Basic Calculus for Engineers, Scientists and Economists
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Lecture – 01
Numbers

Welcome to this course. You must have already heard the 2 minute video in which I spoke about the importance of calculus. But, the fundamental to every mathematic is numbers, without numbers there is no mathematics, so ultimately mathematics has to speak about numbers.

Today we will start by talking about numbers and trying to understand some of their important properties. First and foremost we speak about Natural Numbers.

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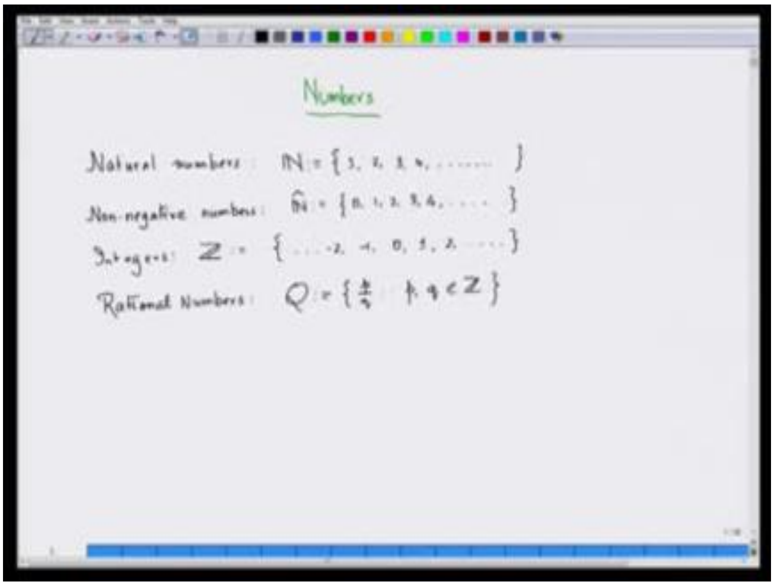

Natural numbers are essentially natural 1 2 3 4 5 6 7 8 9, I really do not have to make an explanation to you about these things. 1 2 3 4 5 6 7 8 9 10 11, whatever as it goes on is the numbers we use to count, we use them for counting everyday objects, we use them for counting money we use them for keeping dates. So, these numbers are absolutely natural and remember this number starts with 1 and with 1 you keep on adding 1 and then go on, it just goes on and does not end at all. It is moving on without stopping is what we say moving towards infinity.

Now some mathematicians would like to add 0 to this sub set of numbers, where 0 is a number which signifies that there is nothing. Then, negative numbers which you are aware of and the collection of negative numbers, the natural numbers and 0 is as the set of integers.

Now, what is the negative number? The question could be very natural because Blaise Pascal, a very famous mathematician about whom which you have possibly heard about in high school when you started studying binomial expansion of 2 numbers. So, you must have heard about Pascal's triangle which gives you the coefficients that comes in by different binomial expansions.

Pascal once might have comment that, for him it is difficult to imagine how you can take away something from nothing. So, the issue of negative numbers became very big rather a philosophical issue but, to understand negative numbers just you can look in to the idea with credit card. If we have a credit card, when you get a credit card you get if for free, there is no account - you have an account, but you have no money there. So, when you go and swipe a credit card you essentially are borrowing money from someone else, borrowing money from the bank whose credit card you hold.

This process of borrowing means that you owe someone some money, at this movement that money is really not your money. To signify that fact that you have borrowed that money and dash sign is putting before that and that introduces what is called a negative number. We saw that you have a negative balance on your account.

It is a very good way of showing what we are short of. So, negative number thus is a very natural extension of our basic working in daily life. Along with, thus negative numbers, positive number and 0, the natural numbers, negative numbers - minus 1 minus 2 dot dot and 0, we form what is call the integers.

After that the most important number possibly is a fraction and we know everything about a fraction, we were taught in schools that you have 5 pieces of cake and there are 6 peoples standing how you will equally divide the cake among all the 6 peoples; so that these sorts of small questions are actually teaching you about fractions.

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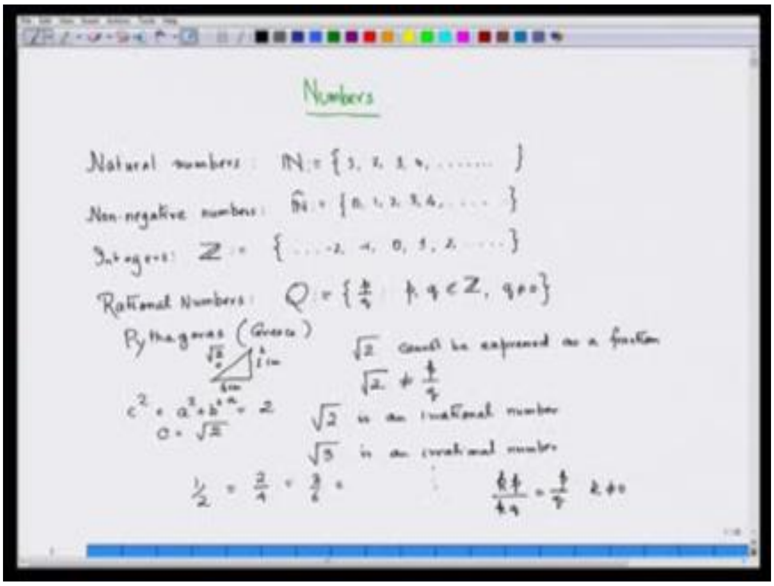

Fractions or rational numbers are essentially numbers which can be expressed in from p by q, where p is an integer and q is an integer. Rational number of course, has both negative rational numbers and positive rational numbers. Long back in antiquity during the time of the famous mathematician Pythagoras in Greece, there was this concept that in this world we just have the whole numbers and the fractions thus nothing else and everything has to do with numbers that was Pythagoras (Refer Time: 04:59), but these are the only things that we have.

Then one of these decibels used his famous Pythagoras theorem in geometry to show that there are some numbers which have beyond the standard fractions and whole numbers that we know. So, consider right angle triangle whose both the perpendicular and the base or a 1 centimeter each. Then you come to a number called root over 2 as the length of the hypotenuse because if this is has Pythagoras theorem says, so this square of this side plus square of the base plus square of the multitude is with the square of the hypotenuse. So, this side I call a and this side I say b and this side is c, and c square is equal to a square plus b square and if a is 1 centimeter, b is 1 centimeter and c square is 2 centimeter. So, c because it is the length we cannot take the negative roots, so it is root over 2 centimeter.

It is 1 of the most important things to know that root over 2 cannot be expressed as a fraction and I will leave you to do the prove in the exercises. Root over 2, you cannot write it in the form p by q, of course there are many other numbers which are not of the form p by q - where p and q are integers and q is not equal to 0. Of course, you can of course ask me this that what would happen if q is equal to 0. So, such a thing is not defined, so you can just improve our writing a bit here.

Now, these numbers which cannot be expressed in the form p by q, or call irrational numbers; so root 2 is an irrational number. Similarly root 3 is an irrational number and so is root 5. So, these are some very basic facts about numbers

Now we are going to talk about the fact that the given fraction can be expressed in many, many different ways for example, if I take half I can multiply both sides - on top and the bottom by 2 and I can get 2 by 4 or I can multiply by 3 and I can get 3 by 6. So, all of these are equal to half. So, basically give me a fraction p by q and if I have a number k and if I multiply both up and down by k then kp by kq is always p by q. So, given a fraction there are many other fractions which are actually equal to this. This is essentially the notion of an equivalence class about which we are not going to speak in detail.

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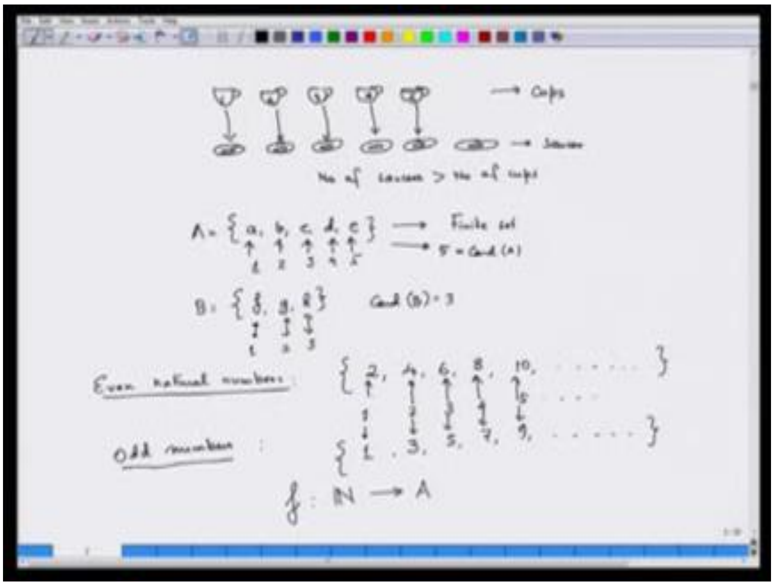

Now, what is the main use of natural numbers? Its main use is counting and you know that we use 1 2 3 4 5 6 7 8, we count everyday objects. Now how do we actually count? If I have a say few cups, see I have a bag which has 5 cups - tea cups and then there is a bag which has saucers means plates, which things I have more, Cups or Saucers? That how do I decide; Even if I do not know to count, how would I do if I do not know about numbers 1 2 3 4 5 6 7 8 9. The best way to do is to put a cup on a saucer.

So, I put this one on this, the third cup on this, the fourth cup on this and the fifth cup on this, and I there in see one saucer is left out which means that the number of saucers or plates is strictly bigger than the number of cups. So, what you are essentially doing this can be marked as 1, this can be marked as 2, this can be marked as 3, this can be the first second, this one marked as 4 and this 1 marked as 5. So, again mark this cup as 1, this cup as 2, this cup as 3, this cup as 4 and this cup as 5.

I am using abstract numeral to signify a cup. So, the same number 1 is signifying a cup is also signifying a plate. 2, is the second cup is also it is signifying, this cup is only signifying this plate. Essentially counting means putting a set of objects in 1 to 1 correspondence with the set of natural numbers - For example, if you have 5 object, a b c d e then how do you count it; a is the first object, b is the second object, c is the third object, d is the fourth object, e is the fifth object, so there are 5 objects. This is of course, you can count is a finite set.

In the same way if I have say a set continuing f, g and h. So, this is the first object that is you are relating a with f, this is the second object, this is the third object. So, you are essentially relating h with c, b with g, but there are 2 objects more in the sets with which I can mark here as a and I can mark here as b.

The cardinality, the amount of elements in a set is called the cardinality of the sets or the cardinality of this set a here is 5 has a cardinality of a. And this set, the cardinality of b is 3. So, cardinality of a is more than cardinality of b, so b has more elements. What is more fundamental is what happens if my set is infinite, it has elements on the number of elements around 10 for example, the set of even integers and this was observed by Galileo, even for natural numbers.

So, what you do it even natural numbers? With even natural numbers that this is a set consisting of the numbers 2 4 6 8 10 and so on and so forth, it does not end - how many objects are there, and then also take the set of odd numbers that is 1 3 5 7 9 and so on and so forth. Now are of these 2 sets countable? Yes, the sets are countable in the sense because 2 is related to 1, 4 is corresponding to 2, 6 is corresponding to 3, 8 is corresponding to 4, 10 is corresponding to 5 and so on. Similarly for the odd numbers 1 is corresponding to 1, 2 is corresponding to 3, 3 is corresponding to 5, 4 is corresponding to 7, 5 is corresponding to 9 and so on.

Both these sets even numbers, set of even natural numbers and odd natural numbers are countable. But the remarkable feature at though these 2 sets are subsets of the set of natural numbers, the elements of these 2 sets the cardinality or the in-cardinality of this infinite set if that has any meaning that cardinality has same as the cardinality of natural numbers. Infinite number of elements that these 2 sets have same as the infinite number of elements that set of natural numbers have, this is very very counter intuitive, but this is what was shown first observed by Galileo and later taken as that very definition of counting in a very basic idea what infinite is if I want to talk about infinite as a number it was first studied by Georg Cantor.

Essentially, what we have done that, when we are tal king about counting we are essentially putting a set in 1 to 1 correspondence with the set of natural numbers that there is a bijective mapping. I have not spoken about bijecting mapping, but you can understand this is very basic correspondence this is like a child's play.

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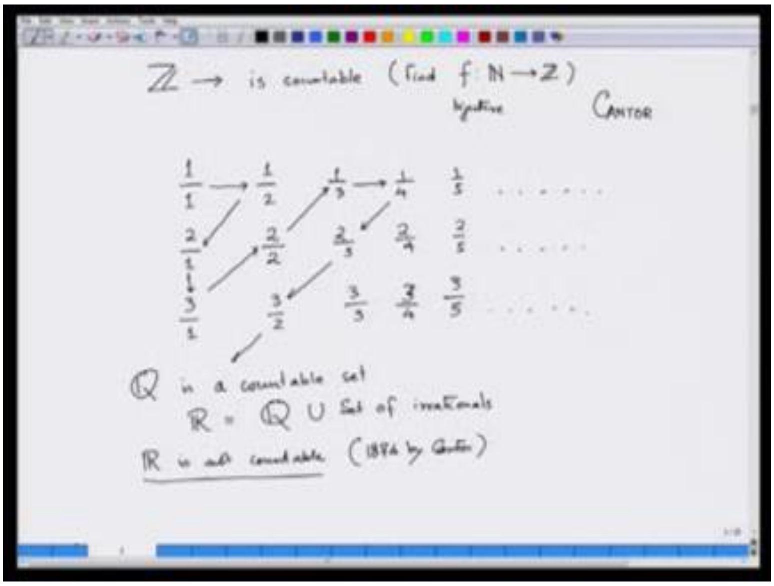

The interesting feature is that, not only this set is countable, but the set z - the set of integers this is countable. So, essentially you have to find a function to show that z is countable; the job is to find f. Here I said that a set A is countable if I can setup this bijective map. Similarly, z is countable; I have to find a map - bijective map from n to z. So, in your homework you will check out, how you setup that bijective map. Bijective map, your 1 to 1 correspondence I most probably would know what bijective maps etcetera they are taught in high school, but we will talk about them in on next lecture.

How do we show that z is countable? The interesting feature of showing z is countable is done in the following way. So, first take 1 write down the numbers in this way, put 1 in the denominator and then just write all the natural numbers - 1 in the numerator and all that natural numbers in the denominator, so 1 by 2, 1 by 3, 1 by 4, 1 by 5th. So, if then you would start with 2, 2 in the numerator and all the remaining numbers are put in the denominator - 2 by 2, 2 by 3, 2 by 4, 2 by 5 and so on.

Similarly, now you take 3 in the numerator and rest in the denominator. Of course, there is repetition 1 by 1 is same as 2 by 2 is same as 3 by 3. A way Cantor taught us to count, I will just write the name is very important to know the name of this famous mathematician whose spend a last part of life in an lunatic asylum as we thought he was mad, because he was talking about actually handling infinite like a number and not just about telling that; it means that you can just grow without bound.

Here the step is the following, you start with this one and go to the right, then come down to the left, go down and go up again, and then go here again and again do the same thing go the right, come down to the left and so on and so forth. So, you have to come down absolutely do the left like this and just go on. From here again go to; from here you go back again. Here from whatever is the number you go back here and here and here like this. So, this procedure that we have just shown it can be used to show that this is countable, so you are not counting 1 2 3 4 5 6 7 8 9 and so on. Basically we have been able to set up this map.

So, what we have now shown that the set of rational numbers q we have done it for the positive parts, so you saying for the negative part, so set of rational or you can put plus minus plus minus it does not matter. So, you can put 1 2 3 4 like this, so q or set of rational numbers is a countable set. The set of rational numbers along with the set of irrational numbers forms what is called the set of real numbers and we are suppose to study about real numbers, because calculus essentially handles real numbers and functions defined over real numbers.

Now, I just would like to stop here by making this famous result known to you that the set odd is not countable, that you cannot set the set R in 1 to 1 correspondence with the set of natural numbers. So, the set R is not countable - this was also first shown in 1874 by Georg Cantor.

With this I end talk here which gives you a very brief idea about what numbers are and in the next talk we have going to talk about functions. To give you what are the very basic properties of functions.

Thank you very much for your attention.