**Basic Calculus for Engineers, Scientists and Economists**
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**Lecture - 13**
**Integration-1**

We are back again for our third week's journey, we are going to talk about the Integral today. The two things, which you always think about - when you think about Calculus - one is Integration and one is Differentiation.

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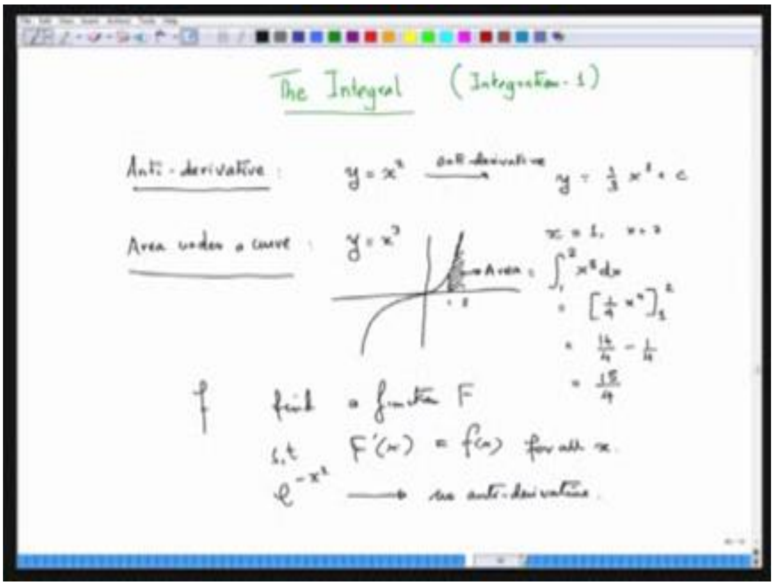


Now, what I want to emphasize is the following; is that one of the key things was integration does for functions one variable is to find an area under a curve. Of course, you can ask me what is the definition of the area blah under a curve and blah blah things. Suppose that you have an understanding, see essentially if you look at the screen these are the two aspects of integration that is usually done in a very basic high school course when you are in your 12th standard.

It talks about the anti derivative which is the first thing that you talk about, that is if I am given a function finds another function, so given a function f, find of function capital F such that F dash of x must equal f. This process is called the process of taking the anti-derivative.

Now, let me look into something very carefully, that is this process is slightly not so trivial. One might thing that gives me any functional I will find its anti derivative, it is not so easy. For example, if you take functions like e to the power minus x square, then I really do not know of you can find an anti-derivative; you really can find an anti derivative. But you should also simultaneously understand the following; that if I can find the anti derivative the integral does of fundamental job it finds the area under the curve.

For example, here I am trying to find the area under the curve y equal to x (Refer Time: 02:27) from 1 to 10, this calculation is absolutely known to you and I just do not want to repeat it.

Now, I want to show that how beautifully this idea of finding an area under the curve can lead us to one of the most fundamental results in calculus, call the Fundamental Theorem of Calculus. So, how do I look at an area under a curve?

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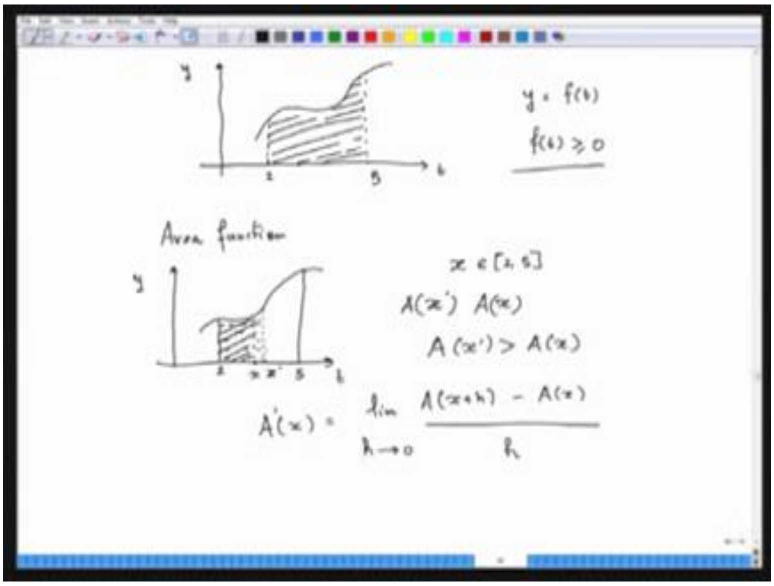


Suppose, I have a curve which is given by y equal to function of t, I will tell you I am taking t now will soon see, t is as just a dummy variable you can take z also. Suppose, I want to take the curve defines a from 2 to say 5, I want to find the area of this curve and this curve has the properties that f of t for the moment is strictly positive or non negative does not matter.

Will soon see just keeping ourselves to non negative function the expending the things will do the job because if the function becomes negative will tell you exactly what to do. Now, how do you talk about an area? Of course, you understand what I mean by an area. Area under 2 to 5 means I am really looking into this, but here we are in the domain of analysis of calculus.

Hence, the first question is can we think of area as a function. We will talk about indefinite integrals also, some people thinking ok why you are coming to definite integrals why not indefinite integrals but we will talk about them, but let us first look at this a picture. How do I start finding the area? So how do I first start finding the area? I will define what is called an area function.

Let us again look it look into this function f which I have draw t and f t here t and y, and this is nice, this is 5. What I do? Let me just consider a point x which is lying between 2 and 5. Of course, you can consider the 5 or 2, so let me just consider any point. For the moment consider an x which is neither x neither 2 or nor 5 and then let us look at the area from 2 to x is this.

Now if I slightly change my x I keep bring it here x dash, then I have a different area. So, you added some bit more or mass to this new area. What happens that, if at x dash and x that a is the area function which give you the value of the area from 2 to x, so a x dash is this. Of course, you can see that a of x dash is strictly without than a x.

Now, if I am trying to find an area by using my function data and by integrating the function f the interesting question is, if I can think out such a function a x, does a x have a derivative? Is a x a continuous function? Of course, by looking at the picture you can immediately make a, immediate guess that a of a of x would be a continuous function, because if I move from 2 to 5 in a continuous fashion my area also changes in a continuous fashion. If I move a very little bit from x, I can x plus delta the area also changes by that little amount.

Question is pertinent that can I find the derivative of the area function, and what is that derivative? If that derivative is f, then a is the anti derivative of f. And so by integrating f we can actually find the area. So, what I mean by the derivative of the area function? you can ask me more question whether it exist etcetera, etcetera. Let me see how to handle it.

This is the definition, now once I have done through let me look into a let me look at the picture.

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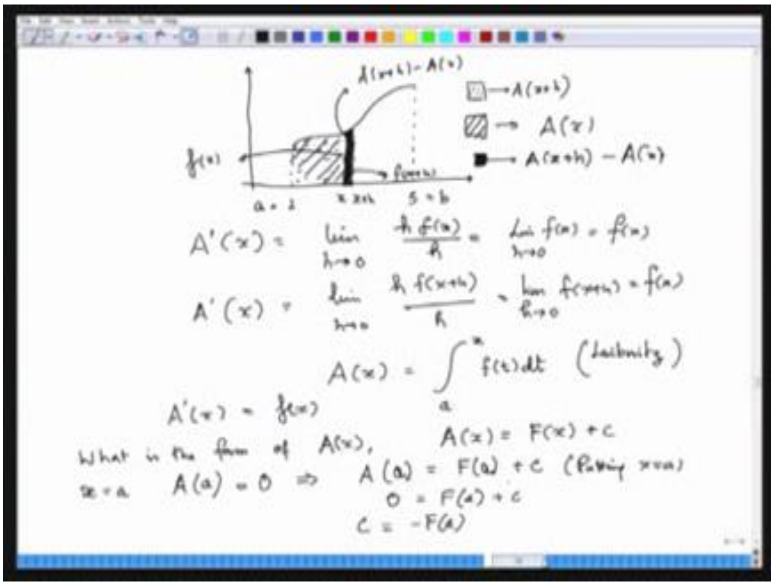

I want to know, what is the area a x plus h minus x? So, I have this function I am just doing it with 2 to 4 what you need not bother much, you can say any alpha to beta. Let me come here x, and then I could go back or go forward does not matter just I am taking x by h testing h positive for this moment you could think negative.

So, first you look at the whole area over up to x by h, 2 to x plus h which is a x is this it is a of x plus h. This block, this is denoting a of x plus h. And now this block is denoting a of x. So, what does that do? This area, this block of small chip of rectangular thing, ok it does not look like exactly a rectangular some curve, but ok because if h is very small we can consider it to a rectangular just by using a Trailer expansion idea that when h is very small then it almost becomes the tangent, we can approximate it to be a rectangular. Approximation is also one of the soul key ideas in mathematics.

This shaded area, this black area, this is your a of x plus h minus a of x. This thing is your a of x plus h minus a x. But how do you compute that rectangle? That is a question. I assume that is hardly much difference between x plus h or minus a of x plus h. Let me take the area is nothing but the base h the breadth and the height I just take it to be f x, but if as between f of x plus h f of x slight is very less. This area which is this a of x plus h minus a x is nothing but h into f of x, because I am telling the difference is so less that I need not bother. So, a dash x is now limit, h tends to 0, h of f x by h and that is now limit f x, h tends to 0. Of course, there is no h here note h dependence.

You can say that, I am not happy with what you are telling. Why did you take your f x to be the length of the game, length of this rectangular type of thing that you are talking about, I can take this to be x plus h has that ok does not matter so it will be now h into x plus h. If you want to take that length, so I took left hand side as the length you can take the right hand y coordinate as a length. Now, this will be the length. So, this was your f x, this was your f of x, this was your f of x plus h this length.

So, I have now limit h tends to 0 f of x plus h. But if f is a continuous function, always in your continuous function you can always pull in the limit inside. These on the key features about a continuous function which have spoken about and we spoke about continuous function.

And when you put h 0 here so that the limit of x plus h when I tends to 0 is 0 you have this is nothing but f of x. So, whatever you do you show that once you want to compute the integral from two to x then your anti derivative is f x. So, Leibniz gave this formal symbolism to it of the area function. I can take it from any a, it can be 2. So, 2 could be a and b could be 5. So a to x, f t d t that is called the area function and that is due to Leibniz.

Of course, do not take me in to issue about what actually an area is and what are those are issues which we have some understanding about we do not want to get into all that. Now you can ask me one important question. Ok a dash x is the anti derivative of f x. What is the form of A x? In general A x would be of the form because a dash x could any way give f x. But it is you have a constant added here, do you want to mean that for the same x you have different values of the area, I said know it cannot have a different value of the area this c has to have some meaning.

Of course you know if I come to this end points say t equal to I will put x equal to a, what is the area value of A a? The area at a is 0 because the length the breadth is 0. So, A of a is 0. If A of a is 0 it would imply if I put a here and a here, so A of a is equal to F of a plus c. Now A of a is equal to 0, so 0 is equal to F of a plus c. I just put x equal to a here putting x equal to A a - which you can understand. Here I have put x equal to A a is 0, so what is c? C is equal to minus F a. So, what does it mean?

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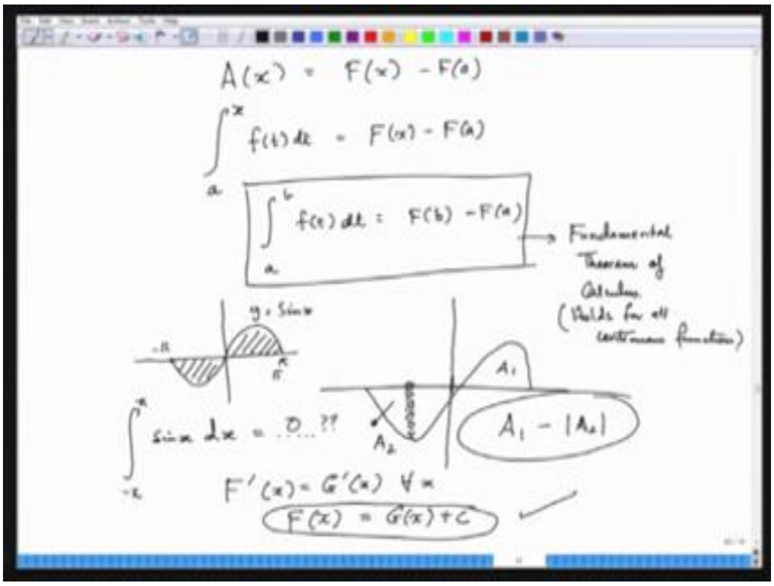


It simply say that A of x is equal to F of x minus F a, and this is what essentially is the fundamental theorem of calculus. When you write it down simply says we are going to take the notation off. See you have a continuous function which is non negative in the intervals a to b then if I take any point x which is say strictly bigger that a tends strictly lesser than or equal to b then I will have. Of course, you can write a to b f t d t is equal to F b minus F a. And what we have reached is, though you can ask there might be some issues about regard, but we have reached is called the Fundamental Theorem of Calculus. You how beautifully and was simply we have arrived at that conclusion.

Now, what about if I take negative function which also takes negative like y equal to sin x minus pi right pi, so what about this area and how to calculate this area. How to calculate such areas? This is one example I am giving, or you can take a more general set up. Now, you know when the function values are becoming negative, note the area function A x when you want to compute the rectangular length if you have this A of x plus h minus A x here. Here, because the function values are negative the area of the rectangle would have a negative sign. Basically, you calculate the area of this curve, the area under this part and you calculate the area under the non negative part and you subtract them.

So, you take the area A 1 and make this area A 2. Basically, when you add they look like subtracting because here the function values are negative so where this is your particular breadth say one centimeter h or whatever. And then, what do have? You have this length which is the function value which is negative in this case, so the area would give a negative sign, when you add them so it will look like as you where to subtracting two areas. So it will be basically A 1 minus mode of A 2. That would be the area when you have function. So, you can try out at home what the answer would be is interesting. Are you going to really calculate?

Now, once you know this you know that the fundamental theorem of calculus can be proved even when functions which are not completely non negative have both positive and negative parts. So, fundamental theorem of calculus now holds for all continuous. Now are you going to work this out and say the derivative of sin x is minus cos x and then you put pi and minus pi, answer is no. The area here is symmetry, they have the same magnitude, so here the answer is obviously 0. Once you have these ideas calculating integrals becomes much more efficient.

Now with this idea about the fundamental theorem of calculus we would like to end our conversation. But we would also state this very important fact if F dash x is equal to G dash x, that is F and G are two differential function if the derivatives are same then they only differ by a constant, then F x must be equal to G x plus c.

If the derivatives are equal at every point and the only thing that differs by is a constant, this is something which we have to remember. With this we will finish today's discussion and then, tomorrow we will going to talk about some interesting way of calculating indefinite integral, we are going to talk and also look at definite integrals, and we will talk about how give a list of some interesting integrals and all those things. That we will tomorrow, you will see that they are just quite fun.

You see how simply we have come to one of the most important result of calculus. You can ask me whether there was a rigour etcetera, etcetera (Refer Time: 20:21) calculation of A of x plus h, it was not that bad as rigour goes but it is alright, but truly one needed what is call the mean value theorem for integrals which I am not going to give you as the course.

I would like you to figure out what is the call the mean value theorem or MVT for integrals of continuous functions. Then using that can you really give a rigorous prove to the fundamental theorem of calculus. We gave a proof which is not bad not very non rigorous gives you a fairly good geometric and intuitive idea and brings you right up to the result.

Thank you very much. Thank you for your patience.