Introduction to Probability Theory and its Applications
Prof. Prabha Sharma
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture - 8
Continuous Random Variables and their Distributions

We will go through exercises on discrete random variables when I will try to give you small hints, so that you can work them out yourself.

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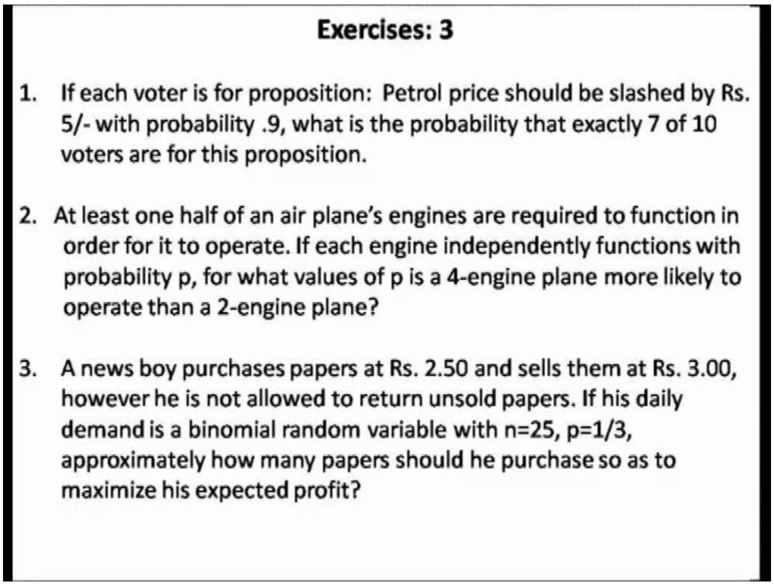

Now, the question 1 says that each voter is for proposition that petrol price should be slashed by rupees 5. So, people are supposed to vote for this with probability 0.9. So, the probability that a person will vote for this that the prices should be slashed by rupees 5 is 0.9. What is the probability that exactly 7 of 10 voters are for this proposition. So, you should be able to guess what distribution you have to use here. At least one half of an air plane's engines are required to function in order air plane's there is a apostrophic there.

So, at least one half of an air plane's engines are required to function in order for it to operate. So, if there are, obviously we are assuming that even numbers of engines are there. So, half of them have to function in order for the plane to be able to fly. So, if each engine independently functions with probability p, for what values of p is a 4 engine plane more likely to operate then a 2 engine plane. So, please first write down the probability of two or more engines working for a four engine plane, and two for two engine planes, it would be 1 or 2 engines working for two engine plane in terms of p, and then write down any quality that you want the probability for the four engine plane to be higher. So, what would be the values of p for which this is any quality would be satisfied.

Question 3: A new boy purchases papers at rupees 2.50 and sells them at rupees 3. However, he is not allowed to return unsold papers. If his daily demand is binomial random variable with n equal to 25 and p equal to 1 by 3, then approximately how many papers should he purchase so as to maximize his expected profit? So, now, see you start with you do not know exactly how many news papers he buys. So, let us say the number is r. That means r successes binomial random variables. You know the probability. What is the probability? It is when this daily demand is the binomial random variable with this, right. How many papers should he purchase? So, essentially what you have to show here is that the expected value would be a function of r, the number of news papers he buys, right and that you have to then maximize with respect to r.

So, that will tell you what the optimum value of r will be because here see when you take the expected value depending on the demand, it will tell you that when he is able to sell a newspaper, he earns 50 paisa. If he is not able to sell a paper, then he loses 3 rupees. So, accordingly you have to write down the expected profit for this newspaper boy and then, maximize it and find out the optimum value of the number of papers he must order.

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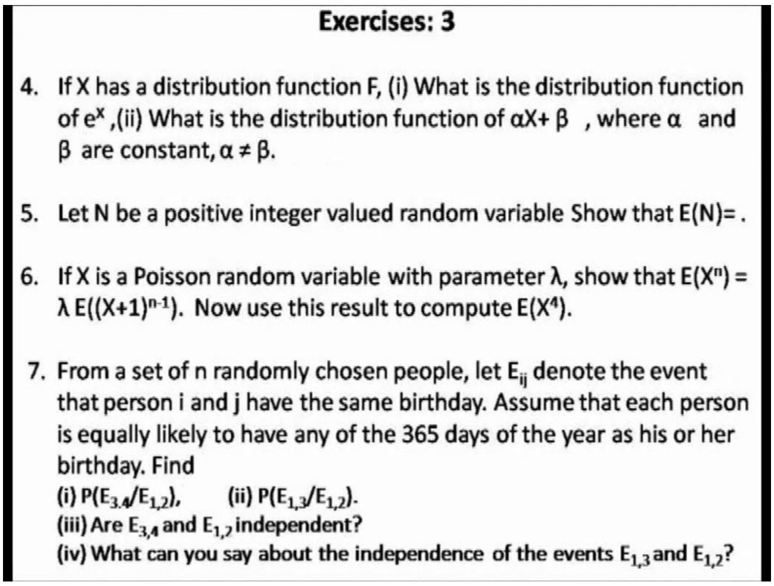

Exercises: 3

4. If X has a distribution function F, (i) What is the distribution function of e^X ,(ii) What is the distribution function of $\alpha X+ \beta$, where α and β are constant, $\alpha \neq \beta$.

5. Let N be a positive integer valued random variable Show that E(N)= .

6. If X is a Poisson random variable with parameter λ, show that $E(X^n) = \lambda E((X+1)^{n-1})$. Now use this result to compute $E(X^4)$.

7. From a set of n randomly chosen people, let E_{ij} denote the event that person i and j have the same birthday. Assume that each person is equally likely to have any of the 365 days of the year as his or her birthday. Find
(i) $P(E_{3,4}/E_{1,2})$, (ii) $P(E_{1,3}/E_{1,2})$.
(iii) Are $E_{3,4}$ and $E_{1,2}$ independent?
(iv) What can you say about the independence of the events $E_{1,3}$ and $E_{1,2}$?

If X has a distribution function F, what is the distribution function of e rise to x? So, you have to find that out. Again apply the definition. What is the distribution function of alpha X plus beta, where alpha and beta are constants, alpha not equal to beta. Then, question 5: Let n be a positive integer valued random variable. Show that expected value of n. So, n takes positive integer values. So, expected n is sigma i varying from 1 to infinity probability n greater than or equal to i. So, it is a matter of writing out the expression for e n and then, rearranging the terms, so that you can get this answer.

Six problem: If X is Poisson random variable with parameter lambda, show that E raise to X n is lambda E raise to X plus 1 raise to n plus 1. Now, use this result to compute E expected value X raise to 4. This is straight forward from a set of n randomly chosen people. This problem I have already discussed with you in one of the lectures. I explained the notation E ij that a person i and j have the same birthday. Assume that each person is equally likely to have any of 365 days of the year as his or her birthday. So, then you have to find these conditional probabilities that I have written out here and then, I am asking the question are E 3,4 and E 1,2 independent.

Then, what can you say about the independence of the events E 1,3 and E 1,2 and you can almost guess what the answers would be. Leave it. Anyway, you have to work it work out, and show that they are that means you have to show that the probability of E 3, 4 intersection E 1, 2. That means, all these four people are having their same birthday will be product of the probabilities of E 3, 4 into E 1, 2 and so on, right.

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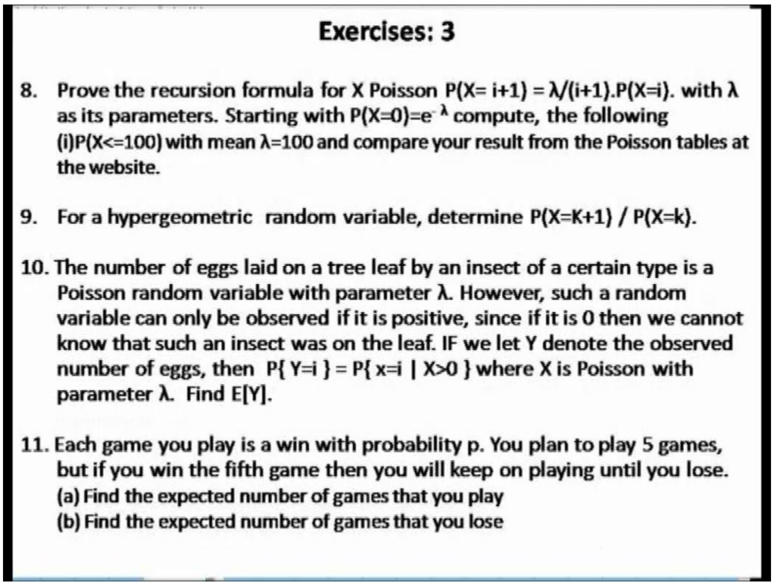

A question 8 prove the recursion formula for X Poisson which is probability X equal to i plus 1 lambda upon i plus 1 into probability X equal to I with lambda as its parameters. So, Poisson random variable with lambda as the parameter, then you have to show this and if you start with probability X equal to 0 equal to E raise minus lambda, then you can compute probability X equal to 1. So, 1 from this recursion formula, then I also want you to compute probability X less than or equal to 100 when the mean of the Poisson distribution is lambda 100, and you can compare your results because Poisson tables are not easily available. So, in the website, you can go and look at the tables.

For a hyper geometric random variable, determine a probability X equal to K plus 1 upon probability X equal to K. So, here also I am asking you to find out the recursion formula. Question 10: The number of eggs laid on a tree leaf by an insect of a certain type is a Poisson random variable with parameter lambda. However, such a random variable can only be observed if it is positive. Since, it is 0, then we cannot know that such an insect was on the leaf because there will be no eggs present on that leaf. If we let Y denote the observed number of eggs, then probability Y equal to i is equal to probability X equal to i, given that X is greater than 0, where X is Poisson with parameter lambda. So, find E Y. So, I am asking you to find the conditional expectation of X.

Question 11: Each game you play is a win with probability P. You plan to play 5 games, but if you win the fifth game, then you will keep on playing until you lose. So, here negative binomial would be used to find the expected number of games that you play. Find expected number of games that you lose. So, I hope you are enjoy doing this exercise.

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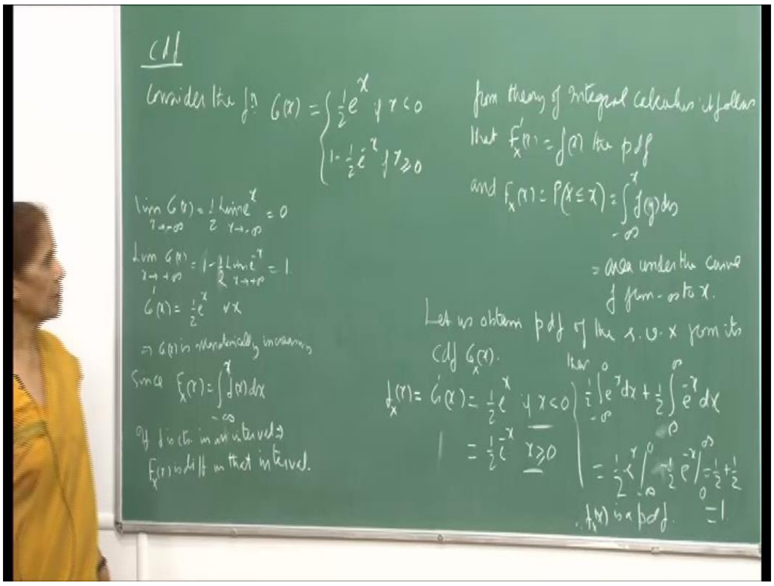

I will continue with the new examples and some more results about the cumulative distribution function because it is a very important concept and also, we go along with special distributions and so, we will be able to get more familiar with the whole idea. So, let us consider the function G x which is defined here for x less than 0 by this, and for x non-negative by this, then we want to show that limit of G x as x goes to minus infinity must tend to 0, which it does because e raise to x goes to 0 as x goes to minus infinity. Similarly, limit of G x is x goes to plus infinity should go to 1, and that also you can see because this is e rise to minus x. So, as x goes to infinity, this portion goes to 0. We left with 1. So, that is fine. Then, you want to show that G x is mono and G x is monotonically increasing. So, we take the derivative here because G is a continuous function differentiable also.

So, I can take the derivative and if the first derivative is non-negative which it is e raise to x half. This is non-negative for all x. Therefore, G x is monotonically increasing. So, G prime x will be half e raise to x for x less than 0, and it will be this is a minus sign. So, there is minus 1 will come from here. So, it will become plus. So, that will be half e raise to minus x for x greater than or equal to 0. So, we see that G prime x is non-negative for all x. This is the non-negative function. So, therefore, G prime x is non-negative for all x which implies that G x is monotonically increasing which is again a property of G that of a cumulative distribution function.

So, we are just verifying that this qualifies to be cumulative distribution function and then, other properties we will check G satisfies all the conditions for being a c d f, right. Now, we also have this result that if f is continuous in an interval, then f the integral of f from minus infinity to x, this is differentiable on that interval. So, wherever whichever region f is continuous capital X, the cumulative density function would be differentiable. So, that means, if you are given c d f and then, it is differentiable, then you can differentiate it and say that it will be equal to pdf wherever the function is differentiable right and also, we have seen it already that since this is equal to probability x less than or equal to x, then this is integral minus infinity to x of f y d y and this defines again the area under the curve from minus infinity to x.

So, now, given this since we have verified that this is the valid c d f, let us now find out the probability density function which we said that we get by differentiating the c d f. So, here you see G prime x is half e raise to x if x is less than 0, and when you differentiate this part, the minus sign becomes plus. So, this is half e raise to minus x which is for x non-negative. So, actually this is very important that when you define the pdf, you have to specify where it is defined because that means where it is non-zero, and where it is 0 because you have to say where the mass is of the random variable.

So, it is very important. Sometimes people just forget this and you simply write this only which is not correct because you must specify the region in which it is defined. That means where the mass exist now. So, therefore, this and this is a non-negative function. You see that right and now, you can verify that this is valid pdf by integrating it from minus infinity to infinity. So, that will be minus infinity to 0 e raise to x, right and then, plus half 0 to infinity e raise to minus x d x, and you can see that the integral would be half e rise to x minus infinity to 0. That will give you, so at 0; it will be 1, right. So, that is half and minus half. So, when you integrate this, it will be minus sign half e raise to minus x 0 to infinity. This again gives you half. So, half plus half is 1. So, therefore, we have verified that your small f x is p d f and also, that this is c d f.

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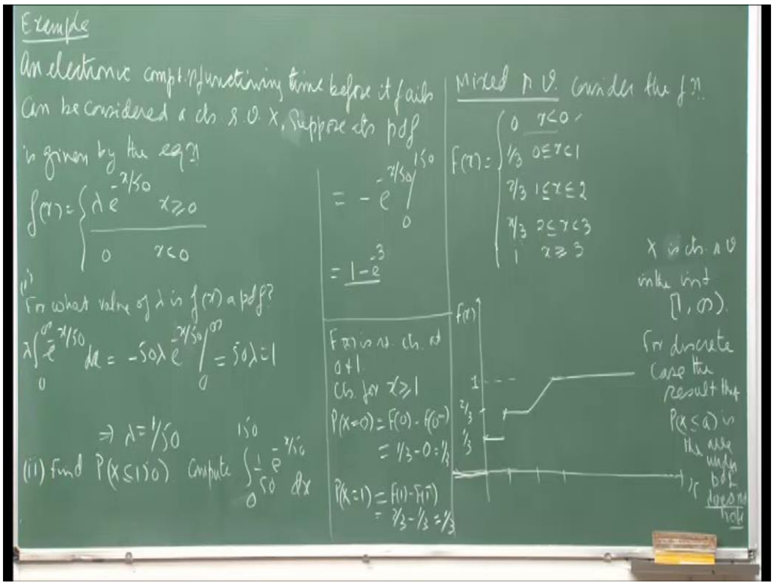

Let us take another example of this cdf. Now, here an electronic component functioning time before it fails can be considered as continuous random variable x, and suppose it is pdf is given by this, so therefore, you want the first question is for what value of lambda is f x? Pdf. Obviously, we apply the condition because it is a non-negative function. So, the second condition is that from 0 to infinity, since here again you see the mass is of x, non-negative for x less than 0, it is 0, right. So, I will integrate the function from 0 to infinity and e raise to minus x by 50. So, this becomes minus 1 by 50 comes to the top to the numerators of minus 50 lambda e raise to minus x by 50 from 0 to infinity. So, at infinity, this is 0 and at 0, it will be 1. So, 50 lambda is equal to 1. This is a condition we need, so that this is a valid pdf. So, we get that lambda is 1 by 50.

So, the first question has been answered. Now, I want you to compute probability x less than or equal to 150. So, you compute 0 to 150, 1 upon 50 e raise to minus x upon 50 dx which comes out to be this and then, it is 150. So, at 0, it is 1 and then, this is minus e raise to minus 150 by 50 minus 3. So, look up the value for this from your calculator and you get the answer. Now, as I said that there are discrete random variables, continuous random variables and mixed kind of random variables. So, I will take up this example for a mixed random variable.

Suppose you have given this example capital F x, right. So, I plot it for x less than 0, it is 0. So, this is a portion. Then, you note that here this is x less than 0, for x equal to 0, and greater than 0 than this part operates, right and therefore, the value jumps to 1 by 3, ok. Then, it continues to be 1 by 3 till x reaches 1, but it is not equal to 1. So, therefore, at this point, it is 1 by 3 and then, it is like this and here at 1 here. So, you see the continuity part is satisfied. There is a jump here again at x equal to 1 because at x less than or equal to 1, less than 1 is 1 by 3. The moment you attain the value x equal to 1, it jumps to 2 by 3. So, here again the jump is of 1 by 3 and then, after that you see what happens is this is x upon 3. So, if x varies from 2 to 3 at 2, see this is 2 by 3 and this is also 2 by 3, x equal to 2, right.

So, it continues like this. This slope is of this line 1 by 3 and then, of course at 3 as long as say this is the line and then, it becomes free. So, you see here you have both kinds that the functions take as jumps. So, it is a step function for some values and then, after that it is a continuous function. So, therefore, that is what I have said that here, and we have already seen that how we compute this difference, but I am saying that probability x equal to 0, this will be f 0 minus f 0 minus. So, f 0 minus is 0, f 0 is 1 by 3 and therefore, the values 1 by 3 probability is x equal to 0, and probability x equal to 1 will be f 1 minus f f 1 minus. So, that will be because f 1 will be 2 by 3 and f 1 minus is 1 by 3.

So, again the difference is 1 by 3. So, the two jumps are of 1 by 3. So, this is what will happen and here again I want to point out. See the result that we have shown that probability x less than or equal to x or of course, even for an interval same thing when I have shown you last time that the area under the strip would be the probability when x lies in an interval. So, here this thing is valid only when f is continuous. So, this is x is a continuous random variable now. So, therefore, for discrete and mixed kind, we will not apply this result.

So, only when the random variable is completely continuous, you can apply that result. So, here this is this result. So, one has to be careful and say keep this in mind. You cannot compute this by area under the curve because see the p d f would not be this. This is your cumulative density function, right. So, pdf would be for these two values. It will be a bar chart and then, it will be this thing. So, therefore, the concept of area under the curve does not apply for mixed random variable or for discrete random variable. It only applies to continuous random variable.

You know computation of cdf is very important. You should check. The idea should be very clear in your mind. How you can go about doing it and the validity you must make sure that when you consider a function to be cdf, it must have all the properties that we have you know said that the cdf must have and so on. So, you know this even was not enough. You should work out many more problems to get familiar with this concept.

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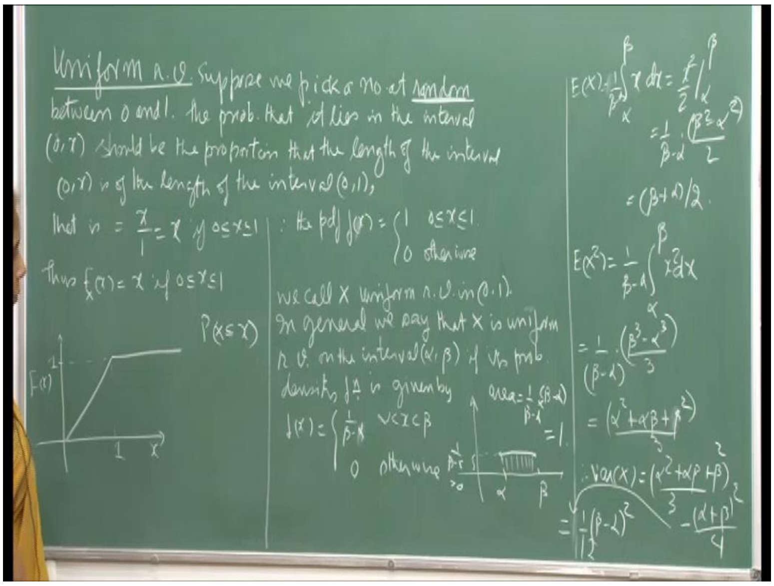

We will talk about uniform random variable now which is one of the simplest and very widely used continuous random variable, and try to get the feeling about this particular distribution. So, now see suppose we pick number at random between 0 and 1. So, this is important we pick a number at random which means that any number is equally likely been using this term, very often describing events and so on. So, we pick a number at random between 0 and 1. The probability that it lies in the interval 0, x should be proportionate. It would be the proportion that the length of the interval 0, x is of the length of the whole interval 0 and 1. This is what we mean by picking a number at random.

I will repeat that if you are saying that I pick a number here and I say that it is lying in this interval 0, x. X is of course number which is less than 1, then the probability that the number lies in this interval should be the proportion that the length of this interval is of the length of the interval 0, 1. And if you can recall that when we in the discrete case, I had told you number of ferule cases divided by the total number of cases. It is sort of an extension over the same concept that if I am saying that the probability that the number that I pick up lies in this, then length of this interval divided by the length of the total interval should be the probability that the number I have picked lies in this interval, and this innocence captures the concept of uniform random variable or what we keep saying a number is equal likely in this interval and so on, right.

So, this will be when we are saying that probability of the number lies here. That means, this actually describes that probability, right because we are saying number that I pick between 0 and 1 lies in the intervals 0 x. That means, the number is smaller than or equal to x. So, it is between 0 and small x. So, therefore, this is f x and what am saying is that the cumulative density function is x upon 1 which is equal to x. If x is in between small x is between 0 and 1, so I immediately get the cumulative density function of this random variable and if you draw the picture, you see it is like this because as x goes from 0 to 1 and then, it stays at 1.

So, you see there is no discontinuity here, no jumps here. Therefore, this represents the continuous this is the graph of cumulative distribution function for a continuous random variable, the pdf. Of course, now we can use this fact that this is differentiable from 0 to 1 and so, the derivative of this function will give you the pdf f x equal to 1 in interval 0 1 and 0 otherwise. It is so simple, right.

So, this is one special case when I said that the random variable is defined on the interval 0 1. So, it is uniformly distributed in that interval. In general, we say that x is uniform random variable on the interval alpha, beta. If it is probability density function is given by 1 upon beta minus alpha x lying between alpha and beta and 0 otherwise, so you see I will again repeat that whenever am defining pdf or a cumulative density function, I have to define the region on which it is specified. So, of course for the cdf, it goes up to infinity because after whatever the values are over, then it stays at 1, ok.

So, therefore, you see here again this is proportionate to the length of this. So, the length of the interval is beta minus alpha, so 1 upon beta. So, here for example if you draw the graph of this pdf of random uniform random variable, this will be alpha and let us say this is beta, then this is the length. So, this height is 1 upon beta minus alpha which is greater than 0, and you see if you look at the area under this curve, this is rectangle height. So, the area is 1 upon beta minus alpha into the length. Length is also beta minus alpha. So, this is equal to 1. So, this is a valid pdf and the area under the graph you know that concepts.

So, now if you are for example, if you if you want the probability that x is less than or equal to some gamma, where gamma is some number here, right then it will be this area under the curve. So, this is how you can simply picturize, not go wrong with it, but just make sure that you write the probability correctly and always validate. You always make sure that you have the right number and then, you want to compute the expected value of general uniform random variable. This will be 1 upon beta minus alpha integral alpha to beta x d x which comes out to be x square by 2 alpha to beta. So, 1 upon beta minus alpha into beta square minus alpha square by 2. That leaves alpha plus beta by 2.

So, very simple way to remember whatever the interval for the uniform, you just add the two end points; divide by 2 and that gives you the expectation. So, it is a middle point expectation. So, what will it be? Yeah, just see here this is the alpha. So, alpha plus beta minus alpha by 2, right because the length of the interval has midpoint of this interval is beta minus alpha by 2 which you add 2 alpha to get to this point, right. The length here is beta minus alpha. So, half of the length, this length is beta minus alpha by 2. Add it to alpha and this gives you alpha plus beta by 2, right.

So, that is the midpoint, right. The expectation x square will simply make it beta cube. So, x cube by 3 and from alpha to beta, this will be beta cube minus alpha cube by 3 and you know expand this divide by beta minus alpha, you get this and then, for the variance, it will be expectation x square minus expectation x whole square which when you simplify comes out to be 1 by 12 beta minus alpha whole square. So, again here it is easy to remember the formula for the variance length of the interval square divided by 12. So, the length of the interval is beta minus alpha, square it up, divide by 12 and that gives you the variance of the uniform random variable.

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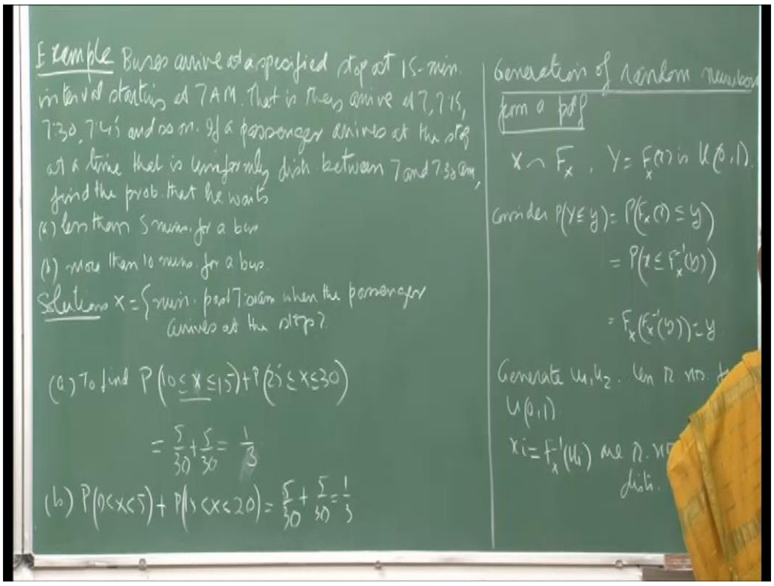

Let us look at this example. Bus is arrived at a specified stop at 15 minutes interval starting at 7 am. So, that is the first bus arrives at 7, then the next bus will arrive at 7.15, then 7.30, 7.45 and so on. So, at interval of 15 minutes, the bus keeps arriving if a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7.30 am. So, this is again an example because you cannot (( )) that time of the arrival of a passenger can be sort of treated as a continuous random variable. You might say that. Now, your clock gives you discrete time, but essentially the concept is that we will treat this as a continuous random variable. So, here the distribution of his arrival time is a uniform random variable between 7 and 7.30 am with equally likely what time he arrives from 7 to 7.30 am, ok. So, then you have to find the probability that he waits less than 5 minutes for a bus. So, he waits for less than 5 minutes for a bus.

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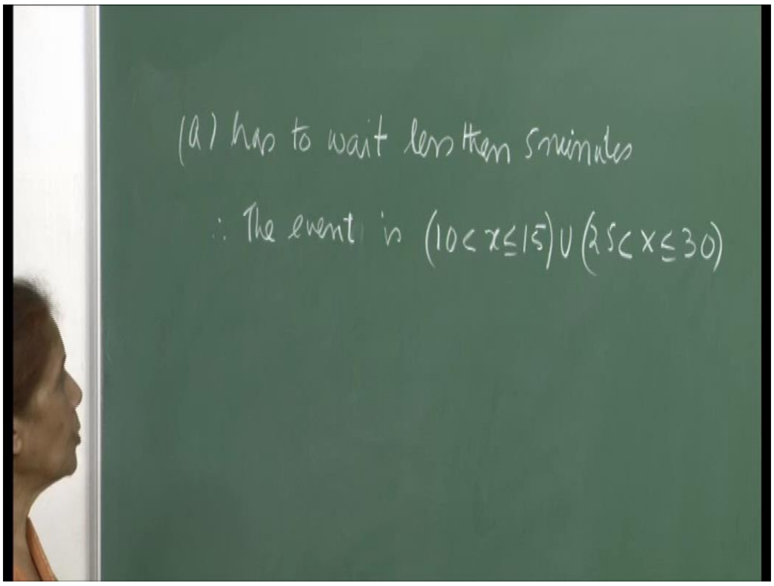

See since the person should have to wait less than 5 minutes, therefore the event at we are asking for is 10 less than x and less than or equal to 15. It should be not less than or equal to because we want that waiting time to be less than 5 minutes. So, equality would have been valid if we had said that the waiting time is 5 or less, but here we are saying the waiting time is less than 5 minutes. So, therefore, it should be 10 less than x. So, make the correction because while computing the probability, I think I have said 10 less than or equal to x. Similarly, here it will be 25 less than x.

So, see here if he has to wait less than 5 minutes, then it should be arriving. So, his arrival time should be greater than 25 and less than or equal to 30. So, please make that correction, then computation. I said the probability part, it does not make a difference, but here it will with the event have to be described correctly. Then, you see he should arrive between 10 and 15. See his time is between 7 and 7.30, right. So, the first bus arrived at 7, the next is going at 7.15. So, if he has to wait for less than 5 minutes, then he should arrive to 10, right and then, it should be bet less than 15 because he has to get the bus. So, this 5 minute interval if he arrives in this interval, then he will have to wait or less than 5 minutes, right. Similarly, it is if he arrives in this time interval 25 and 30. So, this is in minutes and then, again he will have to wait for at most 5 minutes. Is that ok?

So, the event that he has to wait less than 5 minutes for a bus. So, these two events capture this event and since, they are disjoint, I can add up the probabilities, right. This and this are disjoint because the person cannot arrive at both the times, both time intervals. Either he arrives here or here. So, therefore, these are the two events. So, the probabilities add up and here the probability as I told you, this is simple. When you divide I was drawing the figure for you somewhere here. So, it is the length of the interval divided by the interval in which your variable lies just what example we looked at, right.

So, this length interval is 5 and 1 by 30 because his distribution is uniform. Distribution and the length of the interval is 30. So, 1 by 30 is the probability of being, I mean the pdf is 1 by 30. So, therefore, this is 5 by 30 plus 5 by 30 which is 1 by 3, right. Now, the second part is he has to wait more than 10 minutes. So, if he has to wait for more than 10 minutes, then he should either arrive in this interval because if he arrives anytime between 0 and 5 minutes in the see x. So, that is what is important. I should have specified x is the minutes past 7 am when the passenger arrives at the stop.

So, x is the random variable because the arrival itself is a random variable. So, therefore it is random phenomena. So, x, the number of minutes which is past 7 am when the passenger arrives at the stop. So, therefore, if you want to describe this event that he has to wait for the bus for more than 10 minutes, then he should either arrive. That means the x should lie between 0 and 5 or between 15 and 20 because if he arrives at 15 minutes, the bus has just left. So, he will have to wait for the next bus which will come at this thing 7 if he comes at 7.15. Then the other one will come at 7.30. So, he will have to wait for more than 10 minutes, right and up to 20 because if like he arrives at 7.20, then he will have to wait for 10 minutes because the next arrival would be at 7.30, right. I hope that this event is described by these two again. These two are disjoint.

So, therefore, we can add up the probabilities and this will be 5 by 35 by 30 which is 1 by 3, ok. Now, important usefulness of random variables because there will be many variations when we will see how uniform distribution is used. For example, in simulation you need to generate random numbers from a particular pdf and simulation is the order of a day. Sometimes if you cannot get physical data, you try to generate it.

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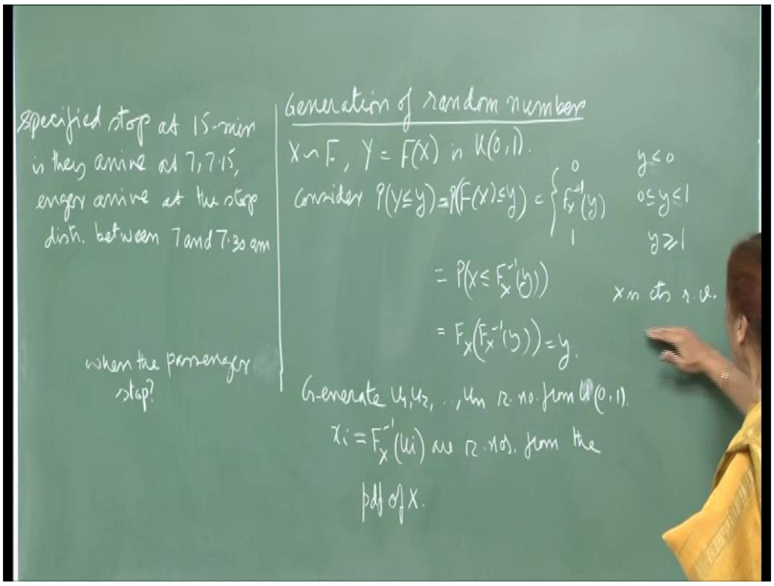

So, here what you will do is, you want to generate random numbers and we can use this concept of cdf very nicely here. So, take X to be a random variable and capital F is the corresponding cumulative distribution function. Now, define the random variable Y which is F X because now whatever the function F, I just place a capital X there. So, this becomes a random variable again and this is uniform 0 1. So, this is the way where we will use the property of a cumulative distribution function. Now, you can show immediately this property because if you consider the probability of Y less than or equal to small y, then this is the probability of F X less than or equal to y, right which you know will be 0 if y is less than 0, right and will be FX inverse y if y is between 0 and 1, and this will be 1 when y is greater than or equal to 1.

Now, I have to use the concept of F inverse here which I did not mention earlier. Now, a whole idea is that here let us take x to be continuous random variable. In that case, your FX is monotonically increasing which we have seen through so many examples also that FX and we proved that also. This is one of the properties that FX is an increasing function and therefore, x inverse FX inverse will exist. So, there is no problem in case x is. So, I am now giving you this property of generating of random number for a continuous random variable, but they are certainly even when x is a discrete random variable. This may not be unique, but you can very usually you know there ways of determining unique value for the FX inverse which is possible.

So, see whenever x is not continuous random variable and is discrete, then for certain interval as we saw, the value of the function FX will remain constant. So, we can decide that the inverse, when we take the inverse, we will take the smallest value of y. So, that is possible. So, we can define, we can determine the inverse in a unique way whether the function random variable x is discrete or continuous. So, this will be valid for both of them, but here I am just now talking about x being continuous random variable. So, therefore, now again this thing can be written as F X of F X inverse y and therefore, from the definition of F X and F X inverse, this comes out to be y which is that means, your capital Y has a uniform distribution 0 1, right.

So, the idea is that you generate random numbers u1, u2, un random numbers from the uniform. So, of course, you might check it how do you do that and there are methods, there are computer methods for generating random numbers, but which are actually pseudo random numbers. So, there are whole lot of techniques and lot of available for generating these random numbers which are actually pseudo random numbers from the uniform 0 1. Once you do that, then you will say that the x i given by f inverse of u i are random numbers from the distribution of the original random variable that you started, right.

So, the process is that you generate random numbers from the uniform 0 1 and then, take the x i which have given by F inverse x u i, and these would be the random numbers from the distribution of x. So, now, you see you can immediate application of your uniform distribution. So, you can generate any number of values from the specified pdf to you know do all kinds of analysis that you want to do about that data.

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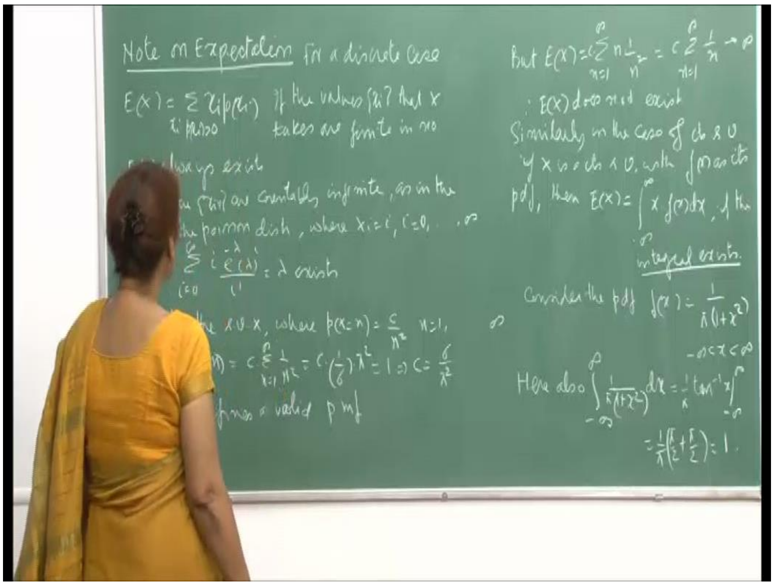

So, I would like to revise the concept of expectation of a random variable now and for the discrete case, we saw that it is defined as summation x i p x i over all x i for which the probability is positive, right and if these values x i that the random variable x takes r finite in number, then this will be the you know finite number because you are adding up p x i are all between 0 and 1. These are finite in number, and then this will add up to a finite number. So, in that case whenever the random variable takes a finite number of values, the expectation always exist, right, but we also saw that in case of Poisson random variable where the values are taken by the variable r countable infinite, in that case the expectation which is the sum of this series, we could add up and show that it is actually equal to the parameter of that Poisson and in fact, it is right.

So, this is also called the mean of the Poisson distribution. So, for this case where the values taken by the random variable are infinite, countable infinite, the expectation exists. So, therefore, there always has to be when we define the function e x, we will say that this is the expectation provided. It exists.

So, now look at another random variable which takes countable infinite numbers. For example, take the probability of x equal to n as c upon n square takes values 1 to infinity. Now, since you want this to be a valid pmf, so the summation n to 1 in 1 to infinity of probability x equal to this should be capital X equal to n is this summation, right and therefore, this is equal to you know this series is a convergent series, and it is known that the sum of the series 1 to infinity 1 upon n square is actually pi square by 6, right. So, your c must be 6 by pi square.

So, once I defined my c to be 6 by pi square, this is valid pmf, but when you want to compute the expectation, the expectation would be c sigma n into 1 by n square n varying from 1 to infinity, but then this sum series 1 by n summation 1 to infinity, we all know is divergent series and therefore, expectation does not exist, right. So, therefore, one has to be cautious and careful and make sure that I mean e x is defined only if it exists.

Similarly, in the case of continuous random variables, this integral may not always exist even if your f is a valid pdf probability density function, and I will give you an example here. This is known as the pdf, where F X is equal to 1 upon pi into 1 upon x square x varying from minus infinity to infinity. This is known as the Koushish distribution pdf and here again, this is a valid pdf because integral minus infinity to infinity 1 upon pi 1 plus x square d x. So, integral of 1 upon 1 plus x, x square is tan inverse x minus infinity to infinity 1 upon pi and then, tan inverse of infinity is pi by 2 tan inverse of minus infinity is minus pi by 2. So, therefore, this becomes plus and this is equal to 1.

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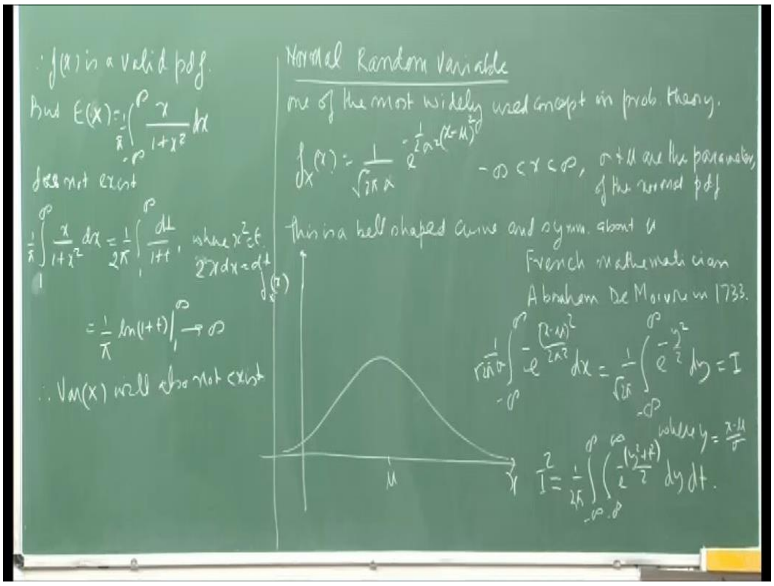

So, this is again a valid probability density function, but when you want to compute the expectation of this random variable, you have to integrate this particular integral and you can show that here you see if I make this substitution that x square is equal to, so actually this is running proper integral and I will just consider the integral from 1 to infinity. Let us see and if this does not exist then, obviously 0 to infinity will also not exist and therefore, this whole thing will also not exist. So, let us put x square equal to t and then, your d x x d x is d t. So, x d x you replace by or there will be 2 somewhere should have because this is then your 2 x 2 x d x is equal to d t. So, this will be 1 by 2 here, right, x d x is 1 by 2 d t. So, this is integral of this is l n of 1 plus t 1 to infinity and you know that l n of infinity is infinity.

So, therefore, this integral does not have a finite value. So, your expectation does not exist and hence, your variance will also not exist. Now, since variance of x would be e x square minus e x whole square and we have just seen that e x does not exist for this particular random variable. So, therefore, variance also will not exist because it has to be e x square minus this. So, if this does not exist, that means, this is not finite. Then, obviously variance will also not be finite. So, we will say that it does not exist. So, I just thought that I will be putting this note here before we proceed with the other theory of probability theory, so that you can find, you know you cannot always be sure that the expectation of a random variable will exist.

Now, another very important or widely used concept in probability theory is that of normal random variable and its probability density function. So, the function is defined by 1 upon root 2 pi sigma e raise to minus 1 by 2 sigma square x minus mu square, where mu and sigma are parameters of the normal pdf and x varies from minus infinity to infinity. Now, you can look at this thing here e raise to minus 1 by 2 sigma square x minus mu whole square. So, it can be shown by people have already drawn the graph for a different values of x and mu and sigma. So, this is a bell shaped curve and it is symmetric about mu that you can see from here because x minus mu whole square.

So, it is symmetric. That means, on either side of mu you take the value, this sign will not matter and therefore, this is symmetric about the value x equal to mu and this distribution was discovered by, was defined by let us say French mathematician Abraham De Marvn in 1733, and can you believe that he was not very, he use to make a living, he used to spend time in a you know at that time dengue gambling house. He would be sitting in the evenings, spend whole evening there and trying to help people because he used this concept of the normal distribution to approximate binomial distribution and binomial distribution.

He was used to help people because it was a gambling house. People would come to bet money and of course, would want to win their bets. So, he would give them the probability of you know winning which bet and so on and so, he actually used this concept for approximating binomial distribution and we will be discussing those approximations little later on. So, this is how, but the concept he introduced is very important and very widely used one, and I think by the end of the course, you will also see how important this concept is to the probability theory and you know for various estimations that we want to make about different events and their probabilities. So, now let us see whether this is actually a valid pdf.

So, we want to integrate this function from minus infinity to infinity and show that the integral is equal to 1. So, here what I do is, I make the transformation. Yeah, I have made transformation y is equal to x minus mu by sigma, so then d y is d x upon sigma. So, d x upon sigma appears here which gets replaced by d y and 1 upon under root 2 pi is here. Then, this is e raise to minus y square by 2. This remains from minus infinity to infinity because mu and sigma are finite numbers. So, then we have to now integrate this, and let me call this integral as i. So, if I multiply this by another integral, this is the notation.

So, i square now will become a double integral 1 upon 2 pi minus infinity to infinity minus infinity to infinity e raise to minus y square plus t square by 2 d y d t. See t and y are dummy variables. So, it does not matter and therefore, I can say that this is also i minus infinity to infinity 1 upon 2 root 2 that becomes 2 because you multiplying them. So, e raise to minus t square by 2 d t. So, this is what you have and now, we want to be able to compute this integral and there this is where we will use polar coordinates. So, let me show you the computations.