Basic Calculus for Engineers, Scientists and Economists
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Lecture – 15
Integration by parts

You know we are all learning something about integral, and we are going to talk about Integration by parts. Most of you must recollect what you have done from your high secondary days, but I would like to say that as most of you have not really understood what essentially goes on behind this technique of integration by parts.

When I was in high school, it was told that if your product of two functions and one of them is integrable very easily and there is something called integration by parts and you can just go through them. The issue here is that the idea actually comes from what happens when we take the derivative of two functions, the derivative of a product of functions.

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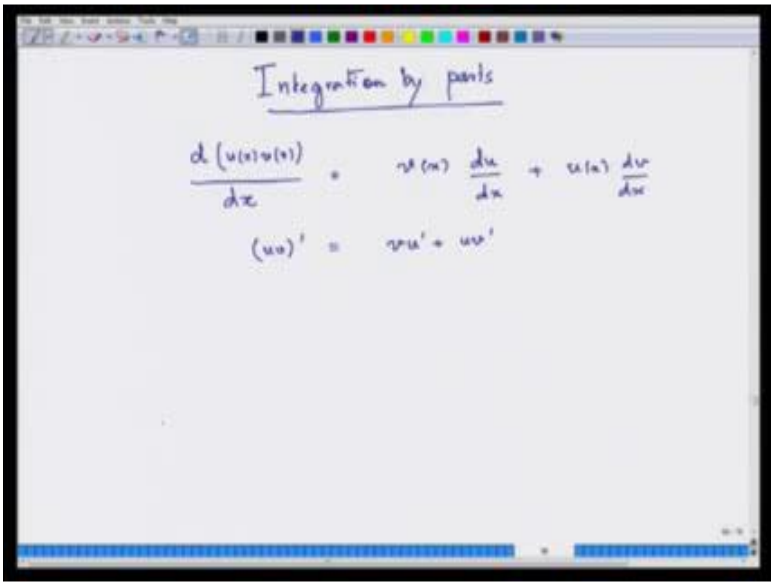

So, you know the d of suppose have two functions u(x) and v(x), I am taking that derivative with respect to x. So, what do we have here, I have v(x) into du dx plus u(x) into dv dx. More generally, you can write u v dash is v u dash plus u v dash.

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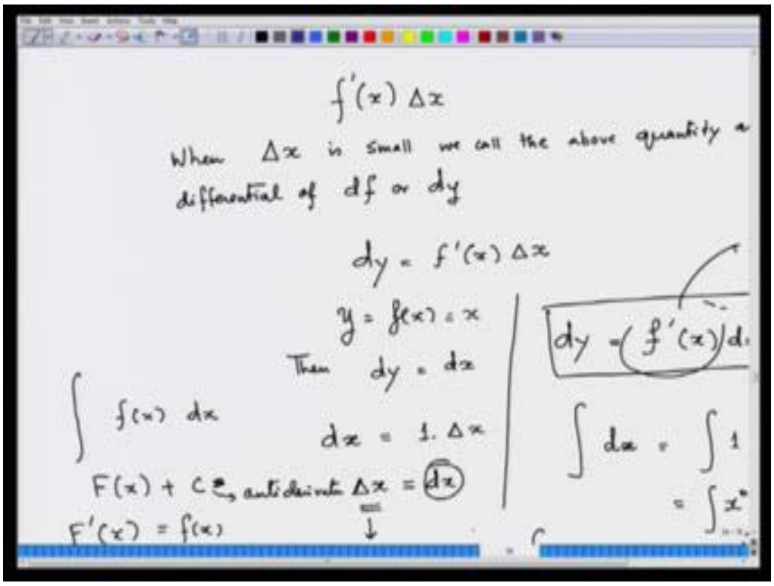

So, again if you adding terms of the differential language which we have taught you in the last class, you just see this.

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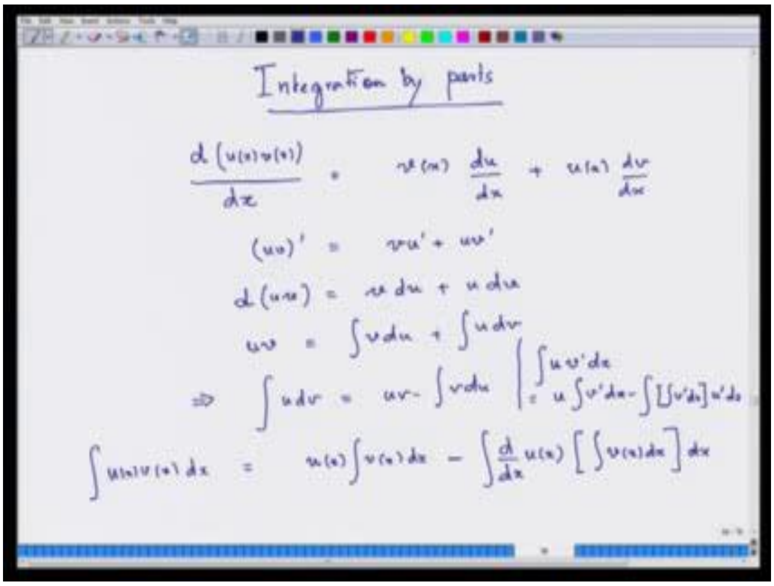

If you go back the differential language then we have the following that d of uv is equal to v of du plus u of dv. In integrates out you have u v is equal to integrals v du plus integral u dv. This would imply that integral of u dv is equal to u v minus integral of v du. Let us start with the simple example and let us see where we really reach.

By the way, I am again taking the examples of this lovely little book which I have personally enjoyed and will recommend to everyone the book, in fact, I want to show you in this series, you do not know, it just call cartoon guy to calculus. Please, it is not a, it is fun book, I do not think, you can learn anything without having fun in it. So, here is a book where you learn deeper thing and have absolute fun and this should be book which every engineer or scientist keep in their desk. This is just fun you get that all basic all the very deeper ideas told in very interesting and fun way.

Now I remember when I was kid, and then I was thought suppose you have a situation like this u(x) v(x) into dx then what you would do is you basically suppose you can easily integrate v(x), so you integrate v(x) and then you subtract which is essentially this. Basically here I have u v dash, so here if I take on a v dash dx, it will give me nothing, but v. So, basically instead of u dv, I am writing dv as v(x) dx. This is exactly the formula.

Now let me see what I will tell you what I was thought. So, you take the integral of u(x) this whole thing into the integral of v(x) dx into dx. This is your du, what is d u basically. So, can I get this formula this is what this was what I was thought in my high school, with I never understood what was happening, but of course, I understood much later that it was just form this whole thing. So, once you have the idea of differential clear, everything becomes clear to you.

Now let me see how it translates this whole thing. Now, instead of this one here I just run so integral u dv. Instead of u dv, I write dv is written as v dash dx. You see I have written here in terms of v and u, but here I am writing the same thing essentially in terms of x, because u and v are the essentially functions of x.

So, u v dash dx, so you see here just remember the formula dv is the differential of the v is the derivative of v with respect to x into differential of x which is dx. That is equal to when you do this u v of course, here u(x) v(x) was the because this u v is, now what is v, v can be in fact, written as integral v dash dx, because v dash d x is nothing but d u when you integral dv. So, when you integrate dv, you get v.

Now what do I have these, now minus, what is my du, du is nothing but du can be written as the derivative u dash x, the derivative of u times the differential of x. And then I have this part I have to look at this part, so what is in v, v has come to v, v can be written as nothing but integral of v dash dx into du, d u is what it is u dash which is du dx which is u dash into dx. See u dash is nothing but u dx.

This is the formula which I was taught by my teachers without giving any reason is exactly this formula which comes out simply from the product formula which I have learnt from the for which is very simple to show, for how to take the derivative of the product to two functions.

Now, once we have this thing in my mind clear, let us give you some simple examples, not simple, but very, very interesting. See these are the problems you start really looking at these problems, so when you do not have, you cannot immediately say I can find a function whose derivative is the given function under the integral sign. No, if you cannot do that you really have to think about integration by parts. Your substitution method does not work; other method that you no possibly I would integration does not work then what are you going to. So, here we are going to really concentrate on taking examples from this lovely book.

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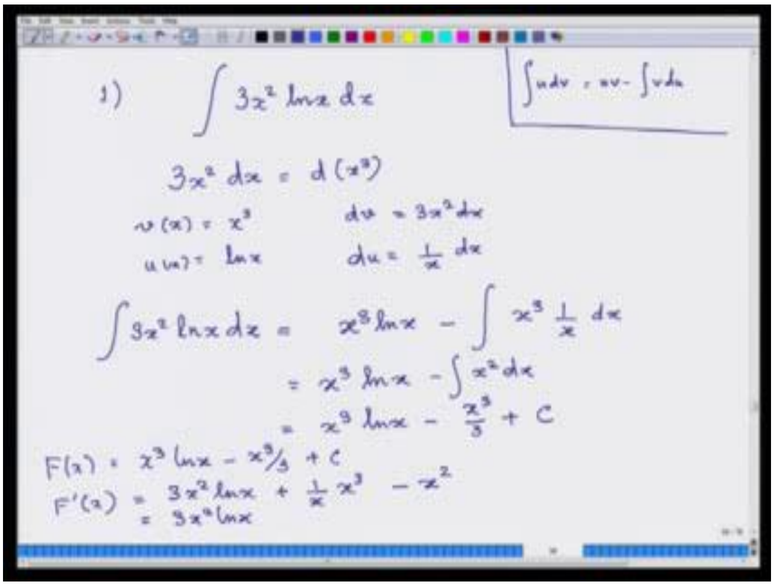

Let us take the first example - number 1. I have to integrals I have to find the integral of 3 x square log x dx. Now if you think of it nothing seems to work. If it is does not work what I am suppose to do. Let me see if I have a candidate of something like dv here. Now if you look at 3 x square dx, 3 x square is the function which is y is equal to 3 x square, y is equal to x cube then 3 x square is the derivative. The derivative of the function into dx is the differential of the function it is d of x cube.

If I go by my formula which I am again keeping on the side, if you forget it is integral u dv is equal to u v minus integral v du. This one once you have this thing, so you know that my choice of v is now known. So, my v must be this one, so my v x is equal to x cube. If my v x is equal to x cube, I have u dv, so I have 3 x square dx is my dv, so it is dx cube. This actually means my u(x) must be log x. So, my dv is of course, already I know 3 x square dx and my du is 1 by x dx.

Now, I am going to apply this formula. Integral 3 x a square ln x dx is equal to now let me see you have u into v. So, you have x cube ln x minus v that is x cube into du that is 1 by x dx. So, you have x cube ln x integral x square dx. That is x a cube ln x minus x cube by 3 of course, this is plus C.

Now that is the whole game, now do you believe this that this result is correct? If you do not believe, it let us take f of x to be x cube ln x minus x cube by 3 plus c. Take the derivative of this. These are product forms. So, you have 3 x square ln x plus 1 by x x cube minus x square, because it will be 3 x square by 3. This will give you 3 x square and ln x and voila, this is the integrand and hence I have done the correct thing this correct one.

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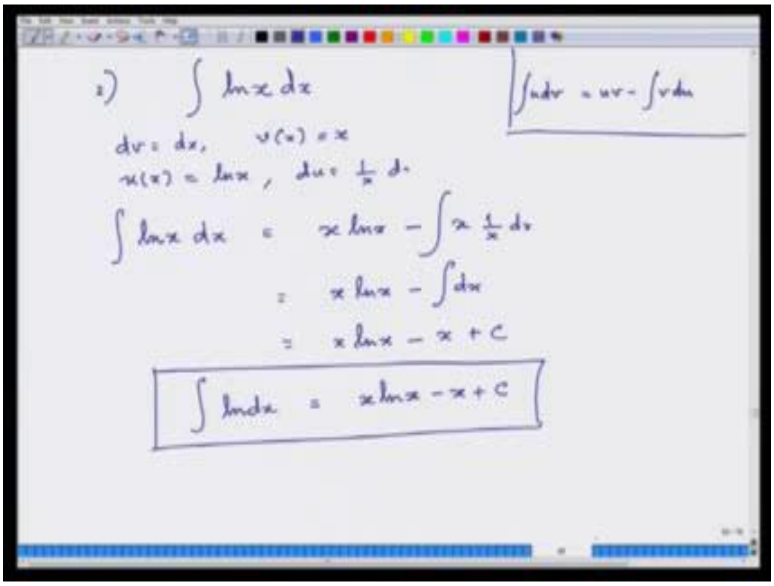

Let us take our example two. This is the funny integral, by (Refer Time: 11:11) it looks pretty interesting. You cannot think our function immediately whose derivative is log x. Of course, those who study optimization conventionally is that are higher level, a very high level; they would immediately realize what it is, because they use the anti derivative of this function day in and day out. So, but where is my dv that is problematic. Let me do. It must be u dv again the formula forgetting, write it down.

Every time, I will write it down integral u dv is equal to u v minus integral v du. This formula should become a part of your game. So, here you have the u may be you take a log out what is dv. Let me just say dv equal to dx basically, I am taking v(x) is equal to x. And u(x) I am taking as log x.

See, if I use this formula integral log x dx is equal to x log x minus integral; what is v, v is x and what is du, du is 1 by dx voila. So, here this x x cancels. So, x ln x minus integral dx and you know what the game is. Take the derivative and see the answer comes. This is a beautiful result. You would appreciate mathematics much better if you find try to find some beauty in this is not it so amazing this C has to be always put in. I do not like this C, it was like clumsy to me, but somehow in mathematics you also have to take into account the truth more than anything else. So, here is this lovely thing

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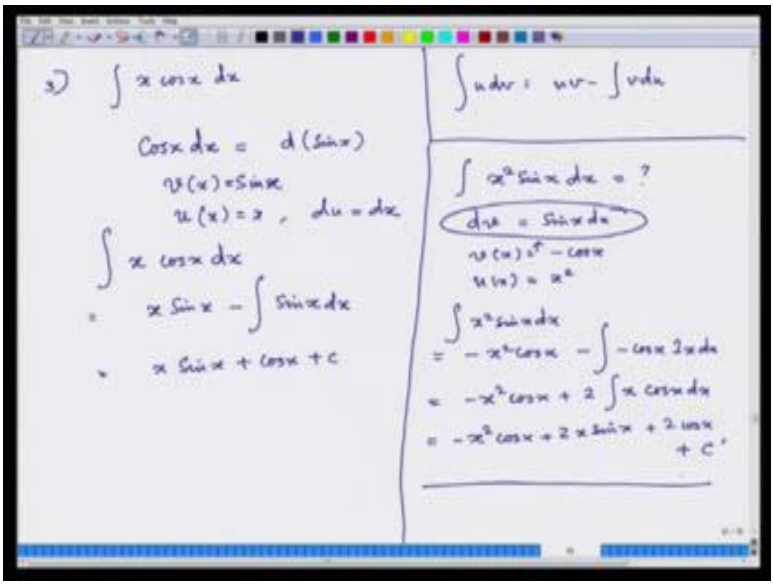

Now get on to the next one. Is in calculus fun, calculus is truly fun. Let us try it out. What about say I will do two integrals, one with the help of another, so example number 3, integral x into cos x. So, what do you want to substitute? So, you want to take t equal to cos x, u equal to cos x and you have so du, du come in terms of sin x. There is no other to substitute anything; there was no way to do any issue. So, what you can do is there no way to do anything. These trigonometric integrals are the hallmarks trigonometric and lowering the integrals are the hallmarks where immediately know that integration by parts is my tool, there is no other tool again really friend of friend for.

Now let us have a look. Integral x into cos x dx, so what I am going to do. If I look at it very carefully, the derivative of sin x is cos x. So, cos x into dx is exactly d of sin x differential of the function sin x. Wow, so this is your dv, you have already know. So, your v is actually a sin x, your v(x) is equal to sin x. So, what is your u, so your u is x, because it dv, so this must be u, u(x) is x. Integral x cos x dx, so again repeat in your mind integral u dv is equal to u v minus integral v du.

Once you know this, you will immediately understand the game. So, how neatly things comes so, you now know which is u and which is v. So, x sin x minus integral v du, you know what is v, v is your sin x; and you know what is u, u you know what is du, d u is equal to dx. It is x sin x minus obvious integral of sin x is equal to minus cos x, because minus derivative minus cos x is sin x. It will become plus cos x plus. Now how does it help in some other integration, let us see.

If you have integral x square sin x dx, now once I do this, I want to find this integral. So, again my question which is my u and which is my dv, so you know that the derivative of minus cos x is sin x. So, dv, if dv, if I take dv is equal to sin x dx, which means what my v(x) equal to minus cos x, my dv is actually this my dv is this one. So, which means my u(x) is equal to x square. It is again integrals u dv. Integral x square sin x dx is equal to very funny what to do; now it is u and v. It is minus x square cos x minus integral v du, your v is again minus cos x; and what is du, it is 2 x dx, so it is 2 x dx, good.

It will become minus x square cos x minus, minus comes out, two comes out. It is plus 2 integral x cos x d x, this is already done here. We have already done integral minus cos x dx. So, again just put in the value we have already known. This will be plus 2 x sin x plus 2 cos x plus 2 C same as C, let us call it C dash does not matter. So, here is the game, so here we have computed.

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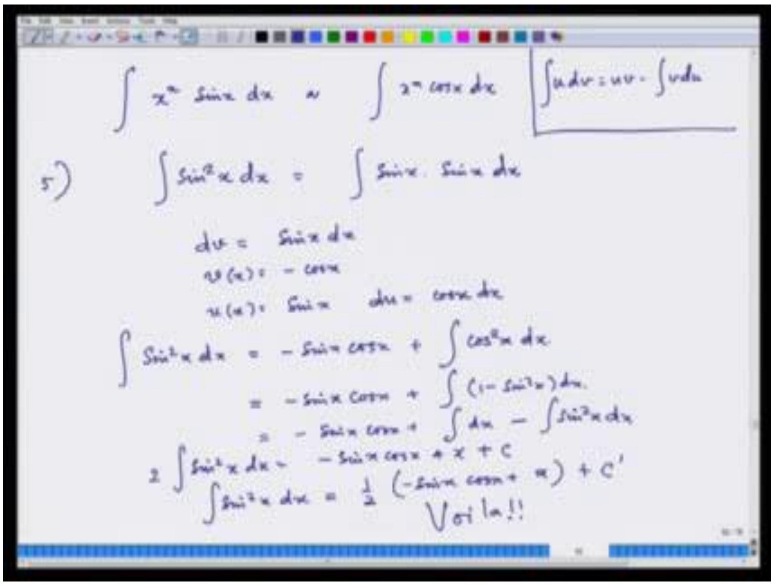

See, what happens that these there are lot of lot of integrals, which come in the form integral x n sin x dx or integral x n cos x dx and then these integrals are one for which integration by parts is a best way to go over. I will end by taking one more example the last one I guess, I think the sixth one. If I have not the numbered the previous one, let me number this as 4, so this were the fifth one.

So, fifth one is integral sin square x this is little tricky. Sin square x what is my u n, what is du. Now how do I think it sins square I cannot think about any function whose derivative is sin square x, can you think about, no, I do not think so. Let us break it up and write it as sin of x into sin of x dx. Let me thing this as dv. Let me think that dv is equal to sin x dx. So, my v what should be my v(x), it should be again just like the previous 1 minus cos x. So, my u(x) is equal to sin x. So, my du is equal to cos x dx. This is the only information I have know, this is the guess work what I am just trying to break it out.

Now, again apply the formula; again if you forget just repeat in your mind u dv is equal u v minus integral v du. If you can repeat this in your mind, you have done a major thing. It is sin square x dx would be minus sin x into cos x and there is minus cos x into sin x plus integral cos square x is dx, oh boy, what to do. Again, I am integral of a similar sort.

So, why not we do write cos square x as 1 minus sin square x dx, so what I will get is minus sin x cos x plus integral dx minus integral sin square x dx. So, you take the sin square x to this side.

Two of integral, so two times integral sin square x dx is equal to minus sin x cos x plus x write plus C. Integral sin square x dx is equal to half of minus sin x cos x plus x plus x plus C, C by 2 we do not have to write C by 2 is some other constant plus C dash. And the French would say voila - that is it.

With this, we are going to end our talk today. And I hope you have enjoyed this little session, we have on integration by parts. And in the next talk, we speak about how to actually effectively used here idea of the definite integral, we give some examples right.

Thank you so much and have a good evening.