Introduction to Probability Theory & its Applications

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Lecture - 9
Continuous Random Variables and Their Distributions

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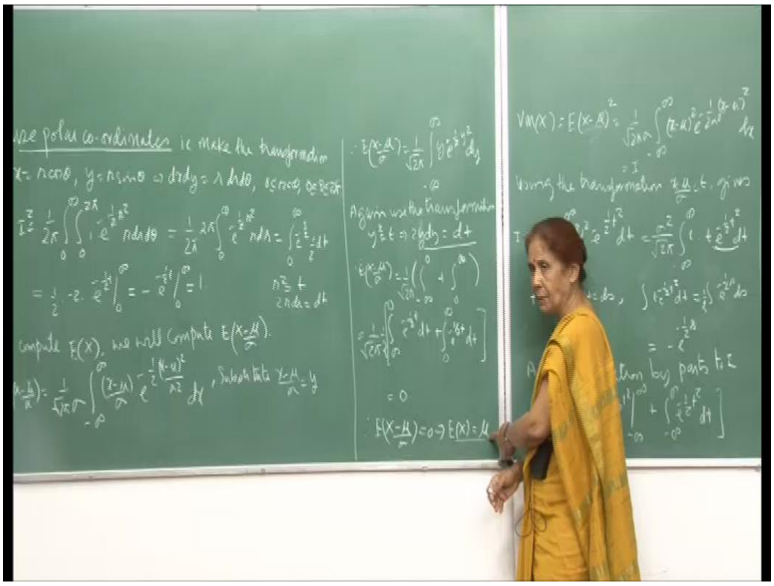

To compute that double integral, I make use of the polar coordinates, that is, I make the transformation x is equal to r cos theta y is equal to r sine theta. Then, you know, that dx dy, the element of area in x-y plane gets transformed to r dr d theta, that is the element of this term. Your r varies from 0 to infinity because r is a non-negative variable and theta varies from 0 to 2 pi. This is your polar coordinate transformation.

So, therefore I square, then becomes 1 upon 2 pi 0 to infinity 0 to 2 pi and e raise to minus half r square because this was x square plus y square will become r square cos square theta plus r square sine square theta since cos square theta plus sine square theta is 1. So, it reduce to r square. And this r dr d theta, right. And again, this is in a nice form because now you see, you have r here and you have e raise to minus half r square.

So, again we will make this transformation, right, that r square is equal to t that is what I am doing here. And so, your 2 dr, dr will be dt. So, r dr will be half dt and so, that is what I have written here, half dt and this will go to e raise to minus half t. So, this is and you write, and see the limits are 0 to 2 pi 0 to this thing.

Now, here what I have done is, already theta does not appear here anywhere. So, therefore, this is simply 1. So, with respect to theta this just integrates. So, 2 pi, which I have written here and then is only integration respect to r. And to integrate this I make the transformation r square is equal to t and so, this helps me to reduce it further. And this is, you know, minus 2 times e raise minus half t 0 to infinity. So, this reduces to this, which is equal to 1. So, we have verified, that the normal p d f, that we have defined is valid p d f.

And then, now the second step is to compute the expectation and instead of computing for e x we will compute expectation of x minus mu by sigma. That will be easier because we know, that the expectation of x is actually mu. So, we will show, that this expectation is 0 and therefore, get the answer immediately. So, here the, only in this integral you get this term x minus mu by sigma and so, again I make the substitution x minus mu by sigma is equal to y and that reduces the integral to this. Again it is of the same form. You see, that is why the expressions may look cumbersome, but the working is not very difficult and it is just a question of little patience and you start seeing where the calculations are going. So, now we have this.

So, again you make the transformation y square equal to t, the same steps, and you get this. And so, so now, what I am saying is, that before I make this substitution I will break up this integral to minus infinity. See, one way I can immediately from here conclude, that this integral is 0 because this is an odd integral, right. y is here, this is here, the change of sign will not matter. But here the change of sign will matter since this is from minus infinity to infinity, this integral will be 0, you know one of the… So, I just thought, that I will show you the steps.

So, what I am doing is, I am breaking it, breaking up this integral from minus infinity to 0 plus 0 to infinity and then, when you make that transformation, that y square is t, then you see, this becomes plus infinity because when y is minus infinity square will be plus infinity. So, this is from, so this would be infinity to infinity to 0 and the integral will be e minus half t, just as we did here and then, this will be 0 to infinity. So, now, it is a same integral expect that here the limits are upside down. So, when you change the limits it will become minus sign. So, there will be a minus sign here and the same integral with the plus sign. So, when you add, the result is 0, ok.

So, I just showed you the steps, in case you are not very sure, but otherwise we could have concluded our computation at this step only and said that this is equal to 0. So, since this 0 expectation of x minus mu by sigma, sigma is a constant, goes out. Therefore, this implies, that expectation of x is equal to mu. So, I had shown you that the normal p d f is symmetric about mu and this is also the mean. And in fact, mu has all the properties, that too now I will show some more properties of mu.

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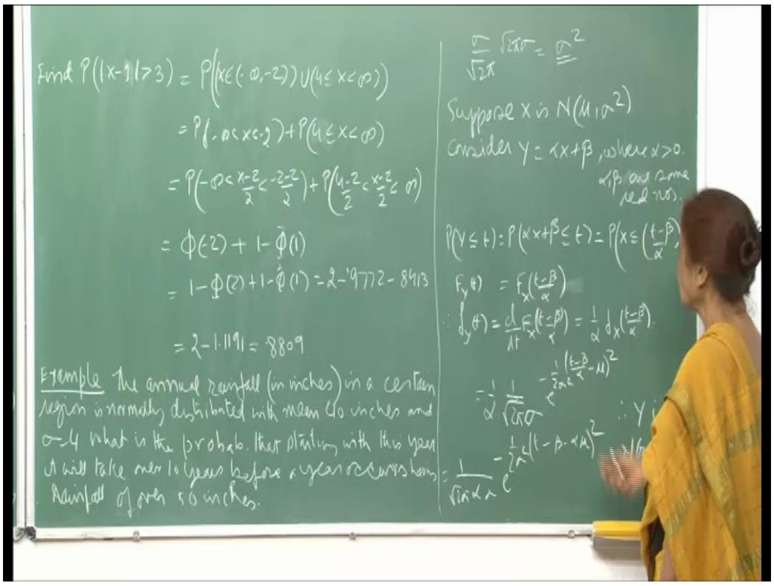

Similarly, to compute the variance of x, which is actually, since mu is the mean, it is actually expectation of x minus mu whole square, right. And this I will write in this form. So, now here what you have to do is, again if you have first make the transformation, that x minus mu by sigma is t, so dx by sigma can be replaced by dt, right. And the limits remain the same. So, this is the simple integral and this way comes sigma square because you have x minus mu whole square. So, this is sigma square t square. This is it.

Now, what I do here is, the t square I break up into t and t because this integral we have handled already, right. Well, computing this thing and so, therefore, we will do integration by parts. This will be my first function. That means, I take the integral of this, multiply by this and then, the derivative of this into the integral of this. This is your formula, right. So, this integral I have shown you is e raise to minus half t square, this t into this thing, and ok, no, no, why am I, ok, I am actually computing this for you. So, I am saying t e raise to minus half t square. dt is half, when I make the transformation t square is s, then 2 dt is ds. So, this reduces to this and therefore, is this right.

Now, applying integration by parts. The integral of this I have already computed for you is, this minus e raise to minus 1 by 2 s and so, this will be t into, when I transform back again substitute for s from here, this is t square. So, then t into e raise to minus half t square minus infinity to infinity, you can see, that this is 0, right, because this is t square. And so, in the denominator t square e raise t e raise to infinity is much, much larger than t in the numerator. So, therefore, this will be 0. You will be left with this

And then, here again this is what, this is your in a normal p d f where your mu is 0 and sigma is 1. So, the 1 upon root 2 pi is missing. So, therefore, this integral will be equal to root 2 pi because with 1 upon root 2 pi this will become 1, right. So, therefore, this whole thing is 1. So, this is, this integral is equal to root 2 pi, so I write root 2 pi, which cancels with this and this is sigma square. So, therefore, the variance of the random variable. So, therefore, the parameters are, now it is very clear what the parameters denote, mu is the mean and we say normal mu sigma square mu is the mean and sigma square is the variance.

So, we just saw, that normal distribution, the parameter mu is the mean and sigma square is the variance. Now, suppose x is n mu sigma square and we consider the random variable y equal to alpha x plus beta where alpha is a positive number and alpha and beta are some real numbers, right. So, I mean, I am continuing with the properties of the normal distribution.

So, if you want to find out the p d f of y, then we start with the cumulative distribution function. So, probability y less than or equal to t is probability alpha x plus beta less than or equal to t, which reduces to this and since alpha is positive, the inequality remains in that. So, this is t minus beta upon alpha, ok. And so, this is your cumulative distribution function y, which is equal to the cumulative density function of x, but the parameter the t replaces is, gets replaced by t minus beta by alpha, right. So, now, if you differentiate both sides, that means, differentiate with respect to t, then this will become the p d f, right.

And this is d dt of f x t minus beta by alpha, which will be 1 by alpha into f x of t minus beta by alpha, right. The derivative of capital F x will be the p d f of the random variable

x. And so, when you substitute, you write down the expression for this, it will be 1 upon alpha 1 upon under root 2 pi into sigma e raise to minus 1 by 2 sigma square and your t gets replaced by t minus beta by alpha minus mu whole square. This simplify the expression, this gives you t minus beta minus alpha mu whole square. Here, you have alpha sigma and this is 2, sorry, the alpha in the denominator comes here. So, this will be, there will be a into, into alpha square also.

I hope you can read it, anyway I am speaking it out, may be let me just rewrite it whole thing here. Yeah, this is 2 alpha square sigma square. And so, by our definition of the normal p d f this will be n, the mean now becomes beta plus alpha mu at the various becomes alpha square sigma square instead of sigma, right.

So, therefore, you see, that if you make this transformation where x is normal mu sigma square, then for y, the expectation will become alpha mu plus beta, which anyway, you can show from here also. This is alpha expectation of x plus beta because beta is a constant and so, this is alpha mu plus beta. And the variance will be, just think because the constant will not matter, since you will see, when you write down the variance, you will write down this minus, I mean, this minus this. So, beta will cancel, alpha will come outside and so, it will become square.

And so, either way you can verify, that the, for the, for this random variable then mean will be beta plus alpha mu and the variance will be alpha square sigma square. So, you can see, that you can carry on the properties of normal variant.

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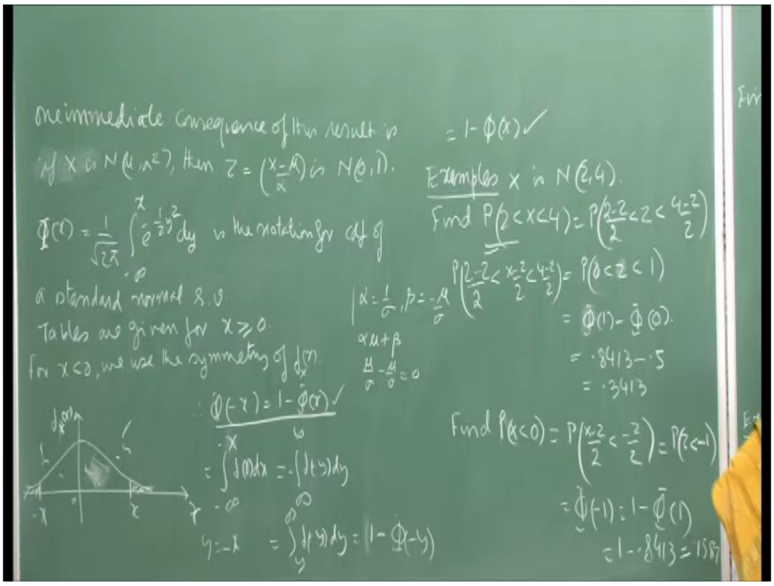

Quite easily, now immediate consequence of this result is, that if x is n mu sigma square, then z, which you write as x minus mu by sigma will be normal 0, 1, right, because what is happening? Your, here alpha is actually 1 by sigma. If you compare it with the expression alpha x plus beta and your beta is minus mu by sigma. So, now if you substitute, because we said, that the mean will become alpha mu plus beta for that one. So, here it will become mu by sigma minus mu by sigma, so it is 0, right. And similarly, you can show, that the variance will be 1. So, the transformation x minus mu by sigma results in a standard normal variant and which we refer to as n 0 1.

Now, we have tables for computing the various probabilities for normal 0 1 and you see, that you can then compute the probabilities for any random, normal random variant through this and that is what, because, because of that transformation, right. And I will work out few examples to show you how it goes. So, anyway this is the standard notation for, for a standard normal variant. This is your probability minus infinity to x. So, that means, this is a cumulative density function. So, this will be 1 upon root 2 pi minus infinity to x e raise to minus half y square d y. So, this probability is given the notation.

Now, the tables are given for x non-negative, you know, for values of x going up to… We will, we will also later on see, that we do not need the values to be tabled for very large values of x. Then, for x less than 0 we use symmetry of the p d f around because this standard normal. So, this is, this is symmetric about the origin, right. And this is your, this is this, right.

Now, the symmetry means, that if you have x here, then the area to the right of this number is the same as the area to the left of minus x. This is what symmetry means. Because this area, this area are equal, therefore, and since this is half area and this is 0.5, so therefore, this shaded portion here same as the shaded portion here, and so the formula is this. So, if you tabled your values for x positive, then for x negative you can get by this and we can verify this formula right away.

If you want to compute phi of minus x, then that is minus infinity to minus x of fx dx. Now, if you write y as minus x, then x becomes minus y. So, the limits go from infinity to y, right, infinity to y f of minus y dy, right, and there is a minus sign here. So, you interchange the limits. This becomes y to infinity f of minus y dy, which is 1 of minus, no, phi minus y, right, because this is now y to infinity. So, therefore, by the formula this is this, but phi of minus y is phi of x. So, therefore, I have shown you, that this is 1 minus phi x. So, this formula has been verified.

So, now you can get values of the cumulative density function for negative, positive, both of x, right. So, let us look at few examples. Suppose x is normal (2, 4), that means, the mean is 2 and the variance is 4. So, then you want to find the probability, that x is between 2 and 4. So, I will use that transformation. So, here, sorry, this should be z, that means, what I am doing is, I show, I will write in detail.

I am subtracting 2 and dividing by 2, so less than x minus 2 by 2 less than 4 minus 2 by 2. So, this probability goes here, right. And now, so this is 2 minus 2, 0. And this is your standard normal variant, right. x minus mu by sigma is normal 0, 1 for which we always have this notation of z, right. And this is 4 minus 2 by 2, which is 1. So, this probability you can compute in terms of the standard normal variant in this way and by our notation this is phi of 1 minus phi of 0.

And from tables I get, that phi of 1 is 0.8413 and phi 0 will be 0.5 because we have said, that the standard normal is symmetric, the p d f is symmetric about the origin. So, this portion of the, so this area under the curve will be equal to 0.5 and this will be also 0.5. So, phi 0 will always be 0.5 because this is the area for phi 0, right, less than or equal to x less than or equal to 0. So, that would be half of the area, which is 0.5, right. So, therefore, this is the probability.

So, therefore, very conveniently, once you have the tables available for this standard normal, you can compute it for any normal variant, right. And again, find probability x less than 0 for this variable. So, here again I make the transformation x minus 2 by 2 less than, so this will be minus 2 by 2. So, this is probability z less than minus 1, so which is phi of minus 1. And then, I use this formula, this formula. So, phi of minus 1 is 1 of minus phi 1. phi 1, I already know, is 0.8413. So, this is 1 minus 0.8413, which is 0.1587, right.

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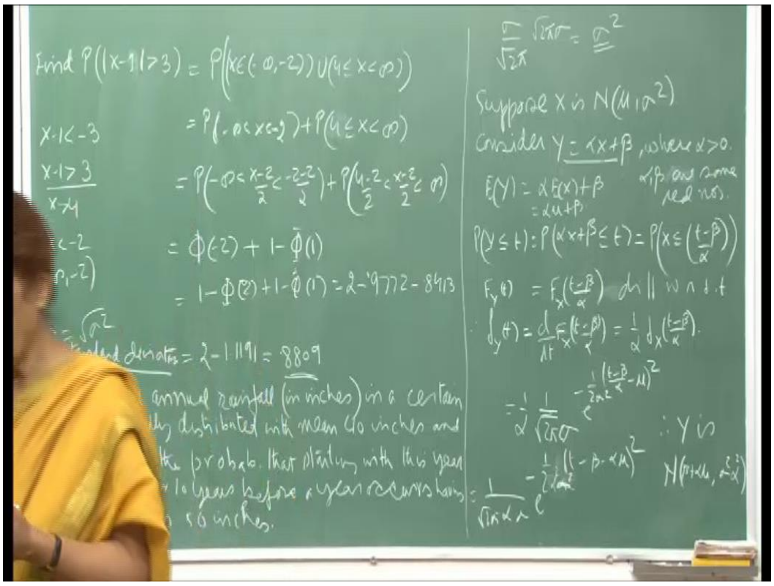

Now, when you want to compute absolute x minus 1 greater than 3, so what are you saying here? You are saying, that your x minus 1 should be greater than 3 and x minus 1 should be less than, that means, x minus 1 should be less than minus 3 and x minus 1 should be greater than 3, yeah. So, from here what do you get, that x should be greater than 4. So, from 4 to infinity. And from here you get, that x should be less than minus 2. So, that means, it should be from minus infinity to minus 2. So, that is what we have done. I have written, that this event is equal to the, that means, x must be either here, minus infinity to minus 2 or it must be in the set 4 to infinity. Well, I have used a different notation here than here, does not matter.

Now, since these two sets are disjoint, you can see, right, minus infinity to minus 2 and this is 4 to infinity. So, I can write the probability as the sum of the individual probabilities and so, I get this. And again, I transform my variable x minus 2 upon 2. So, this remains minus infinity, this is minus 2, so minus 2 by 2.

And here, again I use the same transformation to reduce it to standard normal variant and therefore, this becomes phi of minus 2, yeah, because phi of minus infinity is 0 plus 1 minus, this will be what, this is 1 and this is infinity. So, here phi infinity is 1 and this is minus phi of 1, right. Because yes, you all agreed, that this is phi of infinity, I mean, this phi of infinity will be 1. So, therefore, and phi of minus 2 you can write is 1 minus phi 2, again by our formula for computing negative probabilities in terms of positive this thing and therefore, this is it.

So, I just substitute the values. phi 2 from the tables is 0.9772, this is 0.51 and so, this is the answer. So, one can go on and therefore, the whole idea would be, that you sit down, calculate a few of such probabilities by yourself to become familiar.

Now, let us just take an example here. The annual rainfall in inches in a certain region is normally distributed with mean 40 inches and sigma, that is the standard, we call by the word, I did not name it, but sigma, this is under root sigma square is referred to as standard deviation, this is also known as standard deviation. So, standard deviation is 4 that means, variance is 16. What is the probability that starting with this year it will take over 10 years before a year occurs having rainfall of over 50 inches?

So, let us understand the problem first. They are saying, that the annual rainfall is normally distributed. That means, if every year you add up the total rainfall in that particular region, then those numbers will be fitting a normal distribution, which has mean 40 inches and standard deviation 4, right. So, the numbers that you get as the total rainfall in a year for different years, so that has a normal distribution, right. Now, they are asking for a probability, that starting with this year you go on for 10 years and within those 10 years, no year will have rainfall more than 50 inches. So, it is, you know, actually compounded event, it is not a straight forward. So, let us see how we go about computing this probability.

So, to compute the probability x greater than or equal to 50, I will do the same trick, reduce this probability in terms of a standard normal variant. So, since mean rainfall was

40 inches, so x minus 40, standard deviation was 4, so this now reduces to a standard normal variant. So, therefore, the required probability is the same as probability z greater than or equal to 5 by 2, which is oh, oh, oh this is 1 minus, sorry, this has to be z greater than 5 by 2 would be 1 minus 5 phi by 2, right, and so, this will be 1 minus of this 1 minus 0.9938 and so, this will not be the required probability. We will have to write the number, anyway.

So, what we will do is, yeah, I will continue to say, that p is 1 minus 0.9938 or maybe, I will just do this. So, this is 1 minus 0.9938. So, this is the probability of, is the probability of rainfall. So, the probability would be standard normal variant greater than or equal 5 by 2, which will be 1 minus phi of, phi by 2, 5 by 2 and that is 1 minus 0.9938 because phi 5 by 2 is 0.9938. So, this is probability is.

Now, this we can look upon as the probability of rainfall being more than 50 inches in a year and this we will treat as a success. That means, now you can say, that in a year the experiment is to measure the rain and if the rainfall is more than 50 inches, we treat that is a success, right. So, that is what I said, that the problem requires because the event is complex.

So, first we computed the probability of the rainfall being more than 50 inches in a year, which came out to be this. Now, I want to find out, that in 10 years, rainfall not more than 50 inches in any year. So, that means, if I treat those 10 every year as a trial, that means, the trial is, you add up the total number of rainfall in that year and then you say, you are saying, that in 10 years there should be none of the years, that you count from beginning from this year, the rainfall is more than 50 inches. That means, in other words, there is no success in these 10 trials and so, the probability of no success in 10 trials would be 1 minus p raise to 10, which must be 0.9938 raise to 10. So, again please compute this number because I had made a mistake.

So, here this is, this will, whatever this number, you can now use your calculators to compute this probability to say that that will be the probability, that in 10 years the rainfall will be less than 50. So, this is. So, I am just trying to show you, that how you first use the normal of approximation, and then I mean the standardization and then, you, you use the binomial random variable to compute the actual probability.

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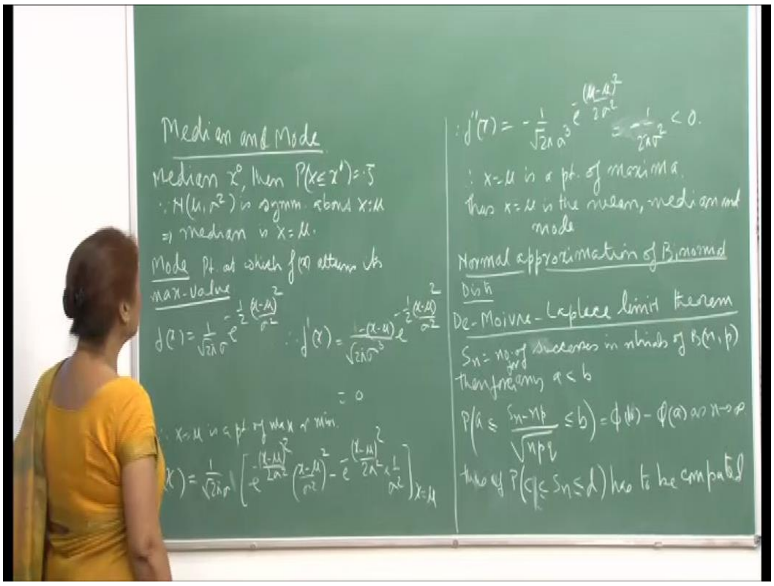

So, median and mode of a normal distribution. This, again see, that is why, I mean, when you look at all these properties of distribution, you, you realize, that it, it is a very interesting one. And this is, suppose, what we mean by the median of distribution is the number for which the, for which your cumulative density function has the value 0.5. That means, the area of the, if you have this thing here when this is the, this is the point where this is x naught. So, this area is 0.5 and this area is 0.5.

And since we have already said, that the normal distribution is symmetric about x equal to mu, so that immediately clear, that the area to the left of mu is 0.5 and area to the right of mu is 0.5. So, immediately we know, that a median, the median is also at x equal to mu, right. Now, to define the, to obtain the mode point, this is the point at which F x, the probability density function attains its maximum value. We did this for the binomial also, remember, and what was the number at which the random variable attains its maximum probability.

So, now, in this case, we will have to find out the maximum, the value, the value of x at which this function attains its maximum value. So, this will be done by finding out the, differentiating it, finding out the critical point and putting this equal to 0. The derivative, now you see, that here this is the derivative. So, this portion is not 0. So, therefore, this is the only portion, which will be 0, and therefore that gives you x is equal to mu. So, x, x equal to mu is the point of maxima or minima essentially, or it could be a point of inflection. But, so now, we have to look at further the second order derivative to determine the nature of this critical point. And I have written down the expression for second derivative, that means, you differentiate this expression again and compute it at, evaluate it at x is equal to mu.

So, then you see this, this portion goes 0 because x equal to mu and here also, when you put x equal to mu, you essentially get this. So, this is again e raise to 0, which is 1. So, you simply get minus 1 upon 2 pi sigma square, which is less than 0. So, if, if the second derivative has negative sign at a critical point, that point must be a point of maxima. So, that much from calculus we can obtain. I am sure, therefore, it is really interesting, that x equal to mu is the mean, median and mode. So, it is all the things combined into 1.

Now, another important, as I told you, I have been, I have mentioned, that de Moivre was used to, you initially defined this normal distribution to approximate binomial probabilities. So, let us, formula is the procedure here. So, this is de Moivre Laplace limit theorem, which is, that if S n is the number of successes in n trials of a binomial random variable (n, p), then for any a less then b. So, here of course, for the binomial the mean is n p and the variance is n p q, right. So, for any a less then b if you want to look at the probability of a less than S n minus n p upon root n p q less than or equal to b.

So, you see, what we have done is, we have standardized this binomial random variable of successes in n trials, number of successes in n trials. So, it will be S n minus n p upon and root n p q right. So, then it is the de Moivres Laplace theorem says, that this probability can be approximated by phi b minus phi a where phi is your cumulative distribution function for the standard normal distribution or for the standard normal variant. So, this will be phi b minus phi a as n goes to infinity.

So, that means, for large n you can approximate this probability by phi b minus phi a and later on, we will show, that this actually is a very general, I mean, you can talk about a more general result when you do not really need to have the distribution as binomial. And for any distribution, this kind of thing, which we call as a central limit theorem, we will talk about it later.

But right now, de Moivres Laplace limit theorem simply say, that if you have a binomial random variable, then if you want to compute the probability, that S n minus n p upon under root n p q. So, this lies between a and b, a strictly less than this, then this can be approximated by phi b minus phi a. This is the Laplace theorem. So, now, we will use this.

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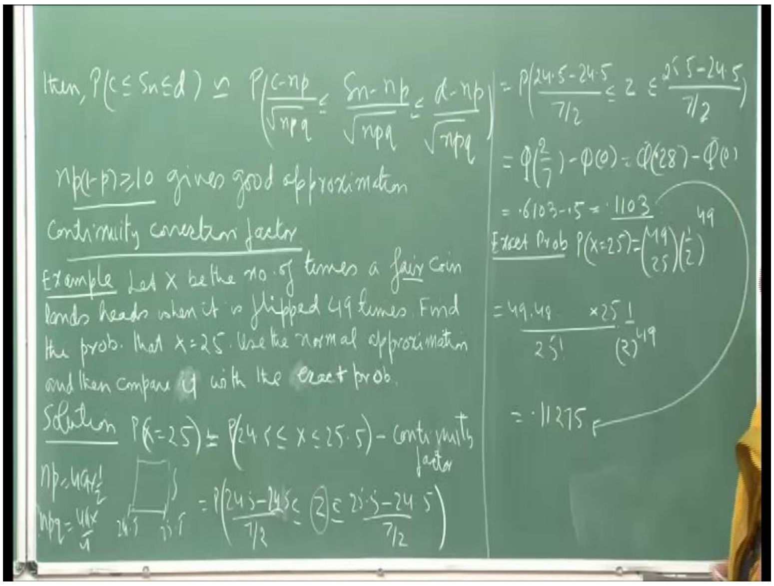

And so we say, that now suppose, you want to compute the probability, that c is strictly less than S n is less than or equal to d and here, of course, it is understood c is also less strictly less than d, then we will standardize the whole inequality in the sense, that we will transform.

This to c minus n p upon under root n p q and this will be less than S n minus n p upon under root n p q, which will be less than or equal to d minus n p under root n p q. And now, you see, this becomes standard normal variant and so, you're a gets replaced by this in the theorem and your b gets replaced by this, right. And so, I can say, that by the de Moivre's Laplace theorem, that this probability, which is the same as this can be approximated by phi of d minus n p upon under root n p q and phi of c minus n p upon under root n p q. So, this is the whole idea.

So, therefore, and these are, these values are tabulated. So, given c d have n n p, you can look up the tables and compute this probability. And of course, if you want to compute the actual probability, then we will see through an example, that it can be very, very cumbersome. So, in fact, this is a very useful theorem and it in a very simple way allows you to compute, approximate these, compute this probability, right.

And of course, then we will continue with the computations for approximating these probabilities and what they said is, that this is good enough as long as your n p into 1 minus p is greater than or equal to 10. That means, you do not require too many very large values of n, but as long as, of course, if this number is larger than 10, you get a better approximation and that you can also for yourself experiment with problems where you try to increase the value of n and then see, that your, the estimate will improve. So, anyway, but this gives good approximation as long as n p into 1 minus p is greater than or equal to 10.

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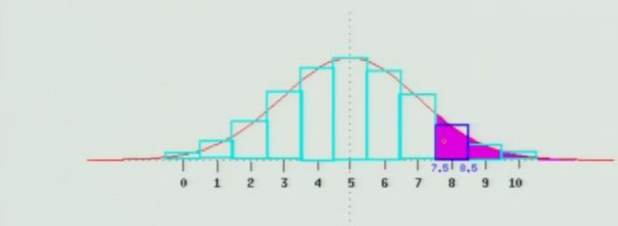

Discrete	Continuous
x = 6	5.5 < x < 6.5
x > 6	x > 6.5
x >= 6	x > 5.5
x < 6	x < 5.5
x <= 6	x < 6.5

Now, there is important aspect to this approximation and that is the continuity correction factor, which I will discuss here. So, you see, look at this curve. So, I have the binomial curve here, the bar chart. And then, the red line curve is the normal approximation and you see, what we are saying here is, that if you are asking for the probability x greater than c. This is the shaded portion that you are looking for.

So, here you want to say, that probability x greater than 7, if you are asking for this probability x greater than 7, yeah. So, then if x greater than 7 means, that x is greater than or equal to 8. So, to get that event, to get that probability, if you want a good approximation, see we are the rectangle, the bar, that represents the probability at 0.8. So, that is starting from 7.5. You want to cover that whole area because you are wanting probability x greater than or equal to 8. So, therefore, you will need to say, that your approximate, approximating standard normal variant should b greater than or equal to 7.5 and not 8.

So, because the, so you will want to say, for the continuity factor you will say, that you will do it from 7.5 because you want to use a, approx, include the area, which is at the point a. The probability, which is represented by, at the point a by this bar, should get added to the, to your estimate, right. And similarly, if you are wanting, let us say I am.

So, I have that table here, just look at the example. So, if you have x equal to 6, that means, a single value, if you have x equal to 6, then you would want it between 5.5 and 6.5 because that is the rectangle, the bar that you constructed at 6 that stands at, from 5.5 to 6.5 and the length, the height is the probability of that x equal to 6, right.

So, therefore, the continuity factor, that you require would be for probability x equal to 6, it will be 5.5 and 6.5. Your x must vary, then because see, the discrete situation, you are now approximating by a continuous situation. Then, as I said, x greater than 6 would be x greater than 6.5 and if x is greater than or equal to 6, then you are including 6, then you will have to go little further down, right. That means, you will have to start from 5.5. If it is x greater than or equal to 6, then you will begin from 5.5. So, x would be greater than 5.5 if x is less than 6. That means, x is less than or equal to 5, then again you will say, that x is less than 5.5. So, that is very clear from this graph. And x less than or equal to 6 would be x again.

See, the moment you have equal, then you have to go a little ahead of it, 6.5 and if it, it is strict in equality, then. So, the rule is very clear. And by looking at this figure you can always… So, if you keep this in mind, then you will not go wrong because you will realize, that you have to, you have to include the area under the bar or the rectangle, which is for that particular value, the limiting end value. And so, you will have to accordingly to give the correction factor, right. So, this is called the continuity correction factor because you are estimating a discrete situation by this thing.

So, if you use the continuity correction factor, now let, let us give this thing, let us give, let me give you, show you the calculations through an example. And so, let me show you an example through an example how we, what I mean by the continuity correction factor.

So, let x be the number of times a fair coin, so fair coin means, that the probability of showing a head and a tale are the same and that equals half, right. So, lands heads when it is flipped 49 times. x is the number of times head shows up when I have flipped fair coin 49 times. Now, find the probability, that x is equal to 25. Use the normal approximation and then compare it with the exact probability. So, I want to compute probability x equal to 25.

And as I told, I just discussed with you, that since this is the binomial random variable, you are trying to approximate it by a normal distribution. So, the bar is actually, when x equal to 25, the bar starts from, 20, 24.5 and ends at 25.5. So, this is the bar, right, and this, the height is the probability that you associate with x equal to 25. So, 24.5 and 25.5. So, therefore I have to change the two approximate again. I should say this, that the approximate this thing will be 24.5 less than or equal to x less than or equal to 25.5. So, I am now approximating this event by this event, right, because of the continuity factor and then, because the mean, yeah, I am sorry, this should not be 49. I have to write it correctly here.

So, now for this binomial random variable, your number of times, you flipping the coin is 49 probability, that your p is half. So, therefore, n p is 49 into 1 by 2, which is 24.5, right. So, it should be 24.5, right and your variance is n p q. So, that is 49 into 4 divided by, divided by 4. And so, under root of that would be and that is why, I choose this 49 to make it perfect square. So, that is 7 by 2. So, this is now standardized. So, x minus 24.5 divided by 7 by 2 would make it standard normal. And so, this is the event that you want to, you want to now find out the probability. So, this is this, right. And therefore, this is equal to 0 and this is 1 upon 2 by 7. So, therefore, this is equal to phi of 2 by 7 minus phi 0.

Now, this is phi of 0.28 minus phi 0, phi, phi of 0.28 from the normal tables is equal to 0.6103 minus 0.5, right. And so, this comes out to be 0.1103. So, this is our probability that we have obtained for x equal to 25 through normal approximation. And remember, the condition was, that your n p q must be greater than or equal to 10.

So, in our case our n p q is how much? n p q number is 49 by 4. So, which is more than 10, right, that is, our n p q is 49 by 4, which is equal to 42, sorry, 12 point something, 12

4’s are 48. So, 0.25. In fact, so this is more than 10, right. And so, we applied the approximation and this is the result that we got, right.

Now, exact probability if you want to compute using your binomial computations, then you see, it is actually 49 choose 25 and 1 by 2 raise to 49, 49 trials and your p and q are the same. If you write out this number, 49 into 48 up to 25 because that will be 49 minus 25 plus 1 and then, 20 factorial 1 upon 2 raise to 49. It took me almost half an hour to compute from, sitting at the computers and this number comes out to be 0.11275.

So, if you just look at this here and that is what I am saying, now you compare it with, so third decimal place, the number differs the value differs and therefore, I would say, that this is a good approximation. And here, you see our n p q was only, 10, 12.25. And if you take larger n, that means, if you had flipped the coin more than 49, then this number would have improved.

And just see the phenomenal calculations you had to do even for n equal to 49, I mean, multiplying these numbers, 24 numbers getting 25 factorial multiplying 2 49 times and then dividing. So, this this the amount of calculation, that you would have to do for the exact probability certainly is not required. If you can approximate this probability by this simple method because this table, these values are already available to you. And so, this is what I am trying to say, that you know, as we go on, you will see the numerous applications, uses of this concept of normal distribution and its computations.

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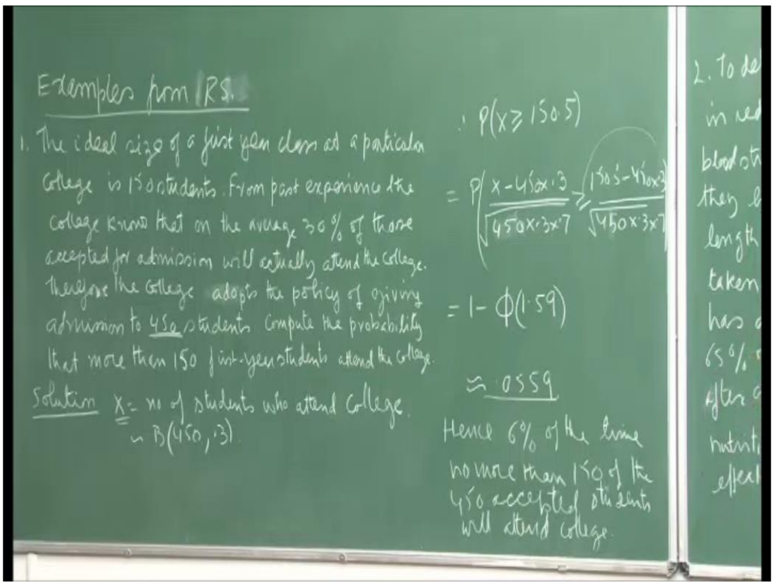

So, I will continue with the examples of you know, binomial, the normal approximation of binomial probabilities and this is, these are two good examples from the book by Ross Sheldon. As I told you, that this is the book on probability theory and at the end of the course I will give you all these references. So, I just thought I will, you know, give you feeling of, you know, some more examples to, to reinforce the idea that the approximations of discrete probabilities can be done by the continuous random variables and in the very effective way. So, let us look at this example.

The ideal size of a 1st year class at a particular college is 150 students. From past experience the college knows, that on the average, 30 percent of those accepted for admission will actually attend the college. So, therefore, you know, people tend to apply to more than one college and then they, of course, decide which is the best one for them? So, the college has the experience, that only 30 percent of the people who have been accepted for admission will actually attend the college. So, therefore, the college adopts the policy of giving admission to 450, 450 students, so that finally, when they, you know, drop out, after the dropout rate, they will still be left with the, with the class of size 150. This is the idea. So, they decide to offer admission to 450 students. Now, compute the probability, that more than 150 1st year students attend the class, right.

So, given that 30 percent is the, so this is the probability of a person attending who has been given admission, will attend college is 0.3. So, therefore we want to find out. So, here, of course, we will say x is the number of students who attend college and so, the value of x will be equal to the, we can treat the person who has been given admission and attends college as a success. So, out of 450 people x will be the number of students who attend college. So, this would be a binomial random variable with n as your 450 and your p as 0.3 right. So, this is binomial (450, 0.3), because we are treating, that the experiment has been performed. That means, admissions have been offered to 450 students.

And out of this those who attend, who all actually come attend the college is a success. So, the number of successes we are saying is a binomial random variable, right. So, therefore, you have to compute the probability. So, actually you want that the class size should be, so you have to compute the probability of more than 150. That means, it should be 151, 152 and so on. So, therefore, when you write probability x greater than or equal to 151, then your continuity factor when you apply, then this will become 150.5, right because you are actually asking for the probability, that x is greater than or equal to 151 because more than 150.

So, therefore, we just standardize this, you know, this variant and that would subtract x minus n p. n p is 450 into 0.3 and then divide it by n p q. So, 450 into 0.3 into 0.7 and do this to the right hand side also. So, you get this and then, of course, the advantage of taking in solved example is, that you have the calculations done for you. So, here this is actually the phi of 1.59. So, this number reduces to 1.5.

So, therefore, this is your standard normal probability from minus infinity to, sorry, this is, yeah, to 1.59. So, we are writing this as 1 minus phi of, so this number is 1.59. So, this required probability is 1 minus phi of 1.59, which comes out to be this, 0.559. So, hence this reduces to 6 percent of the time more than 150 of the 450 accepted students will attend college. So, the college is in good situation, because it is only 6 percent chance, that more than 150 people will actually come and attend the college those who have been offered admission. So, this was one situation.

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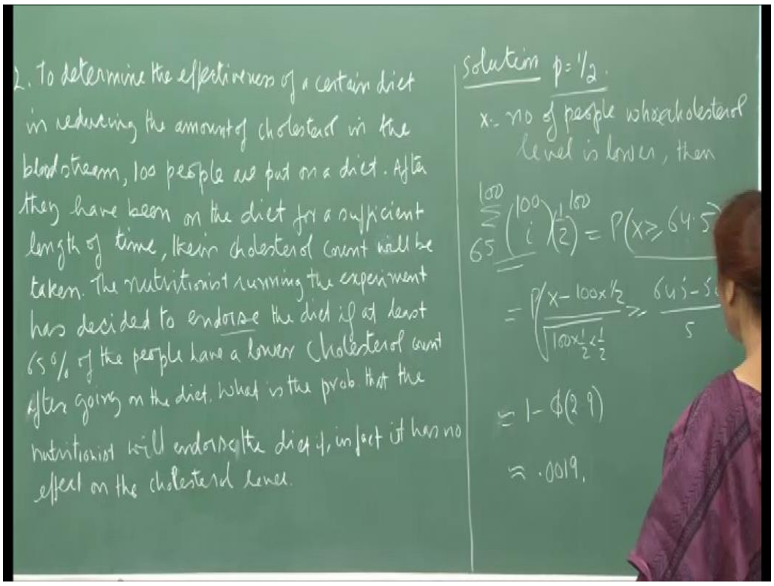

Now, another interesting problem. This is, see to determine the effectiveness of a certain diet in reducing the amount of cholesterol in the blood stream, 100 people are put on a diet. After they have been on the diet for a sufficient length of time, their cholesterol count will be taken. The nutritionist running the experiment has decided to endorse the diet if at least 65 percent of the people have a lower cholesterol count, right.

So, after the trial period, at the end of the trial period, you will again take a, test their cholesterol in the blood stream. And if the count is lower for 65 percent of the people, then after going on the diet, so the nutritionist will, have I made the sentence complete, running the experiment has decided to endorse, so the nutritionist will endorse the diet and say, yes, it is thus proven to lower the cholesterol in the blood.

So, now what we want to find out is, what is the probability, that the nutritionist will endorse the diet. That the nutritionist endorses the diet, will endorse, there I write, will, will endorse the diet if, in fact, it has no effect on the cholesterol level.

So, what we are saying is that suppose the diet has had no effect on the cholesterol, but what is the probability, that the nutritionist will still endorse it. So, therefore, now the way we are arguing out and so, we have to now decide how to model the situation. And the idea here is, that you know, it is either way. See, people may on their own have their cholesterol count come down and so, the chance of that happening is, we know, equally likely, either your cholesterol count goes up or it comes down. So, that is the equally likely situation and therefore, it is being said that ok, you can just take p equal to half. So, therefore we are assuming, that the diet has had no affect on the cholesterol level.

So, but the since we, the count is going to be taken after the trial period and then, if it turns out that 65 percent of people have lowered their count and they will be, then the diet will be endorsed . So, therefore, we will work with p equal to half. So, I hope it is clear. So, what we are assuming is, that chance of the count being going up and down is equally likely. So, therefore, we will take p to be half. And x is the number of people whose, whose cholesterol level is lower, right. Then, see what, what we are looking for is, so the binomial random variable.

And therefore, this is, this is the probability, sigma 65 to 100 100 i half raise to 100, right, because r and n minus r, both are will up to end. This is p is half. So, this is nothing and of course, you can see, that this is ((Refer Time: 51:03)) stupendous task, you know, trying to compute it this way. So, therefore we will say, that essentially, we are looking for probability x greater than or equal to 65 and again add the correction, continuity correction factor, so it will be 64.5 and then n p would be 100 into half and, and p q will be 100 into half into half, which becomes 25. So, under root 5 and this is 50. So, this is what you are looking for and this number comes out to be 2.9.

So, therefore, it is the required probability is 1 minus phi of the normal standard normal probability 2.9 from minus infinity to 2.9 and this comes out to be 0.0019. So, which is very small and therefore, the chance that you know, with p as half the chance, that the 65 people out of those 100 will have their cholesterol lowered is very low and therefore, the diet will not get endorsed.

So, anyway because the diet was not having as, as we said, that probably the diet has no affect on the cholesterol level, so therefore, it does not get endorsed. So, no loss, nobody is lost. So, this is how, you know, one can go on and look at different situations and then try to see. So, I just thought, that these two will also add to your, you know, experience of handling, you know, these problems and also reinforced the idea of, you know, computing, you know computing these, what shall I say, messy probabilities by you know, approximating through continuous random variables and making your task easy.