Basic Calculus for Engineers, Scientists and Economists
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Lecture - 11
Mean-Value Theorem and Taylor's Expansion – 1

We are back and actually I am giving these lectures one after another. So, I am going to talk to you about a delightful result called the Mean Value Theorem. What does it say? It says that take a function a f from a to b, sorry from a b to R, a b is a closed interval.

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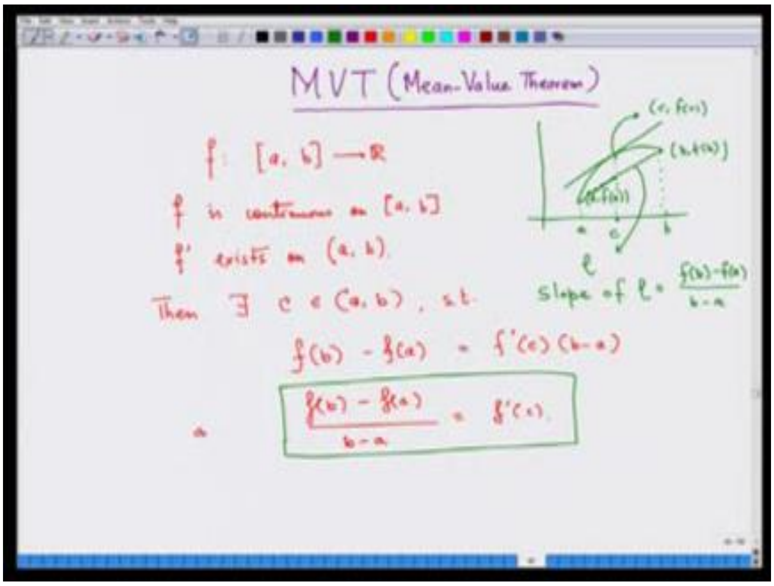

Now, assume that f is continuous on a b. See continuous on a b actually means at the right hand left hand points. We are talking about at the point a, we are talking about right continuity. At the point b, we are talking about left continuity because I do not know anything about thisfunction outside the interval a and b. Then, assume that f dash the derivative exists on the open interval a b and then, I am writing it. So, important they are exists. This is a sign for their existence which is lying in the open interval a b that is strictly bigger than a and strictly less than b, such that f b minus f ais f dash of c into b minus a or the way most calculus books write.

So, what does this signify? Let us look at the picture, the geometry and that will be enough. So, this is f. So, this is a way calculus books write whichever I usually write in this form or you can take this form is up to you, but this form would immediately will give some geometric meaning. Let us just try to see that.

So, I draw sub of function is like this between a and b continuous function. Of course, nice looking and it is differentialat every point means pivot is simply means that you can draw a tangent line at each point, one tangent at each point one tangent totally. So, what is my f b minus f a by b minus say? So, if you look at the point a, I mean at this point the coordinates of this point are - a and f a, now coordinate of these points, this point b and f b, the end points. Now, let us join that these two end points a f a and b f b of the graph by a straight line and then, if I take this line, if I write it as l, then slope of l is equal to f b minus f a by b minus a.

What does it tell you? What does this mean? It tells you it does not matter. I can have a point c lying between a and b. So, there is a point c. So, if I draw that tangent at that point, the tangent to the curveat the point c of c because this point is c f c tangent to the graph of the curve graph of the function at the point c f c. That must be parallel to the line segment joining b f b and a f a and then, that means that we have the same slope, but the slope of the tangent line at the point c f c is nothing, but if does c and that is exactly what it is written, so the geometric gives you the answer.

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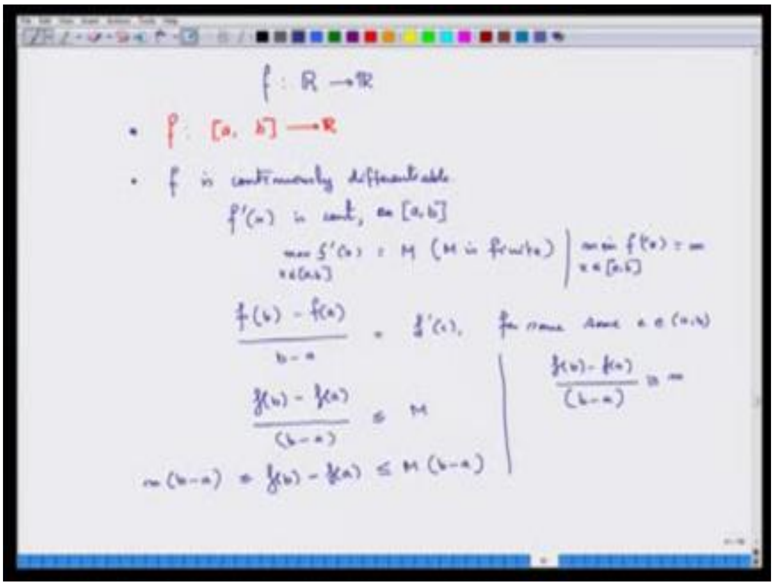

Now, the question would come is, this only done for functions which are on some closed interval. It does not tell me I think if I take functions a form r to r, of course once you have functions of r to r, you are more flexible. So, it does not matter, then you take any two points say a b on our, then the same story can be told that you can always at f b minus f a or b minus a is equal to some f dash c, where c is strictly lying between a and b. If you assume that the function is continuous and differential will throughout.

Now, let us take one by one the consequences of the mean value of (Refer Time: 05:21). Let us look at the first consequence. Let us just keep the definition to f from a b to r the definition which we have started. Now, let me consider the situation where a f is continuously differential. It means that f not only has a derivative, but the derivative of f as a function of x is continuous. So, I repeat function is continuously differentiable if the derivative off as a function of x is continuous. Now I have just told you about continuous differentiable function. Let me assume that f is continuously differentiable. So, f dash x is continuous.

Now, the interesting feature is the following that suppose I take the interval a b, it was interval a b and I say that f dash x is continuous means it is continuous on the closed interval a b. So, I cannot say just continuous from the open interval a b, of course is the continuous from the open interval a b. If you say continuous differentiable, let us assume that f is not only differentiable over the open interval a b. Let us check the f is differentiable on the closed interval a b, but at the right interval at the right side you take the left derivativewith right derivative at a, and the left derivative at b. Let us go on this thing. So, f dash x is continuous on a b.

Now, what does it tell me? It tells me the following fact that if this is true, if f dash x is a continuous function, then if I maximize this function, maximize minimize as a matter because a b is the closed interval, b c is the finite number alpha. It is bound to exist or may be it is better to say capital M, because it is maximum - so these values are finite.

So, what is the consequence of this? Now, the consequence is the following. Now, at the two end points, I apply the mean value of m to get the following f b minus f a by b minus a is equal to f dash c for some c in a b. Now, whatever we are f dash, it does not matter. It has to be less than m. So, f b minus f a is b minus a is less than equal to m. So, f b minus f a is less than equal to m times b minus a.

Now, let us see what happens. So, you have got this fact. Now, if I swap d with a, I am changing b with a. What sort of inequality will I get? Now I make a swap. If I change the inequality, then I can get a reverse inequality that this is greater than minus m times v minus a. Let me tell you now you have got these, but you also know that because of studies about continuous function that the minimum value of these f dash xover x minus a b, these value also exists.

Let this m, but f dash c is then greater than equal to m because m is the minimum value. So, I can again have this. Now, this simply means the following. Let us see what does it mean? It means that no f dash c is here greater than or equal to m. So, what would happen is that we will write now how do I get this? It is f b minus f a by b minus a greater than equal to m. So, you get f b minus f a is greater than equal to m times v minus. This is a fact that you can get immediately from the mean value algorithm, but what does it tell? It tells me anything more the fact is again this.

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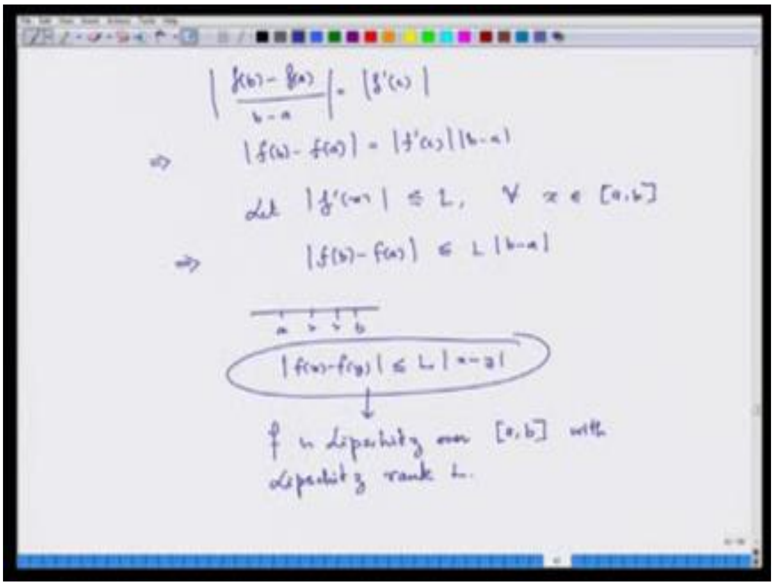
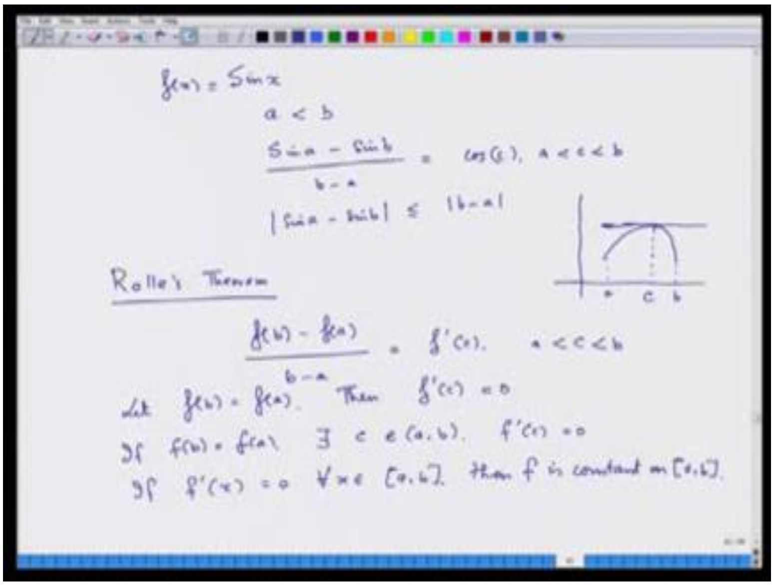


Now, if you have f b minus f a by b minus a, you can write this as f dash c. Let me take the modulus of this quantity. If I take the modulus of this quantity, this implies mod of f b minus mod of f a is less is equal to f dash c, the mod of b minus a, but again because f is f dash c is bounded f f dash x is bounded mod of f dash x is bounded, the absolute value also must be bounded. You cannot blow up.

So, the absolute value also has some maximum value. So, bounded actually means there because the derivative is bounded say f dash x is less than some capital L. So, let f dash x is less than equal to capital Lfor all x element of a b and this implies the following. This implies that f b minus f a is less than equal to l times b minus a. Now, it does not matter. Whatever x and y you choose between b and a, so you take any x and y between b and a. I can always show that f x minus f y is less than equal to L, the same L x minus y.

So, this simply means that this property is called Lipschitz property. So, we say that f is Lipschitz and this is very important property in many aspects of mathematics and engineering, scientists, economists and all those things. So, f is Lipschitz over a b with Lipschitz rank L of very unlikely function which satisfies the Lipschitz is n s over any given interval is the sin function. It is very unlikely. For example, you take the sin function.



So, take any a strictly less than b and you can write sin a minus sin b by b minus a is equal to cos of. So, I am applying them invalid Theorem cos of some c, where c is lying strictly between b and a, but again you can write mod of sin a minus mod of sin b is because cos of c is always less than 1 is less than equal to mod of b minus a. So, with Lipschitz rank L equal to 1, the sin function is Lipschitz in the interval a to b.

Now, very simple application of this is the Rolle's Theorem which usually is taught in calculus classes before the mean value Theorem, but I decided that we should go from the mean value Theorem to the Rolles Theorem and it makes life obvious. Some peoples say no. You should do Rolle's Theorem first because the Rolle's Theorem is used to prove the Mean Value Theorem, but does not matter you can always prove the Mean Value Theorem without Rolle's Theorem.

Let us look at Rolle's Theorem. Rolle's was a very colorful character by the way, but this Theorem is very interesting, same sort of stories. Now, write down the Mean Value Theorem. Now, make this interesting assumption. The assumption is let f b is equal to f a. So, if you assume that f b is equal to f a, then you are showing the following that f dash c is equal to 0 which means if this is holding, so if f b is equal to f a, there exist c which is in the open interval a to b.

So, as that f c is equal to 0 which means I can find the critical point of the function between is lying strictly v to a and b. For example, if you look at this diagram to explain the Rolle's Theorem, so this is a, and this is b. The function of height is same. So, it is like this for example you see it has always one point here where the tangent is parallel to the x axis and that is exactly the meaning of f dash is equal to 0 which we are not told about.

In the last lecture, the tangent parallel to the x axis means that has a zero angle with x axis and hence, the slope value is 0 because tangent value is the tangent of the slope. The tangent is actually tan of the angle at which it intersects, the x axis which is very well known fact. So, here is your c. So, you see how interesting result of the Mean Value Theorem leaves you to several more facts. There are certain things, more things you can talk about.

Now, I will tell you something interesting. Suppose f dash x is 0 throughout the interval a b, what is going to happen? So, use the Mean Value Theorem to show that if f dash x is equal to 0 for all x in a b, then f is constant on a b, I would leave you to work on this. It is just fun. So, how by simple arguments, you can get fantastic ideas of what the nature of function is? So, with this let me stop and in the next talk, we are going to talk about very important thing.

If I want to compute a function which is slightly difficult, if I tell you to compute sin three degrees, you cannot immediately give me the answer. So, any way we compute them, this sort of difficult function by just doing basic arithmetic operations of additions, subtraction, multiplication and division. Now, the amazing easiest we can do a very good approximation and we will talk about that in the next class on Taylors expansion.

Thank you very much.