We shift it you can take within x k plus 1 on x k that also you can take as m k and then the writing would be slightly simpler you could do this that is not a very big issue. Similarly, if you look at the supremum, you can actually construct rectangles with the supremum's. Now what you can do, you with this small once this is the area of the kth partition actually, we have taken x 0. If x 0 means I have to take m k in this form, x 0 is a.

So, taking the whole thing for the kth partition, now you sum this up so k is equal to 0 to n. Now this one is called the Lower Riemann sum, actually it is called Lower Darview sum of f under a partition pi. Where, pi is the partition a is equal to x 0, x 1, x 2 so the n point one points, so there are (Refer Time: 08:54) intervals actually. So, I should start from k equal to 1, then it will give you x 0, x 1 minus x 0 and m 1 if you could take 1 here, so k can be starting from 1 to n, if you take k 1 it is infimum when x is in between x 0 and x 1. This is called the Lower Riemann sum.

Similarly you can construct the upper Riemann sum, if you write m k as the supremum of f x between when x is lying between x k minus 1 and x k. Then you can similarly compute what is called the Upper Riemann Sum or u f pi, pi is called the partition. Then if you once you construct this sum this will become say k equal to 1 to m and m k x k minus x k minus 1. I do not want to leave this page because you finally, have to understand things here, so I just rub this off and talk to you.

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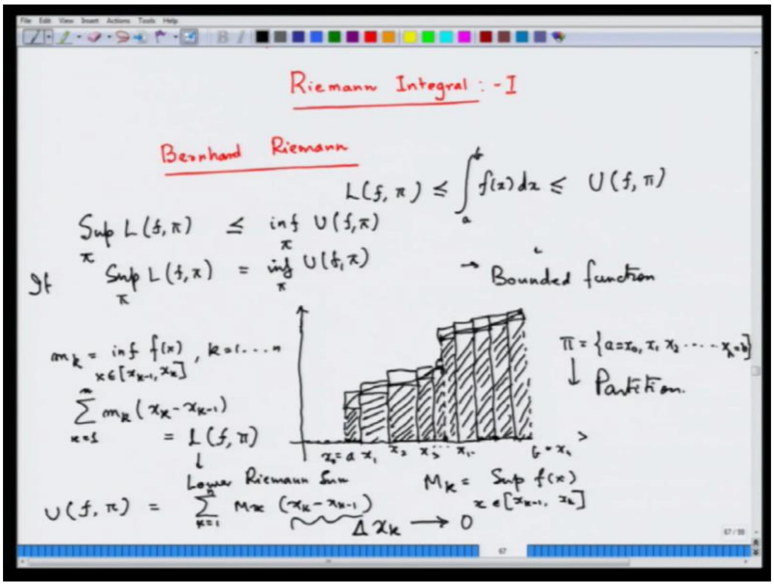

So, what I have now done? If you look at the picture, the actual area under the curve is sandwiched between this area the upper Riemann sum and the lower Riemann sum. So the actual area if we call it the integral, if I call it the integral which will call the Riemann integral very soon is for any given partition pi. This in equality might be strict. Suppose, we take the supremum of L f pi, for all possible partitions pi. And we take the infimum of the upper Riemann sum for all partitions pi. When I say pi it me over this supremum over pi means all possible partitions.

Countably, just there are infinite possible partitions, partition in this way partition this one partition that you can have any other partition. There can be many many many ways infinite ways of partition, countable infinite ways. This is one way of partition there is the second way of partition, the third way of partition and just can go on, countably infinite ways of partitioning.

Then I will make the partition in such a way that I will make the interval length that this x i and x i minus 1 n in x i or x k minus 1 in x k this interval length has made smaller and smaller and smaller. If I call these interval length as delta x k then I will make delta x k go towards 0, I will make it smaller and smaller and smaller and then I will see what is the supremum value p f and so what happens to this. Of course, if you just look at this inequality then it will always be like this, but if supremum of pi l f pi is equal to infimum of pi u f pi.

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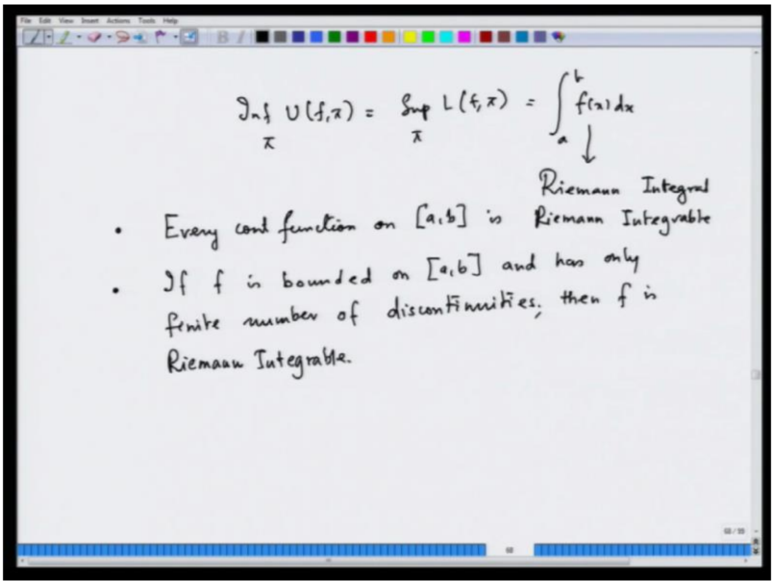

Then we say that Riemann integral exist and infimum of u f pi for all partition pi and supremum of L f pi for all partitions pi is actually called the integral of a to b form f x d x. And this integral is called the Riemann integral. We will not go to too much of a discussion on Riemann integrability. So, let us tell you what sort of functions is Riemann Integrable. Every continuous function on a b is Riemann Integrable. Any function who’s Riemann Integrable exist we call it Riemann Integrable.

Now the interesting fact, what we have just seen that we have done this game with a discontinuous function at just says the one point of discontinuity. If f is bounded on a, b and has only finite number of discontinuities, then f is Riemann Integrable. Is there any function bounded but not Riemann Integrable.

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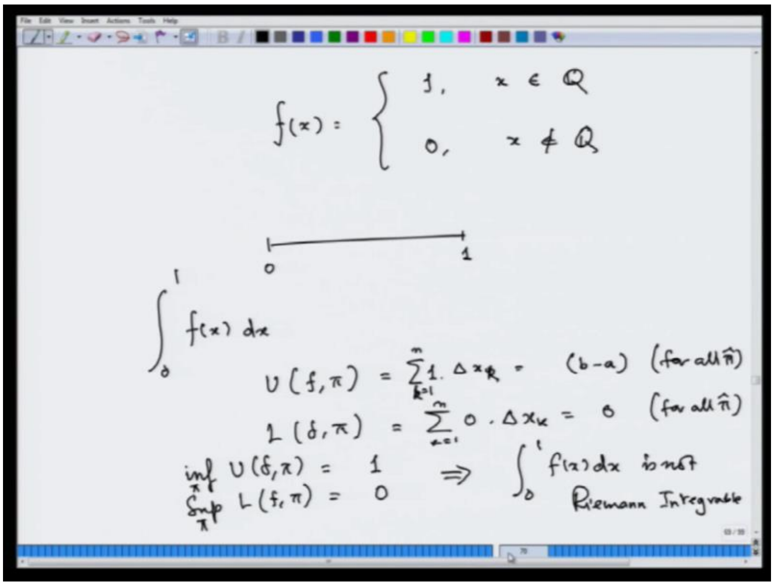

This function was handed over to Riemann by his (Refer Time: 15:38) supervisor (Refer Time: 15:41) and this is the famous (Refer Time: 15:44) function we have already spoken about earlier. F x is equal to 1 when x is rational, that is x in the set Q is equal to 0 when x is not a rational number when it is a irrational number. Now I want to integrate it. Now let me see I will partition it. I will partition it, so let me partition it just do it within 0 and 1. I want to ask what happens to this, for this particular function. Of course, this is a bounded function the values are only 0 and 1.

Now if I partition it whatever is may partition because, they always infinite number of rational and infinite number of irrational numbers inside those partitions my Upper Riemann sum it does not matter for whatever be the partition pi. My upper Riemann sum, u f pi is nothing but 1 into summation delta x i over i equal to or delta x k or k equal to 1 to n. Some of the partition it will give you b minus a.

But when you take the lower partition it will be always 0 into delta x k which is 0. If you take the infimum of the Upper Riemann sum for all possible partitions for all pi. Pi is not the real pi if you are uncomfortable I will just put this partition as pi hat like this, may be people do not want to confuse it with actual partitions and actual pi the number, but it is 0. This is always b minus a and this is supremum of l f pi is always 0 for taken over any pi or possible pi's. So, what happens that because b minus a is strictly bigger than 0 here, it is b minus a basically here b is 1 and a is 0, so this is 1. So, they are not equal. This implies that 0 to 1 f x d x is not Riemann Integrable. I just mention one or two more properties of Riemann Integrability.

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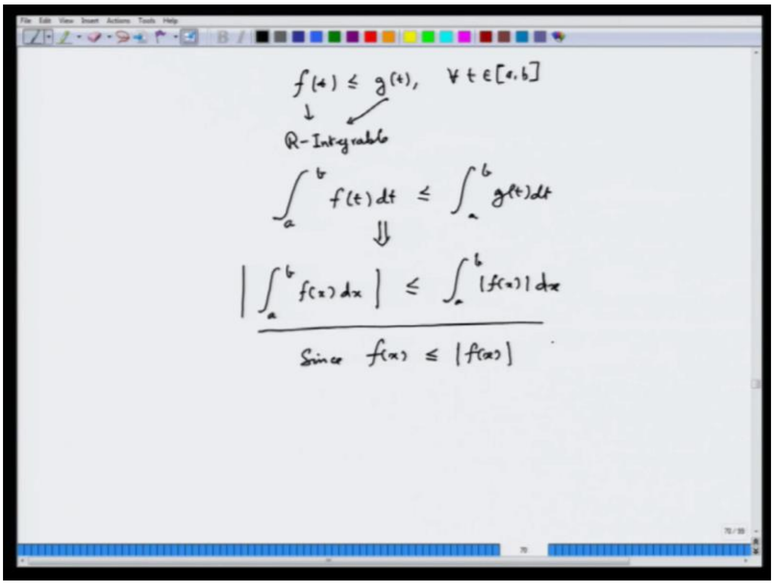

Something like that if f t is less than or equal to g t for all t the (Refer Time: 18:57) of a b where each f and g are Riemann Integrable. We just write for short R-Integrable both of them, then you will have that integral a to b f t d t is less than or equal to integral a to b g t d t you where both a and b f and g are sorry, two functions which are bounded and has this property. And also you will have the property, which is a sequence of this is the following property.

This happens because the f of x it does not matter whether it is plus or minus it is always less than modulus of f x. If f is a bounded function then mod f is a bounded function and if f is integrable then mod f is also integrable, and that is why this happens.

With this very little ideas are all other things will happens for continuous function, the summing up of two functions and integrating the function sum is same as integrating the function separately and summing them up. Those sort of things remains same for all, but those properties remain same in (Refer Time: 20:30) like the once you have learnt for continuous function even in high school. With this, very simple ideas here and it is very simple way of looking at how to integrate functions which are not even continuous and just bounded we stop our discussion of the Riemann Integrable part one.

And in part two, we are going to talk about something called Improper Integral where one of these b or a, are points where b could be just infinity or one of these points are such that the function is not a defined at that point. We will just go into that in 5 minute, in the next lecture.