Introduction to Probability Theory and its Applications

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Lecture - 32
Poisson Process

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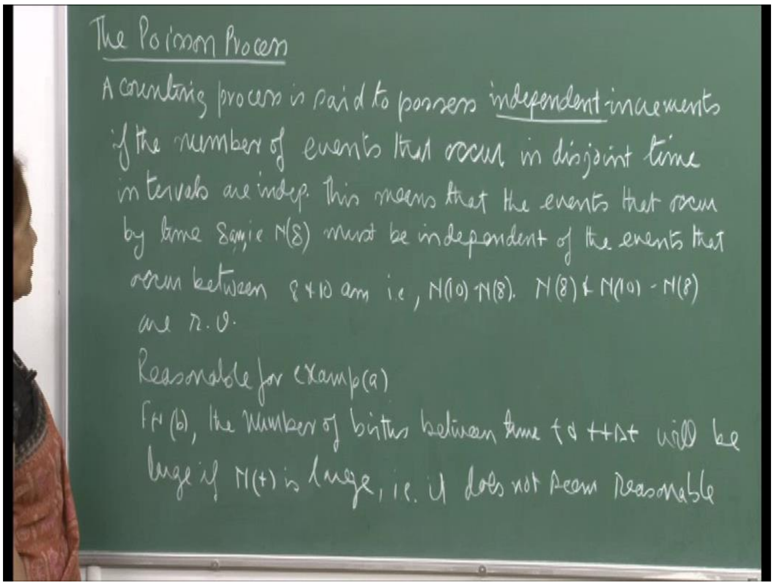

So, while talking of the Poisson process I said we have to first talk about a counting process, and so we said that there should be some norms for the, have to be followed for the counting process. And so the first one is that it should have independent increments; I should underline increment, independent increments. So, that means, that the number of event that occur in disjoint intervals are independent.

So, that means, say for example, if you are saying that up to 8 am you are counting the time, then n 8 would be the number of events that have occurred up to time 8. And so that n 8 will be independent of say, for example, the number of events that have occurred in the between at 8 and 10 am. So, that means, n 10 minus n 8. So, both these are random variables, and so we will assume because the intervals are disjoint this is up to 8, let us say 0 to 8, and then this is from 10, 8 to 10.

So, the 2 intervals are disjoint, and therefore we except we want that these 2, the number of events, that means the corresponding random variables must be independent. Now, for the, for example a, that I see, I gave you 3 examples of counting processes; first one was you know arrivals at a post office. And so example a, this may be a reasonable assumption because we may assume that people as long as the, for the time that the post office is open, people will come in any time. And so people coming say, between say, upto from, if it is open from 8 am, then 8 am to 10 am, the number of arrivals and the random variables indicating the number of arrivals, and say between 10 and 12 the arrival, the random variable, 2 random variables giving the number of arrivals in these disjoint intervals are independent.

Now, for b, which is the number of births in a particular town or a this number of births between time t and t plus delta t suppose we are taking, will be large if M t is large. So, that means, at, if you taking it over a long span then at a particular time when the people, population is large then the number of births will be large. So, here it will depend; that means, it does not seem reasonable that N (t) is independent of N t plus delta t minus N (t) for any t.

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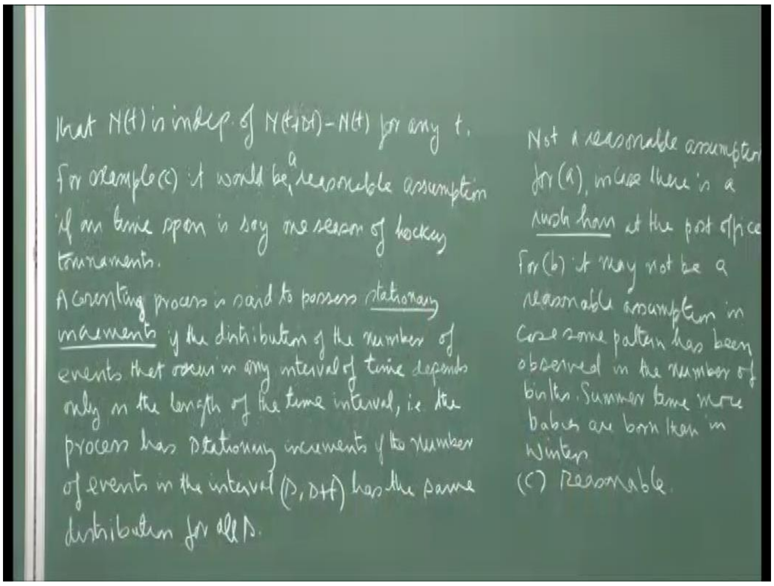

So, it will depend, see, this will depend on the number of births that occurs if you are taking upto; or, in the sense that you are saying that the random variable N (t) and the random variable N t plus delta t minus N (t) need not be independent, if you are, ok. Now, for example, see, it would be a reasonable assumption. So, this example c, refer to the number of goals that are hit by a hockey player.

So, here again if you take the time spent to be one season of hockey tournaments. So, during one particular season we expect the hockey player to be either to have continue to have a good form or not have a good form. See, if he has a good form then he will hit number of goals hit by him upto time t, or from t to s, t to s plus t, should be, the 2 random variables should be independent. So, the time spent is important, right. If I take the time spent to be 2 years then certainly it will matter because one cannot maintain a form for, let us say, upto 2 years or 5 years; so time spent. So, if you restrict your time spent then c would be a reasonable assumption. For c, the independent increment assumption would be reasonable.

The other important assumption for counting process is stationary increments, right. Now, here what we are saying is that the number of increments the, that occur should depend only on the length of the interval. So, here, for example, if you having (s, s plus t) then the length of the interval is t; so it will not matter what value s takes, as long as the interval time has length t, then the number of increments that occur during this interval is just dependent on the length of the interval, right.

Now, here, we can again see whether the counting processes that we wrote down is that reasonable assumption for all those counting processes. For example, for a, it is not a reasonable assumption, why? Because, for a post office, and even, similarly you can consider a bank, there may be a rush hour. If there is a rush hour then certainly you cannot say that this and this are independents.

If this is the rush hour then, you know, that there will be more arrivals, and so the random variables n 8 and n 10 minus n 8 would not be independent. So, if you have the concept of rush hour, but if you sort of ignore the rush hour and then you look at the counting process for a post office then this may be, assumption of stationary increments may be a reasonable assumption.

Now, again for b, it may not be a reasonable assumption. In this case, some sort of pattern has been observed in the number of births. See, somewhat times more babies may be born then during winters, and so again, if the, some pattern has been observed in the town that you are considering, then again b may not be, because it will not follow the assumption of stationary increments.

So, the, it will matter if your time span is during summer, so then, for the same period more babies may be born as opposed to the time span this is during winters. So, for the same length of time, the 2 may not be, the number of events in the interval has the same distribution for all s. So, it may be different distributions.

Then again for c, it will be a reasonable. Here again, if you are saying that if the person, if the, if the player if the hockey player is involved then surely the number of goals that he hits will depend on the length of the time that he is played. And so it will have the same distribution, and will depend, only the distribution of the number of goals that he hits would be dependent on the length of the interval, and not on the, when he hits.

If I am again restricting myself to, let us say, 1 season or may be 2 seasons, if that is considered to be reasonable that a person will continue to be in form for a player will be continue to be in form for 1 season or 2 seasons, maybe sometimes, it depends; whatever the way to look at it in that case the stationarity increment assumption would be reasonable one for c.

And, this is what I am trying to say is that you have to, before you start applying, modeling a situation, for example, a particular counting process, you have to see that certain basic assumptions are satisfied. And in that case, you can, you know, then we will see that based on these 2 assumptions we can now talk about the Poisson process. So, these are the 2 basic assumptions under which we will now formulate our this probabilistic model for counting, for the counting process, and which is which we will define as the Poisson process.

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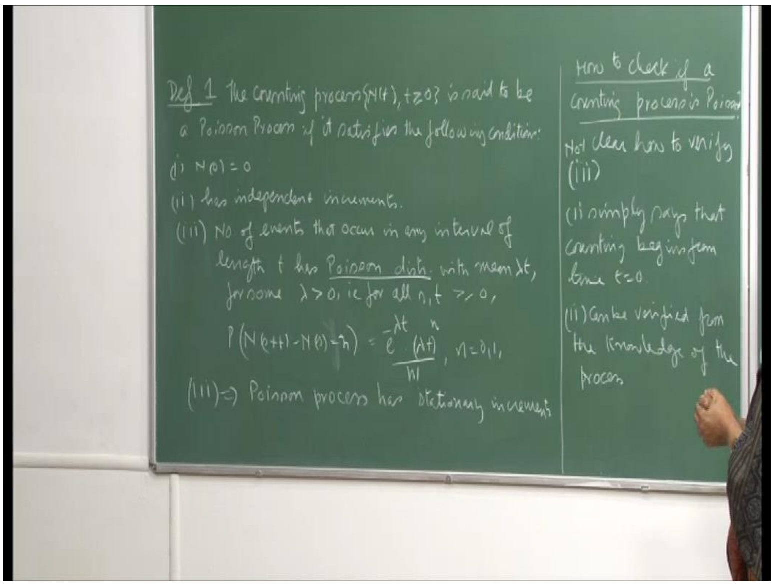

So, after having defined counting process, now I would make a definition of a Poisson process. So, what we are saying is that the counting process N (t), t greater than or equal to 0, is said to be a Poisson process if its satisfies the following conditions. So, as we said that we start the counting from time t equal to 0. So, n 0 is 0, then this has independent increments which we already said that while we define the conditions for a counting process. And then, we said additional properties were has independent increments. Then 3 is, that number of event that occur in any interval of length t has Poisson distribution; see, this is thing.

So, therefore, we are saying that it will be a Poisson process. The counting process will be a Poisson process. If it the number of event that occur in any interval of length t has Poisson distribution with mean lambda t, where lambda is some constant, positive constant. So, that means, what we are saying is that because the interval of length is t, time interval is of length t, so therefore, probability of N s plus t minus, N s equal to n.

Because, what we are saying is that the number of events that occur in this interval, s plus t, s, ok, we can say that the length of the interval is t equal to n will be, e rise to minus lambda t, lambda t rise to n upon n factorial, n varying from 0. So, this is your probability for any value of n, here integer value of n, right. So, now, since we are, since this probability is only dependent on the length of the time interval that is t, it is not dependent on when the counting process is started, the beginning of the interval.

So, therefore, you can immediately conclude that the process that we have defined also has the stationery increments property, right, which we said that it does not matter, it is only the number of events that occur which depend on the length of the interval and not on when you started counting. So, therefore, this would be a Poisson process, that is what our definition is.

So, now let us just look at this definition. And see you can, of course, condition 1 is defined that simply says, that counting begins from time t equal to 0, which we have been saying repeatedly. Then, 2 can be verified from the knowledge of the process that it has independent increments whether it is a valid assumption or not for the process that you are trying to model, then you can tell from the knowledge of the process itself, right.

Now, 3 says, that the number of events that occur in an interval of length of t, has Poisson distribution. So, it is not clear how to verify 3. And this is the whole thing; I mean, if I am calling a process a Poisson process then I am just saying that the number of events that occur here follows a Poisson distribution. So, it is not clear; and therefore, this definition is not very implementable decision, or, you know, you cannot really verify this condition because, right.

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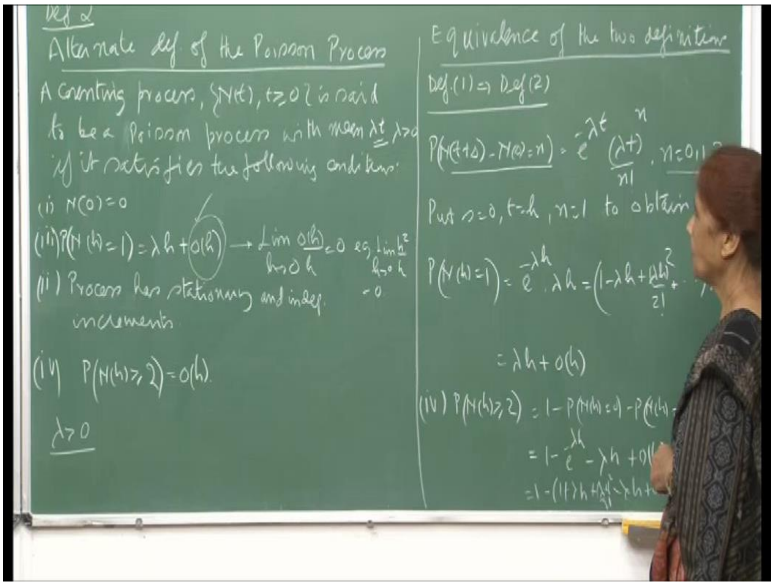

So, therefore, alternate definition is needed to determine whether a counting process is a Poisson process or not, we would be wanting to have another definition which hopefully would be a more easily implementable or verifiable, that a given counting process is Poisson or not. So, let us see. So, here we will say that can the same mean lambda t, if it satisfies the following conditions. So, counting process is said to be a Poisson with mean lambda t, if it satisfies the following conditions.

So, this is same as for the first definition, N (0) is 0. Now, what we are saying here is that probability N (h) equal to 1; that means, in the time interval h your probability that N (h) is equal to only one arrival takes place, 1 event occurs. This should be in this form; that means, lambda h plus order h. Now, this, of course, means that your function here is of value of high order than h, when you say, this is higher order of h. So, this is linear; that means, the terms here would be square, cube powers of h.

In other words, you can say that what this means is, that limit o (h) upon h, as h goes to 0, is 0. So, therefore, you know, this is of higher order than h, square, cube. For example, if you take o (h) to be h square, then h square upon h, limit of this as h goes to 0, is 0. So, this is the idea; that means, higher order terms here so this. And so what we are trying to say is that you can always discretize the occurrence of the event.

That means, if you take, if you make the interval h small enough, then you know, there will be the probability of an occurrence of an event is positive in a small interval. So, therefore, you can discretize. And the second one, I suppose, ok, this is third condition.

The second of course, is the same as that one, that process has stationary and independent increments. So, these 2 properties we require for the counting process to be able to say that it is Poisson process, right. So, this is has to be satisfied.

Now, this one tells you that the probability of the occurrence of event in a small interval is dependent on lambda and that you can separate out the. So, in other words, the, you know, bunching of occurrences of events is not permitted here. That, probability N (h) greater or equal to t, is of order h. So, that means, when h is small, this probability is really very small; of 2 or more events occurring in a very small interval of time h, so that is of order h.

And, since I have told you that this means the higher powers of h then linear; so therefore, this would be, you know, when h is very small, this also be very small. So, given lambda positive, same lambda we are saying here; when I say this, that means, it is understood that lambda is greater than 0 here. So, this is the alternate definition.

And, let us now see that obviously, when we are saying that this also describes a Poisson process, that also describes a Poisson process, there should be, we should be able to show that the 2 definitions are equivalent, right. So, I will do it, 1, both ways. I will first show that definition 1 implies definition 2, and then show you the definition 2 implies definition 1. And this is very interesting and nice, simple.

So, here, see we start with this. Definition 1 says that probability N (t) plus s minus, N (s) equal to n is, e rise to minus lambda t, lambda t rise to n upon n factorial, and varies like this, right. Now, put s equal to 0, t equal to h, because this is for all s, t; and n equal to 1, in this equation; then you obtain that probability N (h) equal to 1 because this is 0; t is h, s is 0. So, N (h) equal to 1 is given by e rise to minus lambda h; then lambda h rise to n is 1, and this is it, right.

Now, just expand e rise to minus lambda h, that will be 1 minus lambda h plus, lambda h whole square by factorial 2, and so on, multiplied by lambda h. So, when you bring lambda h inside, this would be lambda h plus, all terms will be of higher order because this will be square, h square; this will be h cube, and so on. So, I can write in this way, and which satisfies, right.

So, that means, your condition 3 implies, condition 3 here, for the definition, second definition. And similarly, it is also implied from here because when you want to compute this probability N (h) greater than or equal to 2, this will be 1 minus probability N (h) equal to 0 minus, probability N (h) equal to 1. So, N (h) equal to 0, since your definition is saying that when N is o, the probability is e rise to minus lambda h, that is all, right, because your N is 0; so that is e rise to minus lambda h. And this is probability N (h) equal to 1, we have just computed is this thing. So, minus lambda h and order, minus or plus does not matter, whatever.

And now, you see, when you expand this because this would be 1 minus, of 1 minus lambda h plus lambda h whole square by, 2 factorial minus, lambda h plus o (h) right. So, then 1, see, you have a minus sign here, so this becomes plus, right, because minus and then 1 minus lambda h; so therefore, this is, 1, 1 cancels out; lambda h minus lambda h cancels out; and you are left with something because this is order h. So, order higher than h, and this is also o (h). So, therefore, the whole thing is o (h), right.

So, you can see that definition 1 implies definition 2. And now, we will show you that definition 2 implies definition 1. And you see that this, then we would most of the time we are working with this definition of the Poisson process.

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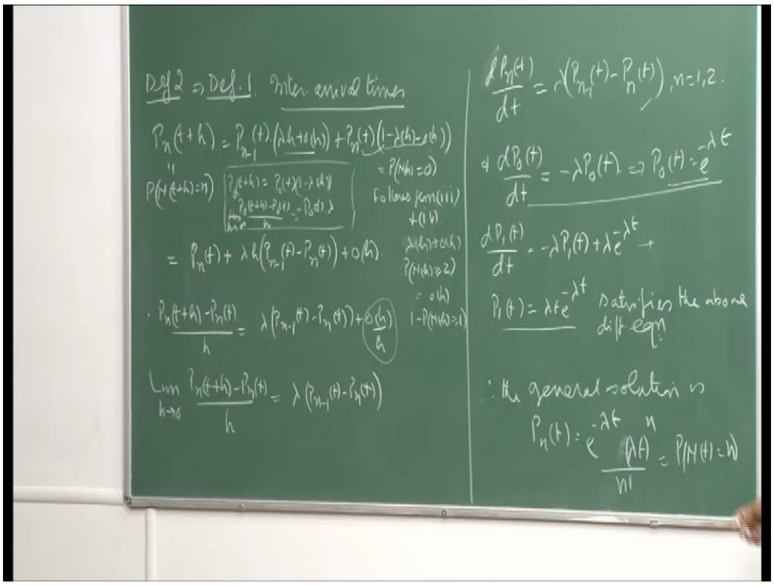

So, let us see; we will try to now show that definition 2 implies definition 1. And of course, then we will talk about inter arrival times later. But, you see here, probability n (t plus h). So, this is, I am saying is the, probability of N (t plus h) is equal to n, this is what we are saying, what we mean by this. So, this is the new notation I have started; so p n (t plus h), that means, number of arrivals are n in, upto time t plus h.

Now, this can be thought of as, n minus 1 arrivals upto time t; and then 1 arrival at, within the interval h, right. That means, see, from t plus, upto t plus h, you want n arrivals or n occurrences. So, upto t, if there are n minus 1 occurrences, then in the interval time length of h you want 1 arrival. And according to our definition 2, this is the probability of 1 arrival in the time interval h, length of interval is h, right.

And, since we are talking of independent increments, so this will be product, that means, I can say this; plus, there is no, that means, there are n arrivals upto time t, and there is no arrival in time length h, this is what; and, this will be probability n h equal to 0 which follows from 3 and 4. See, 3 said that probability of 1 arrival is 1 plus, is lambda h plus order h, right; and, probability n h greater than or equal to 2, was of order h, right.

Now, when you say that there is no arrival, then that means, you want 1 minus probability n h because, right; you want this to be; so if you take the complement, so then, n h greater than or equal to 1, would that be ok? See, if you want n h equal to 0, I, in time, so that means, arrivals have 1, 2, and so on, in the interval h; so 1 minus of that will be, right. So, therefore, yes; so this would be then 1 minus lambda h because this o h and this is lambda h plus o h, so therefore, this is what you have.

So, therefore, probability of n h equal to 0 is 1 minus lambda h minus order h. So, again independence of events, independent increments, so this will be p n t in to this, right. So, this is how you can describe n events in time upto t plus h, by breaking up the event into number of arrivals upto time t, and then n minus 1, one arrival in time h, or n arrivals upto time t, and no arrival in time n h. So, this is how we will write it down, right.

And therefore, this will be, if you simplify this expression, p n t is coming from here plus, lambda h p n minus 1 t minus, p n t, right. So, this is it plus, all terms are of higher order of h, right. Now, just rewrite this. This is p n of t plus h minus p n t, divided by h, so that will become; I have divided by h here, so lambda times p n minus 1 t minus p n t. And this is order h upon h.

Now, when you take the limit you can immediately see that when you take the limit as h goes to 0, this will be limit as h goes to 0 of this, and here this will be lambda, this is independent of h, and this we have seen will go to 0, right; order, higher order of h means that this limit is going to 0, right. So, this is what you have. And then this is nothing but the derivative of p n t. So, this will be the derivative and this is equal to lambda p n minus 1 t minus p n t.

Now, so this is valid for n varying from 1 because you have this n minus 1. And d p 0 t upon d t is minus lambda p 0 t, because when you are looking at n equal to 0, see from here, then you do not have this; you simply have this, right, p n t because I should have n equal to 0, and right, so it will be; if you want me to write it down separately, see here becomes crowded.

So, this will be, p 0 t plus h is p 0 t; so no arrival upto time t, and no arrival in the time intervals, so 1 minus lambda h. Remember, I can ignore that term because that will anyway go to 0. So, then this will be p 0 t plus h minus p 0 t, divided by h. So, limit of this, as h goes to 0, it is not legible, but you can, I am talking loudly, so you can hear. So, this is equal to p 0 t minus into lambda, right.

So, this, therefore, this is derivative, d p 0 t upon d t is equal to minus lambda p 0 t, and so you, this much knowledge you have about the differential equations. So, here, from here it follows that p 0 t is e rise to minus lambda t; and so you have these 2 sets of differential equations. So, here the solution is immediate, and this is n from 1 to t. So, if you put like, for example, n equal to 1, it will be d p 1 t upon d t which will be from here, minus lambda p 1 t plus lambda e rise to minus lambda t.

Because, I am putting, substituting for p 0 t. See, when n is 1 this is p 0 t, which is e rise to minus lambda t. So, therefore, this is it, right. Now, of course, there are methods to solve these differential equations are difficult, but what we are saying is that you, if you just try p 1 t equal to lambda t e rise to minus lambda t, it will satisfy this equation, right. Just differentiate and substitute here, the 2 sides will be equal.

So, p 1 t is the solution here. And then, in general, the solution would be p n t you can very easily verify that you know, for all values of n this is the solution to all general differential equation that you obtained here. And this is nothing but as I have said by definition is that. p n t is nothing but probability of n t equal to n; so n arrivals upto time t. So, therefore, you see all other conditions remain the same in the definition 2 and 1 this was the only thing.

Because we were not sure how we would go about verifying in definition 1 that the number of arrivals in time interval of length t will follow Poisson distribution. So, now, using these properties, you know, probability of definition, of number of occurrences in the time interval h, by now given those, the third and fourth condition of definition 2 help us to show that the number of arrivals, probability of number of arrivals upto time t would be follow a Poisson distribution.

So, a nice way of showing that and because, see therefore, definition 2 is easier to verify, we feel because that you know, to digitize and so on, you can sort of approximate the condition 3. Actually, that is the important 1 in definition 2, and then that is probability of 1 occurrence in time interval length h is of this order that you can supposedly easily verify, there are methods to do it.

So, therefore, most of the time we would be, would now else we have established the equivalence of definition 1 and 2; it does not matter whichever you feel, when needed you can use it in your; you know, when you are trying to analyze certain results, and, or obtain certain probabilities, ok.

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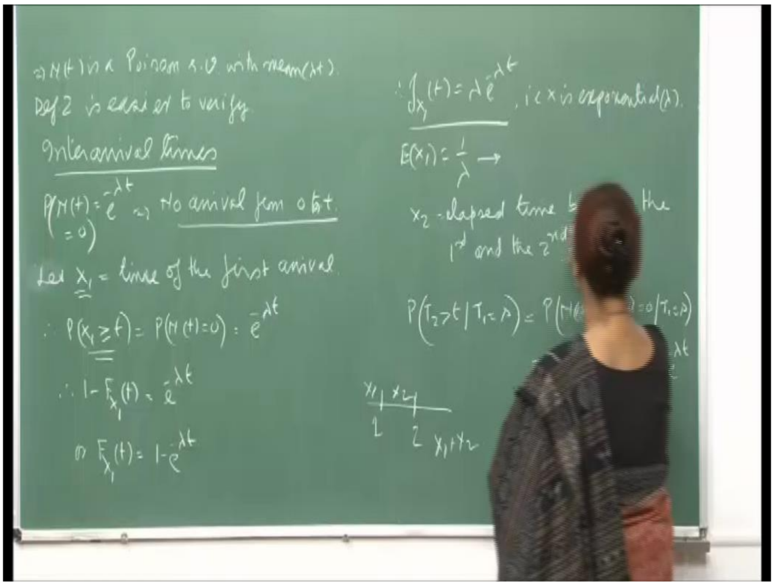

Now, as I have written there, inter arrival times we would now like to look at the probability of, that means, the time. So, here, time between 2 occurrences, because again the occurrence is overall chance events are unpredictable. So, we want to now look at the, if it is possible to determine the distribution of these inter arrival times, right. We have seen that probability N (t) equal to 0 is e rise to minus lambda t. So, no arrival from 0 to time t, this is your.

So, that means, upto, and if you define x 1 as the time of the first arrival, if the time of the first arrival, and then if I want to compute the probability that x 1 is greater than t;

that means, that there has been no arrival in the intervals 0 to t. So, no arrival from 0 to t is the event that N (t) is 0. So, the 2 events are the same; that means, if the first arrival has not occurred, the time for the first arrival, see x 1 is the time of the first arrival. So, if the first arrival time is greater than t; that means, in the intervals 0 to t, no arrival has taken place which implies that N (t) is 0.

So, the 2 events are the same, is just that here we have defined the random variable N (t), and here if the random variable x 1. So, therefore, and this probability is e raise to minus lambda t. So, if somewhere I have written that this event can be taken as x 1 greater than or equal to t then this is not correct. Because, you are saying that the time of the first arrival is greater than t, so it cannot be.

Or, if you are saying that N (t) is 0 then this is equivalent to the event that x 1 is greater than t; it cannot be greater than or equal to t because, in the time 0 t no arrival has taken place. So, any arrival that takes place will be after t. So, this is the important thing. And therefore, 1 minus f of x 1 t; so this is 1 minus of f x 1 t, if f is the cumulative distribution function for x 1. So, then this is equal to e rise to minus lambda t; and so f x 1 t is 1 minus e rise to minus lambda t.

So, this gives you the; and so that shows you that therefore, your f x 1 will be lambda e rise to minus lambda t. So, if you if you differentiate this equation on both sides you will get, lambda e rise to minus lambda t. Now, this is exponential distribution, exponential lambda. So, the distribution of x 1 is exponential lambda; and therefore, the expectation is expected value of x 1 is 1 upon lambda. So, this tells you what? That, you know, 1 upon lambda is the, you know, the mean, mean time of, you know, first arrival, right.

But, actually, now if you look at x 2; x 2 is the elapsed time between the first and the second arrival; that means, here the first arrival occurred here, and the second arrival occurred here. So, I am calling this as x 2. So, that means, if this is x 1, so the time of the first arrival is x 1, then time of the second arrival will be x 1 plus x 2 because this is inter arrival time. So, x 2 is the time between the first and the second arrival.

So, for example, so now, if you look at, that is why I have defined here. This is the elapsed time between the first and the second arrival, or second occurrence; I keep calling arrival, but which also means occurrence. Now, if you want to look at this probability t 2 greater than t, condition that t 1 is s. So, the first arrival took place, we are writing t 2, t 1, but here I have been writing x 1, x 2. So, it does not matter; you can make it x 1, and you can make this as x 2, because this is x 2, so x 2. So, therefore, this will be probability n s plus t minus n s is 0 because this second arrival is not have, has not taken place upto, I mean, this length is not is bigger than t.

So, therefore, s plus t will be this. So, n s plus t minus n s is 0, given t 1 is s. So, therefore, probability n s plus t minus n s is 0, is e rise to minus lambda t. So, this is again, see this will also be 1 minus f x 2 (t), and therefore, you can again see that it will be exponential. So, we will continue with the, you know; so the inter arrival times are all exponential and that you can relate it with the memory less property of the exponential distribution which comes to your independent increments. So, the 2 things are related.

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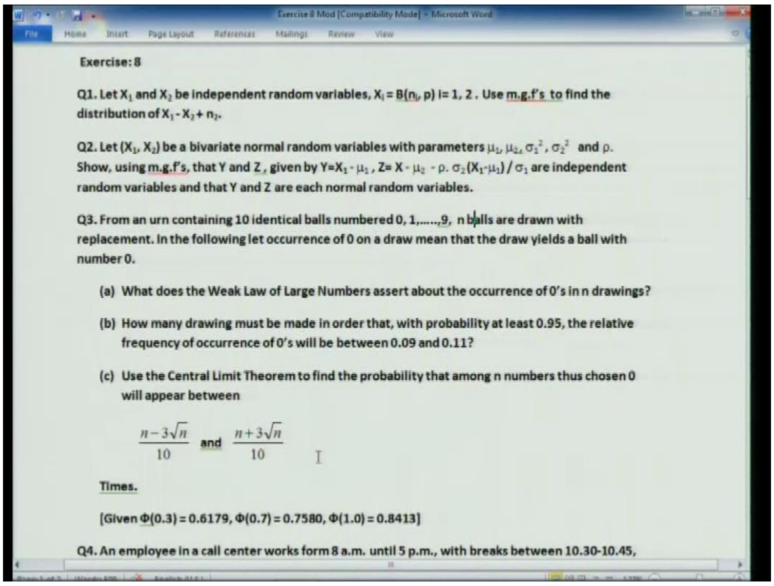

In this exercise 8, I will be going over some problems related with the liber theorems that we had done, and also the Poisson process that we have talked about after that right. So, X 1 and X 2 are, question 1, X 1 and X 2 are independent random variables; X 1 is binomial n i, p, i 1 to 2, right. So, X 1, that means, is binomial (n 1, p), and X 2 is binomial (n 2, p). So, use m g f s to find the distribution of X 1 minus X 2 plus n 2.

So, here, you see, all that you have to show is, yeah; so basically these 2 problems are on the, you know, m g f s joint density functions of random variables, and then add also introduce the concept of m g f for more than 1 variable. So, now, here you see, you can rewrite this X 1 minus X 2 plus n 2 as, X 1 plus n 2 minus X 2. So, since X 2 is binomial n to p, your n 2 minus X 2 will become binomial n 2, 1 minus p, right, is that clear?

Because, you see, the when you consider n 2 minus X 2, so if X 2 has r successes, then x, n 2 minus x 2 will have n 2 minus r successes. So, it will be; so therefore, the, for n 2 minus X 2, the successing of X 2 would be failures here. And therefore, your, see the, at the probability for a failure is 1 minus p. So, that is all. So, once you recognize that you can write this as, X 1 plus n 2 minus X 2, then n 2 minus X 2 is binomial n 2 n 1 minus p.

So, once you know this then you can immediately began, and then they will be independent. So, therefore, because X 1 and X 2 are independent, so X 1 and n 2 minus X 2 are independent, and therefore, you can write down the joint m g f. So, you should be able to do it, right.

Now, question 2, X 1 and X 2 form a bivariate normal random variable; I mean, they are bivariate random normal variables with parameters new 1, new 2, sigma 1 square, sigma 2 square, and row. So, row is your co relation co efficient. Show using m g f s, that Y and Z, given by, so Y is defined as X 1 minus new 1, and Z is defined as X minus new 2 minus row into, sigma 2 upon, sigma 1 into, X 1 minus new 1. So, you have to show that these 2 random variables are independent, and that Y and Z are each random normal variables.

Here the idea is to use m g f s, but actually there is a easier way. And surely, to be able to do this problem using the m g f s, I leave it as a challenge, and maybe when I am discussing, you know, set of miscellaneous examples, we will revisit this. But, right now, way to do it is, see, because Y is X 1 minus new 1, so this will continue to be normal remember, because X minus new 1, the mean of Y will be 0.

Variance will be the same, because by shifting the mean of a random variable the variance does not change. So, therefore, Y is again normal which means 0 and variance sigma 1. And similarly, Z is also, this should be X 2 here that is missing, so X 2 minus new 2. So, here again you have shifted the mean of X 2 by this quantity new 2 plus row into sigma 2 n; of course, this is the full idea. When you write Z like this, so X 1 minus new 1 upon sigma 1, that is the, because you are computing the; so this is the whole idea.

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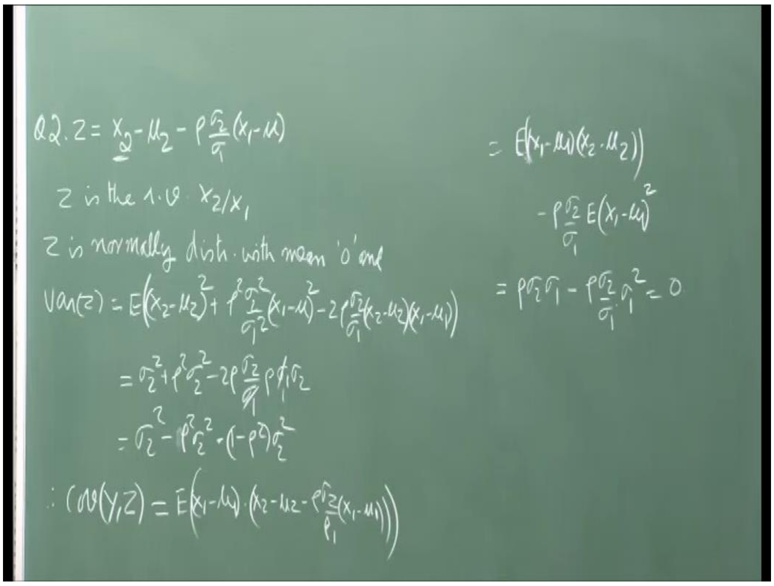

So, here again, the mean has been shifted. And therefore, z is also normal with the same variance sigma 2, and the mean would be well; so by definition of z, we see that z is the random variable which is given by a conditioning x 2 on x 1. And we had to, while computing the conditional p d f s, we have seen that this will also turn out to be normally distributed random variable. And you can immediately see that the mean of z that is the expected value of z is 0 because expected value of x 2 is new 2 and expected value of x 1 is new 1. So, when you take the expected value of z, it will turn out to be 0, right.

So, therefore, we need to compute the variance. And the variance will simply be expectation of z square. So, which I have done here, by writing down the expectation of the square term, and then taking expectation inside as we can do it, and therefore, it will turn out to be 1 minus row square upon sigma 2 square. So, this is the variance. Now, when you want to define the co variance then it will be y comma, co variance of y, z. So, that will be expectation of x 1 minus new 1 into, x 2 minus new 2, minus row sigma 2 upon, sigma 1 into x 1 minus new 1, right.

And again, here we can take expectation inside. So, this will be expectation of x 1 minus new 1 into, x 2 minus new 2; and then, minus row sigma 2 upon sigma 1, expectation of x 1 minus new 1 whole square. And expectation x 1 minus new 1 into x 2 minus new 2 is the covariance the correlation coefficient row into, sigma 2, sigma 1, right. Because, expectation of x 1 minus new 1 into x 2 minus new 2 is the covariance, and, so that can be written as row into sigma 2 sigma 1.

Then, minus row sigma 2 upon sigma 1 into expectation of x 1 minus new 1 whole square is sigma 1 square. So, therefore, when you substitute that value you turns out to be 0. So, y and z are uncorrelated. And now, we use the result that for norm, if y and z are normally distributed then being uncorrelated is equivalent to their being independent. So, therefore, y and z are independent, right.

Question 3 is, from an urn containing 10 identical balls numbered 0, 1 to 9, n balls are drawn with replacement. So, the, draw a ball, and then put it back, you just notice the number. In the following, let us occurrence of 0 on a draw mean that the draw yields a ball with number 0. So, draw whenever the occurrence of 0 means that you draw a ball, you notice the number or you note it down somewhere, so if it is 0 then it is a draw. Means, that the draw yields a ball with number 0, and then we put the ball back, right.

Now, what is the weak law of large numbers assert about the occurrence of zeros in n drawings? So, the mean is 1 by 10 because they are identical balls. So, the probability of drawing any ball is equally likely with any of the numbers, 10 numbers is equally likely. And therefore, the weak law of large numbers will say, that if s n is the number, see we are denoting by s n the number that number of balls that showed up with number 0, right.

So, then s n by n is the relative frequency. So, out of the n draws, you have s n balls have shown up with number 0. So, s n by n, as n goes, is becomes large, will converge to 0 in probability, will converge to 1 by 10; the mean is 1 by 10; so will converge to 1 by 10 in probability, that is your weak law of large numbers. So, s n by n will converge to 1 by10 in probability.

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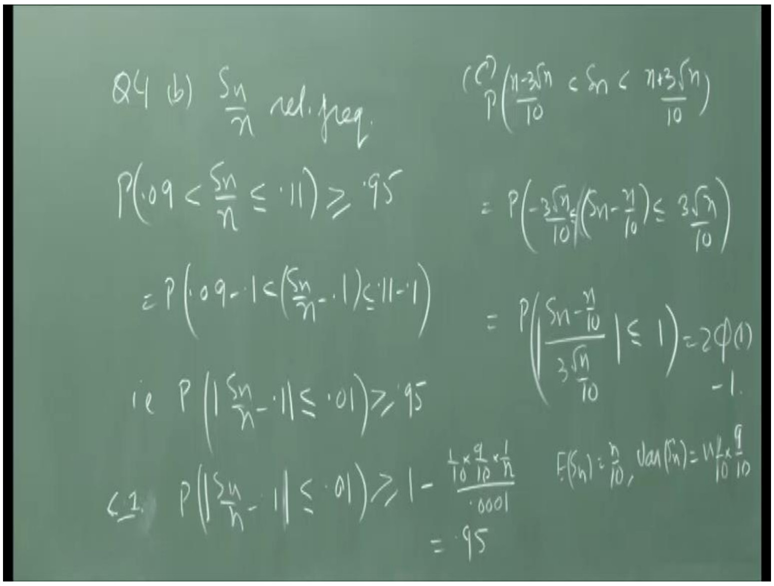

Now, b part is, how many drawings must be made in order that with probability at least 0.95, the relative frequency of occurrence of zeros will be between 0.09 and 0.11. So, that means, you are wanting to know. So, s n by n is the relative frequency, you want to compute this probability that the relative frequency is between 0.09 and 0.11. And this probability should be atleast 0.95, right.

So, therefore, now again I standardize the whole thing. So, this is s n by n minus 0.1, right; 0.1 is the expected value here, and so subtract 0.1 from either side, and so this finally, because the difference is 0.01. So, this is what you have. Now, establishes in the quality this number; this probability is greater than or equal to 1 minus variance of this s n by n which is because this is now binomial 1 by 10 into 9 by 10. So, 0 and nonzero that is how I am treating this is.

So, p, q; p, q into 1 by n because s n by n, the variance, right. So, variance of s n is n p q and that divided by n square. So, this is 1 by n, fine. And so this into divided by. so in establishes the quality 0.01. So, therefore, we will compute the value of n when this is equal to 0.95; and, for higher values of n, it will be higher, right. So, therefore, now you have the equality. You can now get a equation for n, and you will get the value of n for which this probability would be between 0.09 and 0.11. So, you can do the rest of the problem.

Then, part c is, use a central limit theorem to find the probability that among n numbers thus chosen 0 will appear between n minus 3 root n by 10, and n plus 3 root n by 10, fine. So, here again you want this probability using central limit theorem, right; n minus 3 root n by 10, less than s n, less than n plus 3 root n by 10. So, then I, you see, n by 10 is the mean of s n. So, I write it here, this. And then, you see the variance; variance of s n will be n into 1 by 10 into, 9 by 10, right. Because, s n is now binomial with mean 1 by 10 and, sorry, mean n by 10 and variance n p q, right. So, this is this, and so under root of this will be 3 root n by 10; so 3 root n by 10.

So, when you divide, so now this becomes a standard normal variant, and so the central limit theorem says, that probability this less than or equal to, I mean, for n large enough, less than or equal to 1; you want to compute, approximate. So, this is the approximate probability which is 2 phi 1 minus 1, right. So, phi 1, the value is given to you at the end of the problem, and so you can compute this. So, this is good use of central limit theorem.

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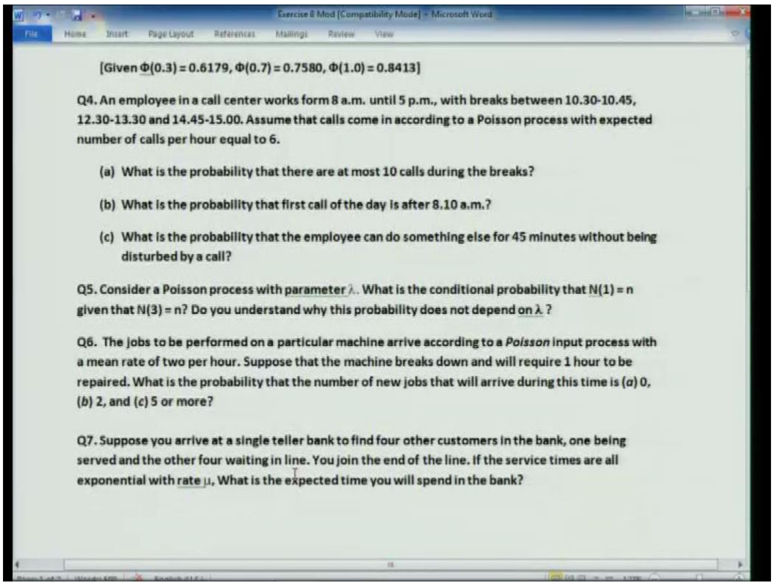

So, now let us go to problem 4. An employee in a call center works from 8 a.m. until 5 p.m., with breaks between, 10.30 to 10.45, then from 12.30 to 11.30, and 14.45 to 15.00 hours. Assume that calls come in according to a Poisson process with expected number of calls per hour equal to 6. So, lambda probability that there are atmost 10 calls during the breaks.

So, here the whole idea is, because you see, it does not matter for the, when the arrivals, the phone calls are coming by as a Poisson process, then the inter arrival time between the phone calls are, is exponential which is memory less. So, that is what you are using here. So, then what we can do is we can just add up the total break up, the break time which is 15, 1 hour and 15; so 1 hour 30 minutes right. So, therefore, 3 by 2 hours; and you want at most 10 calls during this 3 by 2 hours.

So, therefore, you will have to write the probability, see you sum up; that means, the calls can be 0, 1, 2, upto 10. So, your lambda is 6 into 3 by 2; 6 by, 6 into 3 by 2 because 3 by 2 hours. So, lambda t, lambda t becomes your parameter now. And within this time, within the time 3 by 2 hours you want atmost 10 calls. So, you can write down the Poisson probability.

What is the probability that first call of the day is after 8.10 a.m. So, that means, for 10 minutes you do not want any call. So, 0 call. And therefore, again this will be; so your time would be, you have to see, since your arrival rate is given per hour, so you have to convert the 10 minutes to the, to fraction of an hour which is 1 by 6. So, therefore, it will be e rise to minus lambda t, probability that no call comes during the time 8 to 8.10 am. So, that means, 1 by 10, 1 by 6 into 6, which becomes e rise to minus 1 will be the probability, right.

Actually, right, my idea is not to really give you all the answers, but just give you hints. What is the probability that the employee can do something else for 45 minutes without being disturbed by a call? So, here again we are repeatedly using the memory less property. So, 45 minutes can be anywhere; and therefore, you want the break; that means, now your time is 3 by 4 hour. So, you do not want any call to come in between for 3 by 4 hours, and your lambda is 6. So, you can do this now.

Now, consider a Poisson process with parameter lambda. What is the conditional probability that N (1) is n given that N (3) is n? So, here again, this is a good question in the sense that will again help you to understand the Poisson processes. Do you understand why this probability does not depend on lambda? Now, here you see, what is that you have to find?

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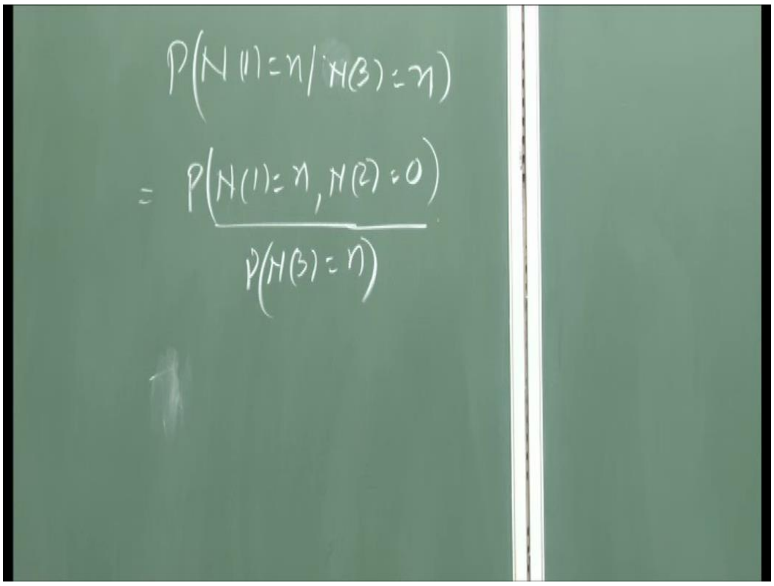

You have to find probability N (1) is n, given that N (3) is n. So, this means, probability N (1) is n. And see, actually you can interpret this as that N (1) is n, and then N (2) is 0, because N (1) is n and N (3) is also n. So, that means, for 2 prime periods, 3 minus 1, there has been no arrival, right. So, therefore, this is this, and then divided by probability N (3) equal to n. This is what you are having, right. And so and you are given, and your arrival rate is lambda, right. So, therefore, you have to write this, right.

So, N (1) is the number of arrivals which is equal to n; n time 1 period and n time 3 period also you have given that n arrivals are there. So, that means, all the n arrivals have taken place between 0 and 1. And so there are no arrivals in the time 2, 2 units of time 0, right. And now. if you right out this, these probabilities, you see that the answer will be independent of lambda. And again I want you to work out the details. So, question 5 is over.

Question 6 is, jobs to be performed on a particular machine arrive according to a Poisson input process with the mean rate of 2 per hour. Suppose, that the machine breaks down and will require 1 hour to be repaired, what is the probability that the number of new jobs that will arrive during this time is 0 to; so that means, essentially you are asking for 1 hour gap; that means, the business of machine is takes 1 hour to be repaired.

So, then in this 1 hour you want to know the probability of 0 arrival, 2 arrival, or 5 arrivals. And this is the Poisson process; and the mean rate is; so lambda is; so the mean rate is lambda which is equal to 2, right, 2 per hour. So, this you can, very simple problem; but again, I just want you to familiarize yourself with all these concepts, and therefore, I have done it here.

Now, question 7, suppose you arrive at a single teller bank to find 4 other customers in the bank, 1 being served and the other 4 waiting in line, the statement is a little this thing because here it should have said 5 people, but anyway, 4 refers to the 4 people waiting in the queue, 1 is being served; and you join the end of the line, so you are the 6 person. So, if the service times are all exponential with rate new, what is the expected time you will spend in the bank?

So, therefore, you see, this is 6 service times because 5 are already there, and because of the memory less property the person who is being served, again the probability of its completing, you know, so whatever service time is over is immaterial. So, therefore, 6 services have to be completed. And therefore, the completion time for the sixth person is a gamma distribution with mean 6 new; it will be 6, new, right; yeah, it will be gamma, 6, new, sorry.

So, it will be 6, new, and therefore, the, you know, what is the mean. The mean would be 6 by new. I think the mean should be n by new, right. If it is gamma n, new, anyway just verify that. So, you will find out the expected waiting time of the sixth person when he joins the queue in the bank.

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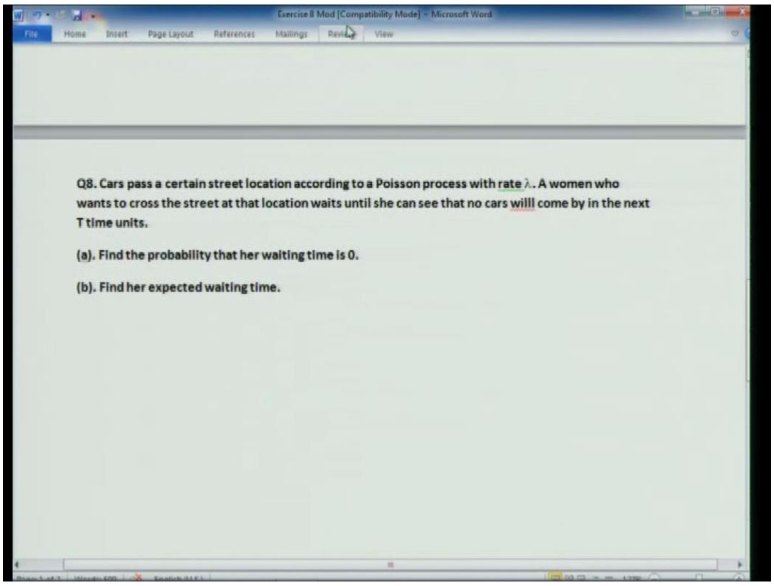

There should have been a, then this eighth one. So, cars pass a certain street location according to a Poisson process with rate lambda. So, a certain location in the, on the street, the cars are passing at the rate of lambda. And a woman who wants to cross the street at that location waits until she can see that no cars will come by in the next t time units. She has some idea that she is standing at this particular place, and then she looks at the side and sees that you know, for quite distance she cannot see any car coming then she will feel free to cross the street, right. And that time according to her is t time units, something like that.

So, find the probability that her waiting time is 0. So, waiting time is 0 means; that means, she comes to the location, and then the probability that there is no car coming for the next t units, right. So, here again because your arrival rate is Poisson with rate lambda, so your inter arrival times are also exponential with parameter lambda. And therefore, you can find out the probability that her waiting time is 0; that means, no arrival in the time 0 to t.

So, find out the Poisson probability that which is e rise to minus lambda capital T, right, no arrivals. And find her expected waiting time. So, now, you have this distribution, e rise to minus lambda t, as the probability that her waiting time is 0. So, you find out her expected waiting time. So, this you can now do by yourself.

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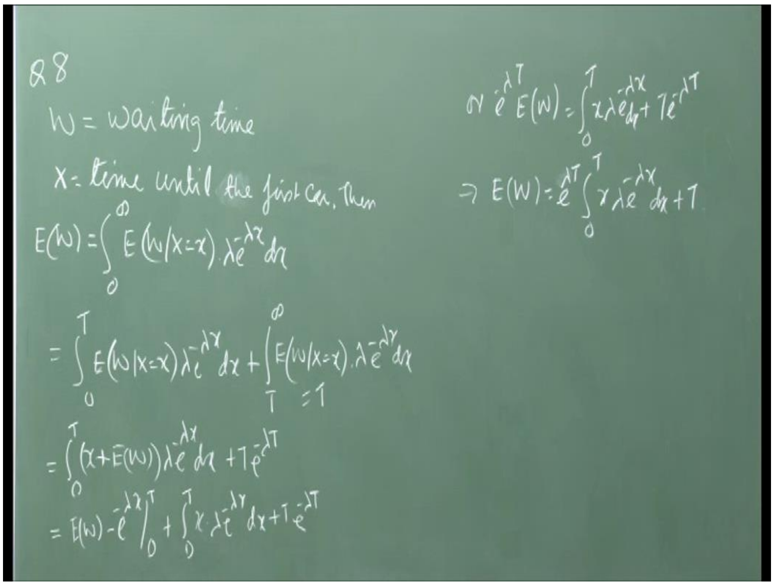

To answer the second part, that is what is the expected waiting time of the woman whose, who waits at the crossing for the cars to come. So, you will have to break up the, you will have to compute the expected value of the waiting time by using conditional probabilities. So, I will just write down the solution for you on the board.

So, I hope, with all these hints, and almost the some of the problems have solved almost completely. So, anyway, I hope you will enjoy doing it. And I definitely will try to come up with the large list of interesting and challenging problems at the end of the course.