Basic Calculus for Engineers, Scientists and Economists

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Lecture - 16
Definite Integral

Today we are going to talk about the uses of Definite Integral. How can you use it to do something effective for it? For example, finding areas of volumes of slightly interesting figures like circles or spheres or cones. That it we are exactly what we are going to do today, finding the area of a circle, finding the volume of a sphere and finding the volume of a cone, of a given height.

Now, I just want to tell you that the idea of definite integral what we do today or idea of integration that integration (Refer Time: 00:59), taking the integral of a function from a given 2 points a to b. I can actual essentially compute the area of underline the curve and the x axis.

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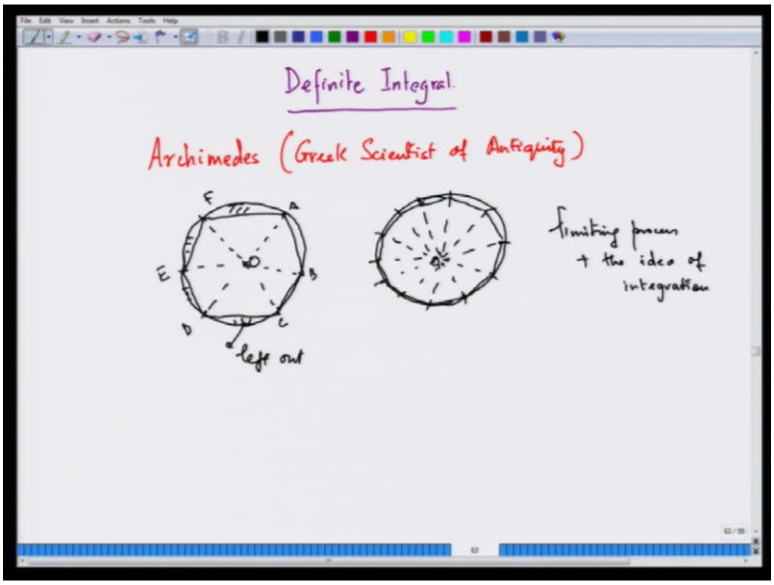

This idea was actually known to Archimedes, one of the greatest scientists of antiquity. So, what he did was that he took the circle. I am taking this circle what I want to find this area. So, what I am taking? I am taking some points, see A B C D E F and this is the center O, what I do is I join to the center O all these points is like cutting of a cake and then you join the adjacent points. Adjacent means from f I can go to a, but f I can directly cannot go c if I walk over the arch or walk over the circumference, if I move from f and go towards c either I go through a and b or I go to e and d I cannot. These e and f and c are not adjacent or if b or not adjacent while if any arc.

So, what I do is now calculate the area of these triangles and then basically calculate the area of this polygon. So, we have now covered a good amount of area of the circle with these little parts left out, left out parts. The brilliant idea which Archimedes thought about was, I take the area of the circle, the area of the polygon and is store it. Now on the same circle now increase my (Refer Time: 03:22) 6 points, now may be I will make some 12 - a 2 3 4 5 6 7 8 9 10 11 12. Now if 12 points, and now I am trying to the same operation let us see what happens and then, what I have done? I made the side of these polygons, the length of these sides smaller and smaller.

Now, if I take this new polygon and compute the area of the polygon, I have left out much lesser area than I had left out earlier. So, you keep on increasing the number of points of the circumference which is called partitioning the circumference with point and then you try to find the area of the hexagon. As you move the areas go increase and increase and increase and gradually go towards the area of the circle. So, Archimedes somehow had introduced the limiting process plus the idea of integration.

For example, we are now talking about one thing, how do I use calculus to find the area of a circle and I will go back and use the very interesting books of Larry Gonick the cartoon one of course, and tell you how to do this stuff.

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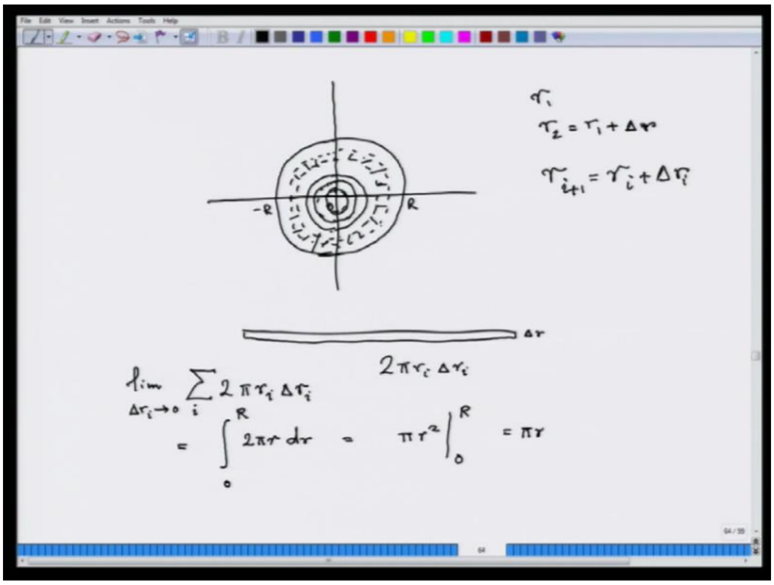

So, you take a circle. Radius R, does not matter, so you taken a circle of radius R. Now, once if you have taken a circle of radius R - this is R this 0, this is R, minus R. What I can do is I can take small circles of radius r i say circle number 1, circle number 2. I am basically dividing up the original circle into pieces of the thin circles, each having a radius which is delta r amount of fixed amount more than the radius of the inner circle. See if I have r 1 if I start with this smaller circle inside the smallest one, if it is r 1 then the next 1 is the radius r 2 equal to r 1 plus delta r.

Similarly I can have the circle with radius r i and radius r i plus delta r is particular dotted region. Basically it is consisting and this whole circle is consisting of some strips like that, if I can find the area of those strips and add it up I can essentially get the area of the circle. Let us see what is the area of this part is interesting. So, what I do is I essentially, I will take my delta r to be pretty small. So, strip is hardly difference between r i of delta r i. When r i and say, so r i plus 1 is r i plus delta r i. This difference between r i plus 1 and r i is really small. So, what I can do is I can cut this particulars strip here and when once I cut it I can now opens it up like this.

These are the thin radius r, but the length of this side 1 of the sides is very very clear. The length of 1 side are cut it a bit, it is a very thin rectangle. Is nothing but the circumference (Refer Time: 08:09) over circle of radius r i. The area of this is 2 pie r i into delta r i, now what I need to do? What I need to do is now I have the area of this particular sigma to sum it out. I have to really sum this up from I equal to 1 to whatever slices you have made. I did not really count the slice.

But I have to take the limit of this as delta r i goes to 0, when delta r i become very very small. Delta r i is actually represents the differential of r i's. Essentially finally, the differential of the radius r we can just instead of telling r, r i we can just take r and r plus d r. This limit is nothing but the integral. You have taken the sums of the areas. So, you get an area, but what are you trying to do, you get an area you take the individual strips and you get an area, but what you are trying to do is you know that here this area that you have computed there is a slight (Refer Time: 09:35) because one side it is the radius value is the length of the circumference is 2 pie r i and the other side it is 2 pie r plus d r.

So, you have to take into account of this difference and this actually which means that you have not completely got the area of a circle, so there is a over estimation or under estimation. So, as you make delta r i smaller that is d r i, this delta r i is very very small. There is when it as you make r i approach r i plus 1, then you actually get what you are doing, then this ideas is correct.

Then that is exactly equal to the integral from 0 to r, and then this is nothing but r square by 2; pie r square 0 to r and this is pie r square. So, how do you find the volume now of a sphere? Now we have to understand one very important thing, as sphere I will just show you is something like this. So, how do I look at this sphere? I basically want to cut the sphere into small; here I cannot take slices like this. So, basically I can cut; it is like an fruit and I can cut it slice it up into very thin pieces and basically then add the so called volume of this thin pieces and add them up.

But if you take a very small thin strip and cut it out then you will realize it is not really a part of; if this the volume of that element is not really a cylinder. Though it looks like a cylinder, it is not, because it looks as if it is a part of a truncated cone. So, because there is difference between the radius one side of the strip and the other side of the strip; but we can imagine them that the difference is so small that we imagine them as cylinders.

So, because we are imagining them as cylinders, we cannot say that when we add up those cylinders the volume of the cylinder we get is exactly the volume of the sphere.

So, what happens is that we have to go to the limiting process of telling that the difference between the 2 radiuses, because when you take a strip as we will just see there will be small difference, that difference goes to 0 and then you will actually reach the actual areas. This is the very basic limiting idea of Archimedes which we are still using now. You see what happens is what we are doing is the following.

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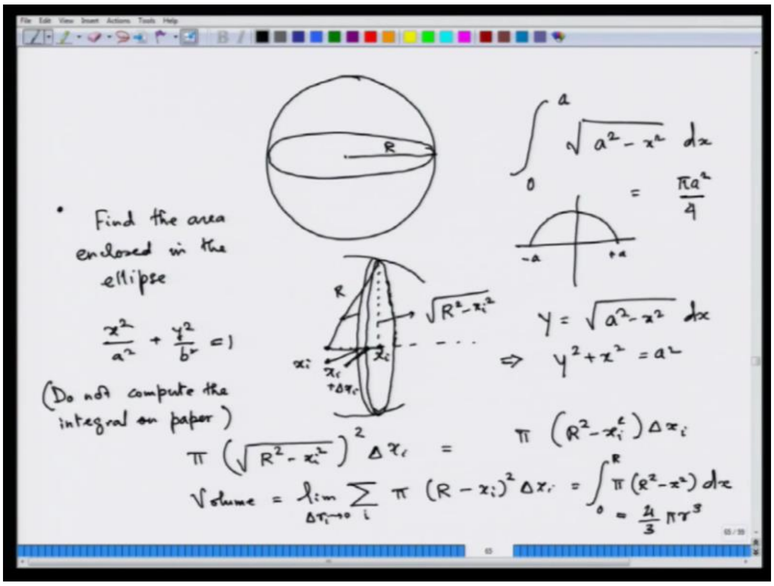

Basically here is the sphere, in three dimensions of radius r. I want to find its volume. So, what I have done is that I am looking at 1 section of this sphere and what I am doing is that I am looking at a very thin strip cut out from it. I am looking at the volume of this strip. This is your x i which corresponds to the first trip. So, you have the radius say r i and then r i plus delta r i, this part if you go from the center of the sphere has radius r i and the reaming part has a distance of r i plus delta r i.

Now what do I have to really do. This is the most critical thing. This is your x i. I really need to find; this look like a thin cylinder. Though really it is not a cylinder, if you really slice up the sphere it is not a cylinder, it is a truncated cone, there is slight slanting on the

2 sides. These sides are slanted you see, you cannot say it is a cylinder, but because we are assuming the delta to be small we assume that is a cylinder. Basically we are thinking that we are putting up some cylinder then it should look like a sphere, it would not; only in the limiting case it would and that is a whole (Refer Time: 14:46) idea of the limiting cases, the charm actually.

So, basically you have the radius r and then you have this x i here sorry this x i is here the horizontal line. Now, what should be the vertical length which I will find by Pythagoras theorem; and that will give me the radius of this infinite decimal cylinder or elemental cylinder. This is nothing but the root over r square minus x i square. Now, what is the area of this circle? The area of this circle is actually root over r square minus x i square whole square into pie and the volume of the cylinder is nothing but this little distance which is actually delta r i.

Basically then we are having pie of r square minus x i square delta r i. So, what you do? So, if sum of all the cylinders, so basically x i some point which is essentially lying in between these 2 n's of the cylinder or the elemental cylinder. It is r minus x i square delta r i, so i is whatever number of cylinders you have pieced up the sphere in 2. Now once you take the limit of this. The volume of the elemental cylinder is this, when once you take the limit of this as delta r i go to 0 that will give the volume.

And what is this? You write this is nothing, but 0 to r pie r square minus x square into. I will instead of writing del r i, I should write here del x i plus here I am looking at the change here it was x i, here it was x i plus del x i. The actual change in the radius, whatever I am writing as r i, here is actually is not really r i. This part, it is x i this is my x i at this end and then this part at this end. This coordinate is x i plus delta x i and this coordinate is x i. I am taking as if the x i thing that this part. I am taking this x i thing and I am calculating the area of the cylinder.

Of course, there is another way of taking an x i between this slab and that also does not matter and then you can take 2 piece of delta x i. This length to be some delta x i, you can call this delta x i, does not matter, you can take any x. So, anyway this I should not write delta r i as this symbol should not be same, then I should write r i here instead of x

i. This is delta x i.

Please check it up, it is just, I confuse usually with symbols. Here, this is this, and this will give you which you can compute out easily. Now this is very interesting. Suppose you are asked to compute an integral of the form 0 to a root over a square minus x square d x, how will you do it? Because if you are doing it, if you look at this, what does this equation tells me; if this is the function then it is; this will give you y square plus x square is equal to a square.

This function represents the upper part of the circle from minus a to plus a. And the area of this is pie a square by 2 and we are looking at half of that because of the symmetry. This is nothing but pie a square by 4, the answer is pie a square by 4. If you know the areas you can actually compute up integrals like this. I will leave you as an exercise that without really integrating find the area enclosed in the ellipse, do not compute the integral by hand.

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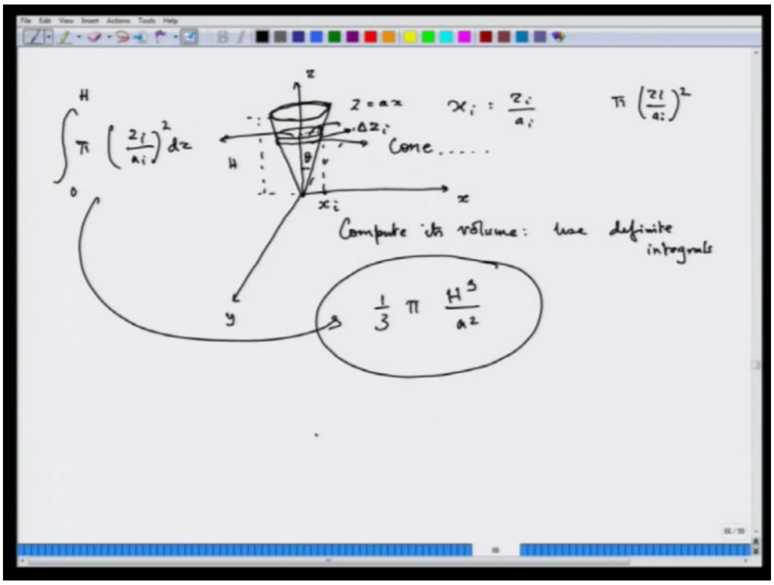

Another important problem which I really want to talk about is, if you take a cone an ice cream cone which all of you eat and take an ice cream cone and basically a cone is three-dimensional object. Take this is x axis, this is y axis and this is z axis. So, you take some sort of a line, y is equal to ax or z is equal to ax. So, you take this line z equal to ax. This is the (Refer Time: 21:02) z equal to sort may be ay I should say. Let me put this as y and this as x and then, you rotate this line along the z axis keeping a fixed angles say theta.

Then what do you generate finally, is called cone these are called solids of revolution, as you are revolving some lower dimensional object and getting higher dimensional object. This is the cone; question is how do you compute its volume? The answer is if this is my height. I give a fixed height to the cone, this is my height h. Answer is 1 third pi h cube by a square. Now, can you find this by calculus; use definite integrals. Many things I would definite integrals we are not spoken about, but this time is very short, this essentially at tremendously tremendous crash code. Again you cut it into the cylinders and then take the limiting process.

You know that this is really not a cylinder it is slightly like a slanted thing, it is like a partially part of the cone, and it is not really a cylinder. It is truncate cones sort of thing, but then you really need to figure out what is the radius here, what is the radius here, find the area of the circle and then multiply by this distance which is del z basically. You take the z i and z i plus del z i and then del z i into what should be the. If you look at this radius, this radius is nothing but this distance x i. This radius is the distance x i. So, x i is the radius, is obviously coming from this equation x i equal to z i by a i.

Once you know that what area is. So, area is pi into z i by a i whole square for this elemental circle. For this particular this elemental circle you know the area is pie times z i by a i whole square, and that you really know that you have to multiply with del y so it is essentially this, and that you have to integrate from 0 to h and that will immediately take you to this answer, that not d y its d z. Because it is delta z i, this distance is delta z i. This distance what I am now marking by pen is delta z i, so d z and that will exactly take you to this. And that ends our discussion over Definite Integrals.

Thank you very much.