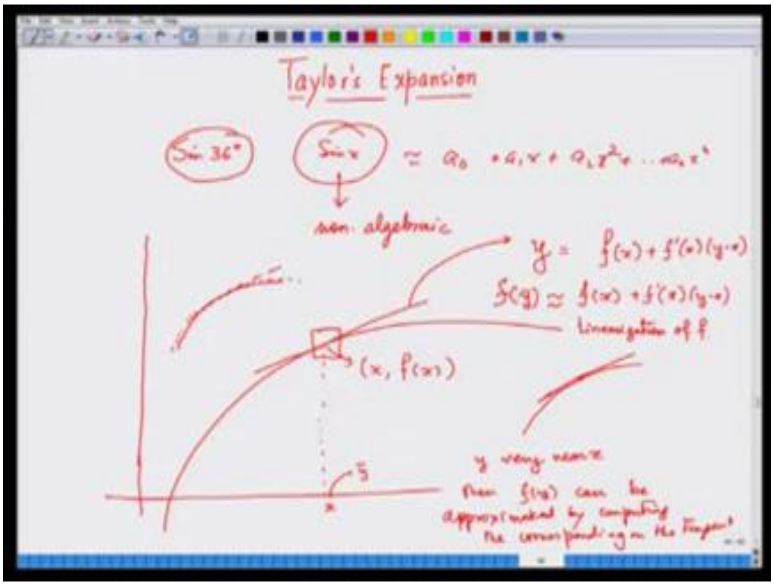
Basic Calculus for Engineers, Scientists and Economists
Prof. Joydeep Dutta
Department of Humanities and Social Sciences
Indian Institute of Technology, Kanpur

Lecture – 12
Mean-Value Theorem and Taylor's Expansion-2

Today we are going to talk about Taylor's expansion. Who was this Taylor? You can see the spelling t a y l o r, who was he? There many (Refer Time: 00:27) this sort of Taylor names in the scientific literature.

His name was Brookes Taylor and he was the student of Isaac Newton. He was listening to the explanation Newton was giving about the binomial expansion that is a plus b to the power m which you have learnt in high school. Now, he thought whether this binomial expansion idea can be used to make computations of functions easier. For example, if I had asked to you to find sin 36 degree.

(Refer Slide Time: 00:59)



You will not find it easy because you have few angles - sin 0, sin 30, sin 45, sin 60, sin 90, these are the things given in the table, but if I ask you sin 36 or sin 1 degree or 2 degree or whatever you will you find it little problematic.

So, given such sort of functions which are not algebraic in nature, so this top of thing the sin function or sin x is non algebraic. It is because this is of non algebraic in nature, is it possible to approximate them through an algebraic function or a polynomial function. Where everything is written in terms of powers of some unknown quantity for example, can I write this as sum a 0 plus a 1 x plus a to x square plus, so a n, x n something like this.

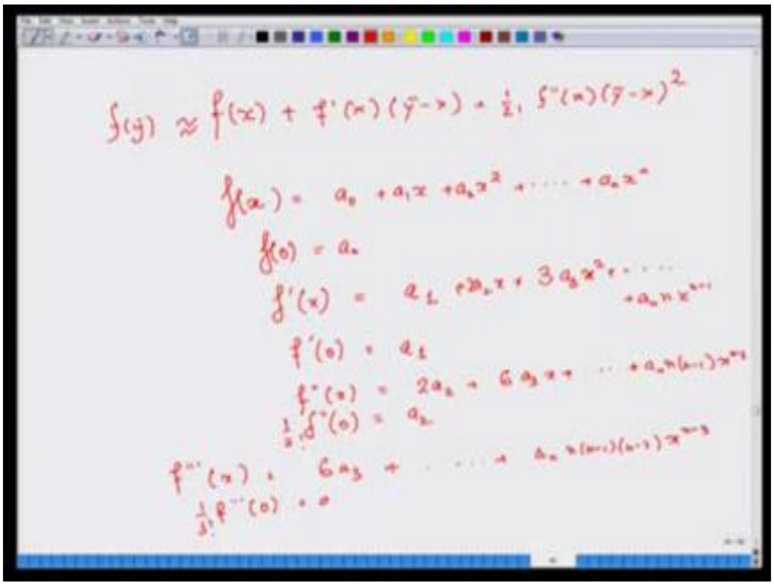
So, again I can approximate it. So, what does it mean is that I cannot compute the value of sin x, when x is some arbitrary number like sin 36, by doing algebraic manipulations like or using the 4 basic arithmetic operations like addition, multiplication, division and squaring and all those things. So, by approximating I mean able to immediately give you an approximate value of sin 36 or sin 20 or whatever. Today we are going to talk about how Taylor thought about this idea, to approximate functions which I have not easily computable.

Look at a function like this, nice differentiable function. Let me draw it bigger, so it magnifies. Look at a function like this and let me take a point here x f x and I am drawing the tangent line. Now, this is your x and this point that you see is x and f of x. If I zoom in here what do I see? If I zoom in here, I will see that at this point the tangent is almost touching, almost very near the distance between the tangent and the curve becomes smaller and smaller as I approach x f x, whether for this side or this side.

Which means that when if I take x, another say y very near x then f of y can be approximated by computing the value of y on the tangent. It can be approximated by computing the point on the tangent, the corresponding point on the tangent. If you look at the equation of the tangent, if this is a tangent and it has slopes say m; does not matter, which is of dash x. Tangent equation for this curve is y, is equal to f of x plus f dash x into y minus x. You take any y here, all were near f of y is slightly higher than y, but may not be very high. I can approximate f at any other say y hat y (Refer Time: 05:11) hat. This is y hat or y bar sorry y bar. As I am approximating is not equal to f of x plus f dash x y minus x. This is called the linearization of f this is called the linearization of f or the first order Taylor polynomial of f.

Now, when you know about the linearization of f, you feel that it is not possible in a great idea. If I move away a little, then I will lose this approximation. This approximation would be very bad and crude. Let me in look at the graph, it is graph more magnified at this point. This is the tangent and I move away little. Instead of the tangent now if I had made the tangent little curvaceous then I could better approximate the function.

(Refer Slide Time: 06:30)



For example, what I want to say is that now, if you add to say f of x plus f dash x y minus x or y bar minus x; here it would be y bar please note that. So, what I have now made is curve instead of a line.

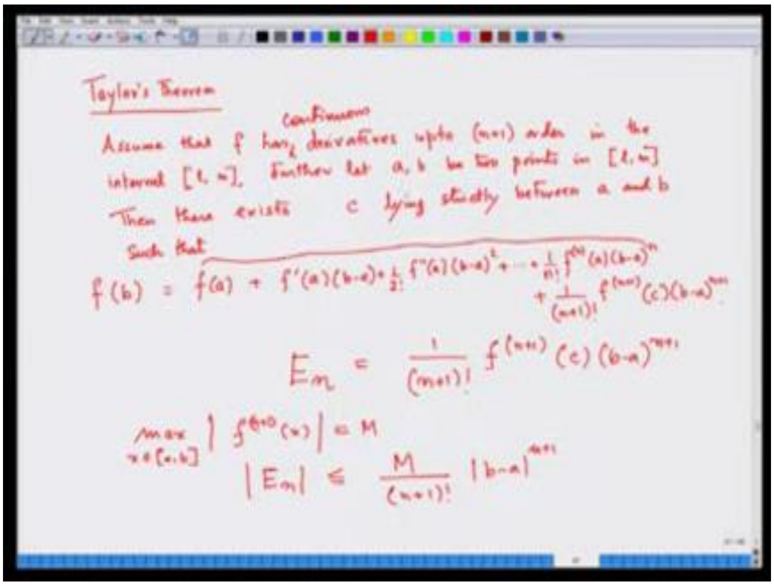
It is a quadratic curve, it is some sort of a parabola and it is approximating f much better that you can see from the following diagram here that it is approximating f much better. I get a curve. So, you can thing that I can do something with, this will be squared. I can do something more. This is the quadratic term y bar minus x square. It is a coordinating curve term in y bar. This f of y bar now is a better approximation it gives a better approximation of f of y bar do not equal to it once you have (Refer Time: 07:30) a quadratic term.

Now, essential idea is a following that OK, let me say that what would happen if f x is actually a polynomial. Suppose I write that f x can be made not; I want to approximate f x with a polynomial. You can observe that these are polynomials, I am not defining a polynomial because there is something very well known. It is you know you write this is a function of this form where a n is not equal to 0, then it is say another degree polynomial.

So, what I want to say is that suppose f is this and f is differentiable in times then what is your a 0? What is your a 1? Now, let me do one thing, use the f of 0 here is a naught, right. Now what is f dash of x here? f dash of x here is a 1 plus 2 a 2 x plus 3 a 3 x square plus so on and so forth plus a n n x n minus 1. Now f dash of 0 is a 1. So, you have been able to compute the constants in terms of these coordinates.

Then I can compute, I want to compute f double dash x, f double dash x here is 2 a 2 plus 6 a 3 x plus a n n minus 1 x n minus 2. So, f double dash 0 by half or factorial 2; I am just writing, will soon see why - is a 2. Let you try out f triple dash x which is now 6 a 3 plus a n n minus 1 n minus 2 x n minus 3. So, which means remaining all will have x, so there will be zeros, f triple dash 0 by c 1 by 6 which is 1 by factorial 3 is a 3.

(Refer Slide Time: 09:59)



So, give me any function f x which may or may not be a polynomial then the function g x which I can write as f 0 plus f dash 0 into x plus 1 by factorial 2 f dash x square plus 1 by factorial 3 which is 6, f triple dash 1 by n factorial f nth degree derivative – nth derivative at 0 x n. This g x now, this g x is called the Taylor polynomial of f at x equal to 0; earlier what this was the second order Taylor polynomial of f at x equal to x.

Now our job would be, if you want to talk about a Taylor polynomial at some alpha. Then you have to write g x is equal to f alpha. So, at x equal to alpha the Taylor polynomial at x equal to alpha. So, plus f dash at alpha x minus alpha, same thing 1 by 2 factorial f double dash alpha x minus alpha whole square plus 1 by n factorial f n alpha x minus alpha to the power n this is call the Taylor's polynomial at x equal to n alpha. That is just the wonderful stuff.

Now, we are going to formally state what is the Taylor's theorem. Once we state that it would be easier for you to understand what we are trying to do. Now, you observe that if I want to compute the value of function at a given point by taking base as some other point then I can approximate it by a Taylor polynomial. If you look at up to this part or what I have written down here, this part is a Taylor's polynomial.

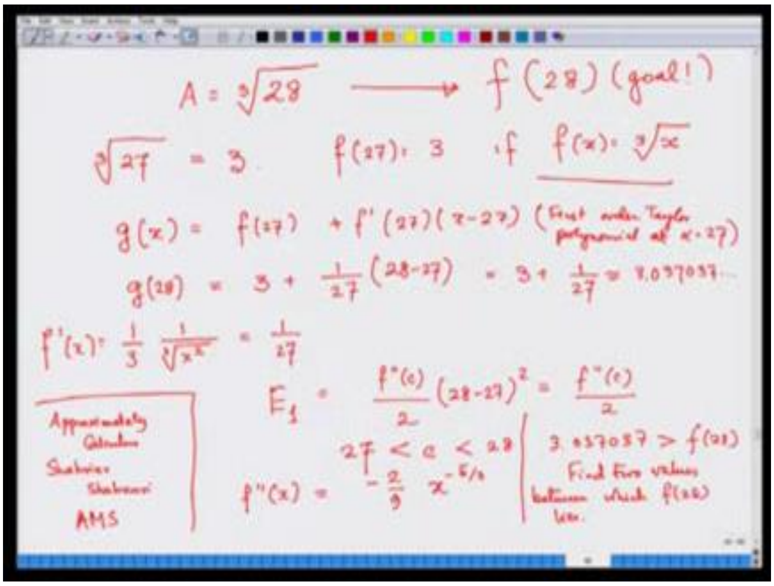
But what Taylor did was to also compute how much error that it would have. If I have the function value at f b and I take its approximation is in the Taylor polynomial this part, the part which I have actually given a line across; not the part which I have written down. Then Taylor is able to calculate the error. So, what he says is that if you have derivatives which are up to n plus 1 with order in the interval l m and rather maybe I would just say continuous derivative. I would like to add continuous derivatives.

If I assume continuous derivatives up to n plus 1 with order and let a and b be 2 points in that interval l m then what Taylor shows was at there would be c which is lying strictly between a and b. It depend whether a is bigger than b or b is bigger than a. Then, c would be greater than or bigger, strictly lesser than b if b is bigger than n just opposite if a is bigger than b. Then f of b is equal to the Taylor's expansion (Refer Time: 13:43) order let us mention expansion plus and error term. This term is called the nth level error.

Now, because I have assumed that f has continuous derivatives up to n plus 1th order then I have something interesting, because once I say that I have a nth order derivative which is continuous up to the n plus 1th (Refer Time: 14:25) which means that this value f n plus 1 x this is bounded over to close interval a b, whatever it is. It could be a, it does not matter. This is say m, you can say that the error e n, this m is bounded by n plus 1 factorial in to mod b minus a n plus 1.

The error can also be shown to be bounded in this particular case and that is very very important because suppose b and a, the distance between b and a is small, so small that it is less than a. Then as n becomes bigger this part will go to 0 and this part will also go to 0. So, as the n is becoming bigger your approximations can become very better and better and better. If b and a are very close to each other than by increasing the degree of the approximation a and n plus 1 n plus 2 as you make the n higher and higher and higher your functional values become closer and closer and closer. That is the key idea what the Taylor's theorem is. So, what you can for example do some little bit of calculations - For example, I want to calculate a equal to cube root of 28.

(Refer Slide Time: 16:00)



I want to compute a, is equal to cube root of 28. Now if I want to compute a is equal to cube root of 28 then is not so apparent to me, what should I do? but I know that cube root of 27 is 3. Now, what I want to do if I know f of 27 is 3, if I said if f of x is cube root of x, if I said f of x is cube root of x then f of 27 must be 3. Now I would like to take 27 as a base and expand in Taylor expansion.

Let me look at the Taylor's expansion g x add up x with base 27 for this function f x equal to cube root of x. That will become f 27 which is 3 plus f dash 27, I will not compute a values of f dash 27 just stating here. I have already done the calculations. So, I will not redo them in front of you which you can calculate. So, f dash 27 is nothing, but 1 by 27.

Let me for the time being not go to higher dimensions and higher orders and just considered on the first order expansion. So, first order Taylor polynomial at 27. First order, now once I have done that I have to look into the following. So, what do I have? I have here 3 plus 1 by 27 which is f dash 27 because what is f dash, somebody is uncomfortable with that let us just do it.

What is this? It is x to the power one-third. It is nothing but 1 by 3 into x to the power 1 minus two-third, right. So, basically two-third means 1 by cube root of x square. Once you do this you will again see that you get, x to the power minus two-third. So, you put x is equal to 27. What is 27? 27, so you take one-third of 27. So, what is one-third of 27? One-third of 27 is 3 and then you take square of that, that will be 9. And you bring it down because you have a minus 2 and hence you multiply with the 3 in the bottom. That will give you 1 by 27. Please check it out yourself it is a very simple calculation.

Now it is 1 by 27. Let me just compute the g of 28. That is exactly what I want. Here my goal is now, if I have given the function like this my goal is to compute f of 28, this is the goal. So, f of 2 1 is now f of 28 is now this 28 minus 27. So, you see how nice it becomes. It becomes 3 plus 1 by 27 which is 3 of course, cube root of 28 would be slightly bigger than 3 because cube root of 27 is 3. It means that when this is approximately 3.037037.

This example I have picked up from a lovely book called Approximately Calculus. This is probably by American mathematical society and written by Shahriar Shahriari. It is a lovely book, which really shows you the power of the ideas from calculus to do numerical approximations. This is what you have. But what is the error term here. So, your error term here at the second level, e 1 is f double dash c by 2 into 28 minus 27 whole square. It is again nothing but f double dash c by 2, at c is some number between 28 and 27.

Now, what c I should take that is the question. So, you might say all things uses because they do not know your c. So, what did a Taylor do? See you really can try doing some worst case scenarios with it for example, you can say that - let me take, what should be the value of c which would give me the worst value. For example, f double dash x in this case is minus 2 by 9 x to the power minus 5 by 3. So, for any x which is lying between 28 and 27 these values would become negative.

That is the very very important conclusion. If this value becomes negative, so if I add these term then that would be exactly equal to f of 28. So, whatever you have, you have to subtract some quantity from this thing to get f 28. The immediate conclusion you get is whatever you have calculated 3.037037 is actually strictly bigger than f of 28 because you have to subtract this quantity for some c that you want to put to get to f 28. This information you immediately get.

One can say that let me see, if I take 27 what would happen? What I can do is you I can I would leave you to do an experiment at home put 28 here instead of c and c what is the value. Put 27 here and see what is the value. That would give you two extreme ends, right. So, as I move from 27 to 28 this quantity will continue to remain negative and it will go down and so as the result of which you can actually say between which 2 values f 28 lies.

This could be a little home work, find 2 values that will give a better estimate of f 28. So, where does f 28 exactly lie, you can pin point it. So, find 2 values between which f 28 lies. So, we have some idea of handling the Taylor's theorem and then we will start integration in the next week.

Thank you very much for your attention. Just have a practice with your assignments. And keep on looking at the lectures, if you forget something go back to the lecture and look at it. If there are still problem put your things in the forum of our TA and I will take care of them and answer back to you.

Thank you very much.