**Introduction to Probability Theory and its Applications**
**Prof. Prabha Sharma**
**Department of Mathematics and Statistics**
**Indian Institute of Technology, Kanpur**

# Lecture - 30
## Reducible Markov Chains

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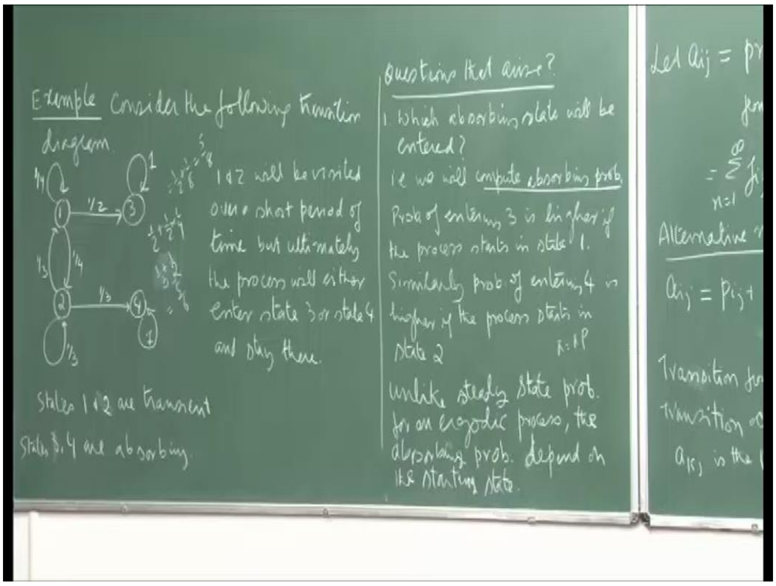


So, I will continue the discussion with the transient and absorbing probabilities. That is your reducible Markov chains. Now here for example, look at this transition diagram. So, there are four steps. And you see that from 1, you can either come back to 1 or go to 3. But then 3, once you reach 3, then probability of returning to itself is 1. So, therefore this is a certain event. So, once the process from 1 goes to 3, the process will stop there.

Similarly from 1 it can go to 2 and then from 2, if it transitions finally to 4 and 4 is the absorbing state. And so your process will again stop here. And similarly 2; so you can see that from 2 also you can go to 1 and 3 or 2 to itself and then 2 to 4. So, 1 and 2, no, I am sorry, 1 and 2. This should be 1 and 2 are transient and 3 and 4 are absorbing. That you can immediately see just by, you know looking at the transition diagram at once.

And of course, you can see that over a short period 1 and 2 will be visited, because they are transient. And so for a while, the process may go from 1 to 1 itself or from 1 to 2, then 2 to 1. That may go on for a while or 2 to 2, but then the moment, the process transitions to either 3 or 4, it stops. So, therefore over a short period of time 1 and 2 will be visited. But ultimately the process will either enter state 3 or 4; and then stay there. So, the process will end.

And, so now using this, I want to talk about absorbing probabilities. So for example, the questions that arise; so I have just; I have written the first question. Even though this is plural, we will talk about other questions after this. So, which absorbing state will be entered? Of course, here again we will only talk in terms of probabilities; which absorbing state or what is the probability.

Now, just to point out the difference between the absorbing probabilities and the steady-state probabilities, you see, now just look at this example. So, what I am saying is that we will answer this question by computing absorbing probabilities. And, so before I start talking about the method for computing the absorbing probabilities, let me just give you a feeling what we are talking about. See, here if you are in state 1, then the probability of transitioning to 3 seems higher than transitioning to 4; because, well, you might say that the one. If it happens in one step, then this probability is half or then it can happen that, you know, you can go to itself and then transition. So, that will be also. Then, that will be; so this will be half plus half into 1 by 4.

So, let us just compute this in two steps. And then if you; so transitioning, if you start in 1, then transitioning to t 3; let say in two steps. So, half plus half into 1 by 4, but if you are in 2, so computing the transitioning; so if you are transitioning from 2 to 3, then we cannot do it in one step. So, the two step transition probability will be; so the path will be from 2 to 1 and 1 to 3. And so this would be 2 to 1 is 1 by 3 and 1 to 3 is half. So, 1 by 3 into half will be 1 by 6.

So, this is the probability. Two step transition probability of going from 2 to 3. Fine. I mean, again the example that I am trying to take is, and so half plus half into 1 by 4 would be how much? This will be 1 by 2 plus 1 by 8 which comes out to be 5 by 8. You are connected to 3 from 1 directly.

So, this is what we are trying to bring out through the computation of the absorbing probabilities. That it looks like that transitioning from 1 to 3; the probability would be higher, if you are in state 1. And if you are in state 2, then the probability should come out to be lower than that. So, in other words what we are trying to make a case for is that the absorbing probabilities are not independent of the starting probabilities; that means the starting states. Whereas, the steady-state probabilities we saw. So, unlike steady-state probabilities Ergodic process, the absorbing probabilities depend on the starting states. See, steady-states were independent of where the system started. And therefore, we would compute, you know, simply pi equal to pi P. And, it did not matter where the system was; because remember all the rows of the matrix became identical. But in the absorbing probabilities, it will depend on where you are starting and this is what we want to make sure.

Like, this will; now we will ultimately, when we compute the probabilities, we will show that the absorbing probabilities when you are in state 1 to 3 is higher than when you are in state 2. And similarly for 4; it will be higher when you are in 2 and then when you are in 1. So, we will show because here we will have to then let us say, I take three steps, four steps, four transitions, then I can show you that the numbers. Here when you are in state 1, transitioning to 3 will have a higher value than transitioning from 2 to 3. So, let us see. It will come out.

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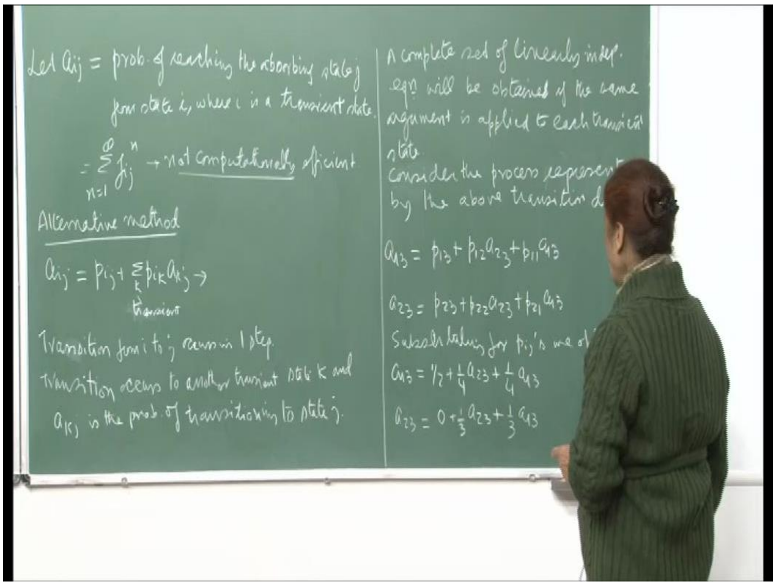


So, let me make these definitions. a i j is the probability of reaching the absorbing state j from state i; where i is a transient state. You want to compute this. Now, of course one way is that you can use the first passage probability. So, f i j; that means, over the first time you are transitioning from i to j in n steps and then you sum it this up from n equal to 1 to infinity. Now, this is not computationally efficient. Right; because we will have to compute all the higher powers of the transition matrix and then we want to compute the f i j n s. So, an alternate method. And, this should look familiar because we have already used this kind of argument.

So, now what we are saying is that if you want to compute a i j, then the transition either takes place in one step. That means, in that case the probability is p i j. So, this is the probability of transitioning from the transient state i to the absorbing state j. So, then this is p i j plus or i transition from i to another transient state k. So, that probability will be p i k. And then a k j will be ultimately transitioning from the transient state k to the absorbing state j. So, this will be; this number with this probability and this is the probability of transitioning in one step from i to j. So, this is the argument.

Now, a complete set of linearly independent equations will be obtained, if the same argument will be applied to each transient state. So that means, here I have done it for i. So, whatever the number of transient states, from each of them I will try to find out this probability of transitioning to the absorbing state j. And then I will have a complete set of linearly independent equations and we will then solve for the a i j s and then we will get the... So, we will be able to answer the question that which absorbing state will be entered. In the sense that you will say that a particular absorbing state will be entered with this much probability starting from the states. This is the way we will be able to answer that question.

So, now let us just take the transition diagram. So, we are considering this process. And, if you write out the equations, when you see from one transition state 1, you want to go to absorbing state 3. Then, if you write out these equations, it will be p 1 3. Either, you do it in one step or you go to the transient state two; p 1 2 and then a 2 3. Or, you come back and follow the loop p 1 1 into a 1 3. So, this is your equation when you want to write down for a 1 3. Similarly, you can write it down for a 2 3. So, this will be p 2 3 plus p 2 2 a 2 3. From 2, you can either transition to 3 in one step. p 2 3; p 2 3 in our case will be 0. Or, then you will have to go from 2 to 2 loop and again a 2 3. There will be a 2 3. So, you come back to itself and then a 2 3; the probability of transition from 2 to 3. Then or you go from 2 to 1 and then you go to from 1 to 3. So, these are the two equations.

Now if you substitute for the p i j s, we obtain these two equations. And, you can see that here the variables are a 1 3 and a 2 3, right, because this is 0 as I said. p 2 3, there is no arc from 2 to 3. So, you have two unknowns and two equations. So, we should be able to solve for a 1 3 and a 2 3 here. So, see if you look at these equations from here, if you bring a 1 3, here it will be 3 by 4 a 1 3 equal to half plus 1 by 4 a 2 3. And here because this is zero, so you bring a 2 3 here. So, this will be 2 by 3 a 2 3 which is equal to 1 by 3 a 1 3.

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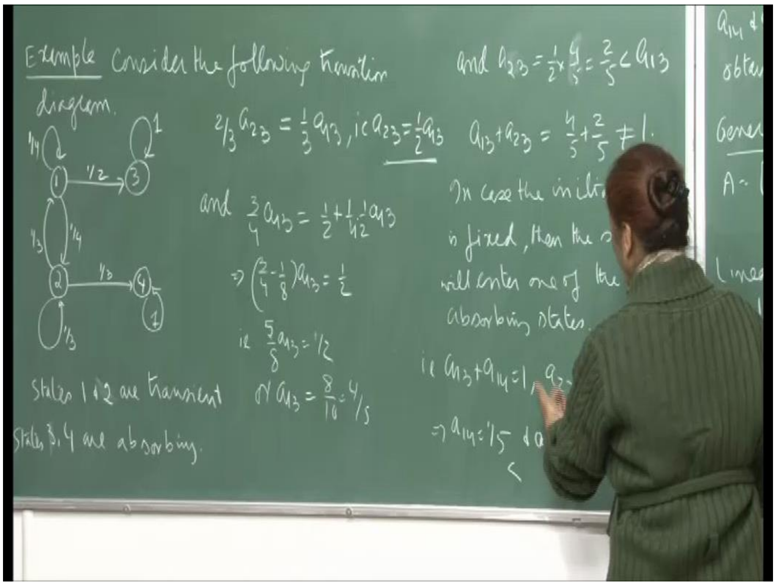


So, that immediately gives you 2 by 3 a 2 3 equal to 1 by 3 a 1 3 gives you a 2 3 is half a 1 3. And, now if I substitute for a 2 3 in terms of a 1 3 in the second equation, which is 3 by 4 a 1 3 is equal to half plus 1 by 4 into 1 by 2 a 1 3. So, then I get the equation for the value of a 1 3, which is 3 by 4 minus 1 by 8 a 1 3 is half; that is 5 by 8 a 1 3 is half. So, therefore I get the value of a 1 3 as 4 by 5. So, once my a 1 3 is known as 4 by 5, I can compute a 2 3.

So, you know, even from here only one could have concluded. What I was trying to show you, but of course, it what I needed at least 2 to 3. You know computations of three, four paths to get you to this. That a 2 3 would be half less than a 1 3. The probability of reaching the absorbing state 3 from 2 is smaller than the probability of reaching 3 from 1. And, you know, maybe you can say that the intuitive feeling looking at the diagram because you have a direct connection here. And here, you will need at least two transitions to come here. So, therefore this was the feeling which you get now sort of validate by doing these computations. So, therefore a 2 3 comes out to be 2 by 5; which is less than 4 by 5. And therefore, it is less than a 1 3.

Now, certainly a 1 3 plus a 2 3 will not be equal to 1; because we are not sure whether we will be; yes, so a 1 3 plus a 2 3. Yes. Because we do not know whether we will, from 1, we will definitely reach the absorbing state 3. You may also reach the absorbing state 4.

So, therefore these two probabilities will not add up to 1, not necessarily. But if you fix the initial state, then the system will enter one of the absorbing states. If you fix the initial state, that means, if I simply say that my system is in state one, then of course a 1 3 plus a 1 4 will have to be 1. Or, if my system is known to be in a state 2, then I must transition to either 3 or 4 ultimately. So, therefore a 2 3 plus a 2 4 will have to be 1.

So if you know this, then I can immediately compute because I have already computed a 1 3 and a 2 3. So, then a 1 4 will be 1 by 5 and a 2 4 will be 3 by 5. And, so here again the same thing gets validated that your a 1 4 is less than a 2 4. So, the probability of transitioning from 2 to 4 is higher than transitioning to 4 from 1. Now, let me just make this 1. This is the whole idea.

Now a 1 4 and a 2 4, I could have also obtained just as I wrote down the equations for a 1 3 and a 2 3. So, the same way I would write down the equations for a 1 4 and a 2 4 and I would compute them. So, this is what now; this is the method for computing the absorbing probabilities. And, as we have seen that these probabilities will depend on where the system is and unlike these steady-state probabilities, which are independent of the starting state.

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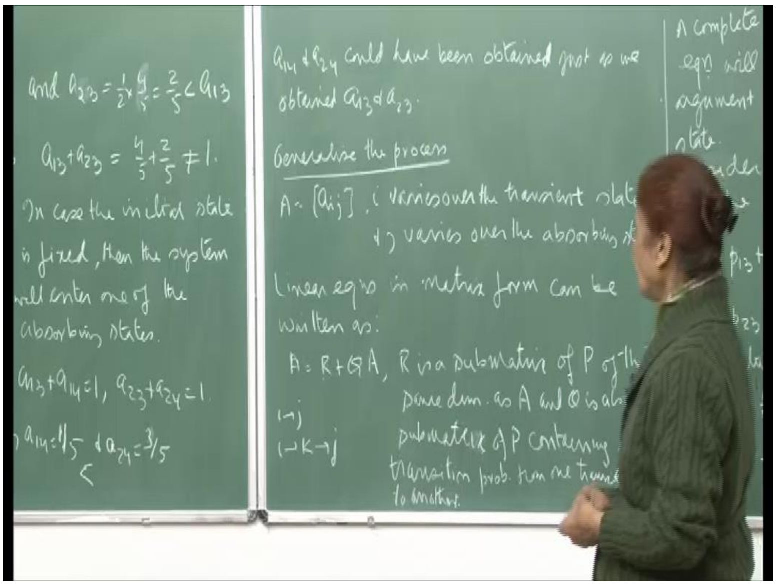


Now, let us just generalize this process. If I now call the A as a matrix of these transition probabilities, then here your i will vary. I mean, transition probabilities from transient states to absorbing states. Then i will be the rows of the matrix. You know, these vary over the transient states. And, j varies over the absorbing states. So, the columns correspond to the absorbing states. And this, so for example, in this case it is two by two.

So, in this case your matrix will be two by two. And, you can see that if I wrote down the equations for a 1 4 and a 2 4 also, then I will have the matrix here corresponding to all these absorbing probabilities that I want to compute. So, then in that case the system you would have been written as the matrix A and this will consist of, remember the R is the matrix consisting of transitioning from a transient state to absorbing state in one step. You are writing so. p i j s; so therefore, R is a sub matrix of your transition matrix p, which has the same dimension as A. And the rows, of course correspond to the transient state and the columns correspond to j because you are transitioning transient state to absorbing state in one step. So, therefore R is exactly the sub matrix of p of the same dimension as A. Then, we wrote down Q A.

So, remember this will consist of the transition probabilities from a transient state to another transient state; because we were writing. So, either from i to j you did in one step. Then, those probabilities are here or you go went from i to k another transient state and then from k, you went to j finally. So i to k, this is transitioning from a transient state i to a transient state j k. So, Q will be that matrix. And, since the columns here are transient states, so and the rows are transient states, so this matches. So, this is compatible and this will be your system of a linear equations written in a matrix form when you want to compute the absorbing state probabilities. So, let us see. So here for example, if you bring to this side, it will be I minus Q times A equal to R. So, we continue with the computation. Through this matrix, we will look at the entries of what do we mean by. So, in other words it will be I minus Q A is R. So, A will be I minus Q inverse R. And, I minus q inverse will exist.

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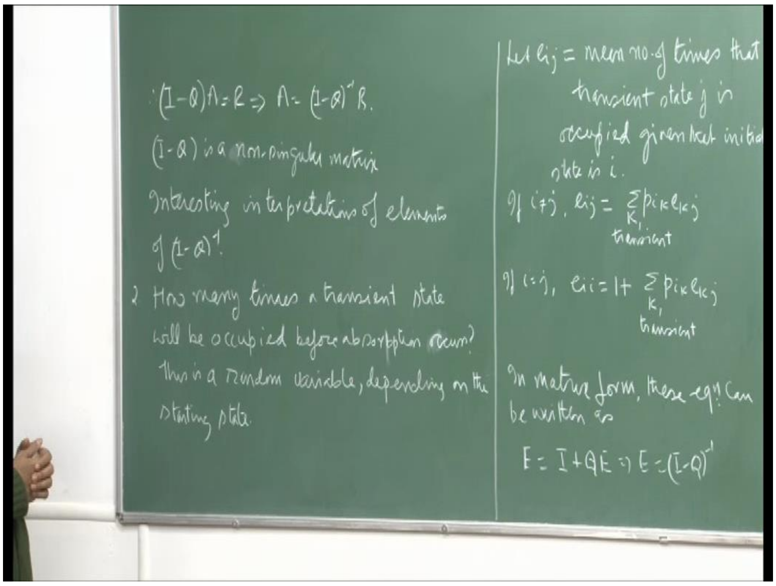


So, I minus Q A is R. So, this implies that A is I minus Q inverse R. So, you see, here A represents the probabilities of reaching an absorbing state from a transient state. And, so when we setup the equation a valid one, then there must be a solution to this system. So, that is what we are saying. So, therefore I minus Q; and, it is a unique solution. So, therefore I minus Q inverse exist. That is what we are trying to say.

So, I am giving the argument in this way. Normally, you try to first show analytically that the matrix is a non-singular. And, hence the solution is exist. But, here we know that the system will ultimately settle down into one of the absorbing states. So, these probabilities are finite and therefore the solution exists. And hence, I minus Q must be invertible. So, this is the whole idea.

So, the stars equation has a unique solution. So, I minus Q inverse must exist. And, now we have interesting interpretations of the elements of I minus Q inverse also. And as I said, the question that keep arising; we will go now answering them. And then in between we will also look at the interpretation of the elements of I minus Q inverse.

So, the second question that arises is how many times a transient state will be occupied before absorption occurs. So, now the second question is how many times a transient state will be occupied before absorption occurs; because obviously, this is, you are talking of a reducible Markov chain. And, so you want to know how many transitions on the average will go continue before absorption occurs and the system stops. So, this is the question mark. And, so for each particular transient state, we want to answer the; answer this question. So then and surely this is the random variable depending on the starting state. How many times transient state will be occupied? Again, it is a random variable and it will depend on what is the starting state.

So, let us define e i j as a mean number to times that transient state j is occupied, given that initial state is i. So, remember when computing your m i j, is which was your mean time for a transition from i to j in the Ergodic case, then we also talked about the same thing while computing your f i j s. And now and then the absorbing states, the probabilities of going from a transient to an absorbing state a i j s. We were using the same argument which we will use here again to compute to right side. Write down your, you know the equations relating these various e i j s, which are mean number of times that the transient state j is occupied; given that the initial state is i. So here, this is of course a transient state. i is a transient state again; so i to j.

So, when i is not equal to j, then of course you will go to i to k and then k to j. So, this will be the mean number of transitions that you need for transition from k to j and this is a probability of transitioning in one step to i to k, when i is not j and this summation will be over all transient k s. So, this is one set of equations. The other would be if i is equal to j, then you are computing e i i. So, it will be one step. You can go from i to I; you transition from in one transition from i to i. So, therefore the number of transitions is 1 plus, you would again transition from i to k, e k is transient and then from k to j; so the mean number of times.

So, the same argument continues. And, so in a matrix form these equations because now we have these relations for all i to j and this is from i to i. So, the same dimension as Q. So, the relationship is that E is equal to I plus Q E. And so here now your E is actually nothing but I minus q inverse. And, this is what I want to say that, you know, you can relate the elements of I minus Q inverse and give them the meaning.

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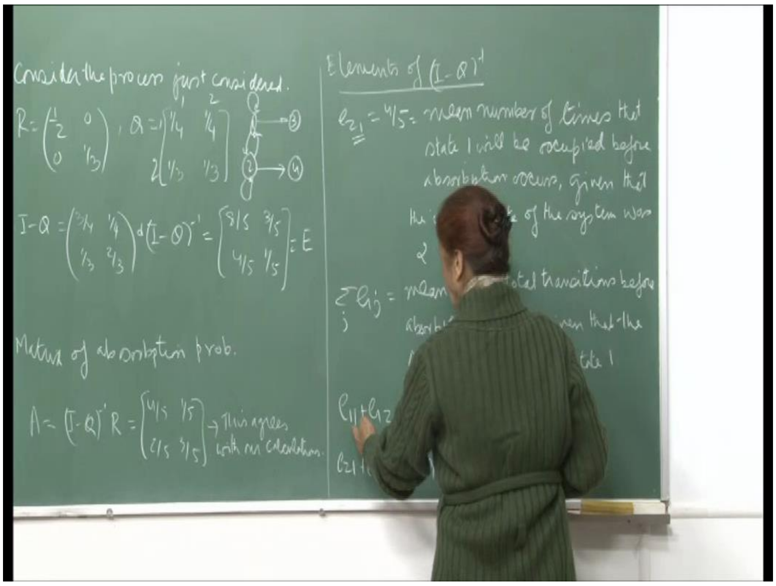


And, let us just look at the entries of I minus Q inverse for the example that we have; we would discussing just with four states; where two were transient and two were absorbing. So, in that case your R was half and 0 and 0, 1 by 3. This was, you know, going from one to one and this was going from three to three, sorry, two to two. One and two were transient and three and four were absorbing. And then Q was the matrix of transitioning from transient state to a absorbing state. So, the diagram; you can just refer back to the diagram. And, so this will be 1 by 4 and 1 by 4; that means from one transient state, you could go to 3; if the probability is 1 by 4 or you could go to; sorry I will just repeat this. Q is basically from transient. So, this is 1 and 2 and this is 1 and 2. I am sorry. So, 3 and 4.

Maybe, I will just again draw the figure. So, this was 3 and this is 4. Yeah. And, so you had this. You had a loop here, then you could come here and you could go there. This was your diagram. Sorry. So, I should you have been careful. Yes. So, our Q is a just the matrix consisting of whether rows and columns both correspond to a transient states.

And so in this case, this loop had probability 1 by 4 and then you could go from 1 to 2 also with probability 1 by 4. And from 2, you could go to itself with 1 by 3 and this was also and then you could go from 2 to 1, probability 1 by 3. So, you can just verify these numbers. Ok.

Now, so I minus Q will be this matrix. And therefore, I minus Q inverse will be 8 by 5, 3 by 5, 4 by 5 and 1 by 5. We can; and then we also computed. We made these computations. So, you have matrix A, which is I minus Q inverse times R is this matrix. And, this agrees with our calculations that we had done. 4 by 5 and 2 by 5; so we said that this is higher than this and then this is higher than this and these computations are all there. So that is there, but now let us look at the elements of I minus Q inverse. So, look at e 2 1 for example; because this is our matrix E. So, e 2 1 is this; 4 by 5.

Now, this is the mean number of times that state 1 will be occupied before absorption occurs, given that the initial state of the system was 2. So starting from 2, you want to find out the probability that you would be going on; finally, going to an absorbing state. And, but in the meantime the mean number of times that state 1 will be visited before absorption occurs. So, that is 4 by 5.

So now, you know it will be interesting. You can just take up any physical process; that, you know, can be modeled as this reducible Markov chain, which has only transient states and absorbing states. And then you can try to give meaning to these numbers. And then if you add up now; for example, e 1 j where, yes, this is mean number of total transitions before absorption occurs, given that the system was initially occupying state 1.

So, now you want to compute these mean number of times, total transitions before because remember you computed 2 1 here. Then I could have also computed e 2 2. e 2 1 or for example, here if your starting state is 1, then you go to 3 or to 4. So, this is… So, we said that e 2 1 is 4 by 5; which is the mean number of times that state 1 will be occupied before absorption occurs, given that the initial state of the system was 2. So starting from 2, then I will occupy 1; on the average, 4 by 5 times before absorption occurs. And similarly, now if you add up e i j s, sorry, e 1 j over j which is again transient. So, this will give you the; that means, starting from state 1, and then both the transient states will be occupied before absorption occurs. So, this will give you; yes, so therefore, for example when this is 1, then in this example it will be e 1 1 plus e 1 2. So, mean number of times.

So, state 1 is occupied or state 2 is occupied, both are transient. So, this total will give you the total transitions before mean number of total transitions before absorption occurs, given that the system was initially occupying state 1. So starting from state 1, how many mean number of transitions can occur before absorption occurs? So, here this was a particular element which is 2 1. That means starting from 2, how many times will mean number of times 1 will be occupied before absorption occurs.

Now, you are saying starting from 1, what is the total number of a mean number of a transactions or transitions which will occur before absorption occurs. So, it will be e 1 1 plus e 1 2; because either 1 can be occupied or 2 can be occupied before transition, before absorption occurs. So, that will be 11 by 5. And, in the case when you are starting from 2, then 2 1 plus e 2 2; that probability is 1, sorry, the mean number. Again, it is not probability. It is a mean number of total transitions. So when starting from 2, it will be 1 and this number is 11 by 5. So, starting from 2; that means, it is faster. The absorption occurs faster; mean number on the average. That is what we are saying. right

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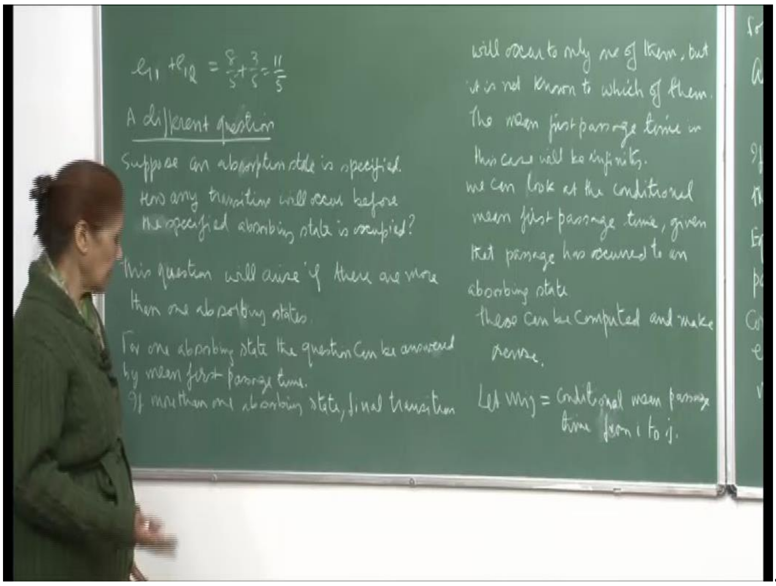

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So see, the question that we have answered was that when we are starting from state 1 which is one of the transient states, how many transitions before absorption occurs? So therefore, as when we were looking at the elements of I minus Q inverse, so this is e 11 plus e 1 2 because either absorption will occur from state 1 or absorption will occur from state 2. So, therefore you add up the two elements I minus Q inverse and this comes out to be 11 by 5. Similarly, you can find out. If you started from state 2, then how many transitions occur on the average before you transition, before you go to absorbing state? So, that sum will be e 2 1 plus e 2 2. And therefore, so that was, you know, I thought I will just emphasize again the interpretations of the elements of the I minus Q inverse. ok.

Now, a different question. This is, suppose you specify an absorbing state and then you want to know how many transitions will occur before the specified absorbing state is occupied. So, this is also because that will give you ideas to how long the process will continue. And because once the absorption occurs before, once you occupy the absorbing state, then your process is over. Or, it will just continue to stay in that same situation; ok, same state. So, and of course, this question will arise if there are more than one absorbing states. Because otherwise if there was only one absorbing state, then you know that ultimately your process will reach that state and your process will stop or will come to an end.

So, if there are more than; if there is only one absorbing state, the question can be answered by a mean first passage time. So that means, you will ask the question that starting in state i in a transient state i, how many transitions on the average before you reach the state j? j maybe in any states. So, if it is there is only one absorbing state, you can find out your first mean first passage time f i j. And, that will be the answer to your question. But, if there are more than one absorbing states, then final transition will occur to only one of them. But, you do not know. It is not known to which one of them. And therefore, the mean first passage time can be infinity in this case because we do not know to which absorbing state you will go.

So, and once you reached one absorbing state, the other one will not be visited ever. And therefore, the first mean passage time will go to infinity. So, therefore here when you have more than one absorbing state, then you need to compute; make some computations to be able to answer this question that how many transitions will occur before this specified absorbing state is occupied.

So, what we will do is, in that case we will compute the conditional mean first passage time; that means, given that you are in a particular state, then we want to know. So, mean first; given that passage has occurred to an absorbing state. So, that is what you are saying; here you are specifying the absorbing state and then you are... So, the mean conditional; the conditional mean first passage time is what we need to compute. So, given that you are going to be occupying the absorbing state, let say j, then you...

So, here you were defining this m i j as the conditional mean passage time, mean first passage time. It should be mean first passage time from i to j. So, let m i j denote this number and this is different from of the m i j; that means number of transitions required for going from i to j, when you are considering the Ergodic process; when all states for recurrence. So, that m i j is different from this one. So, here we are saying that this is the conditional mean first passage time from i to j. So, you are occupying state i and you know that you want to transition to state j, which is an absorbing state.

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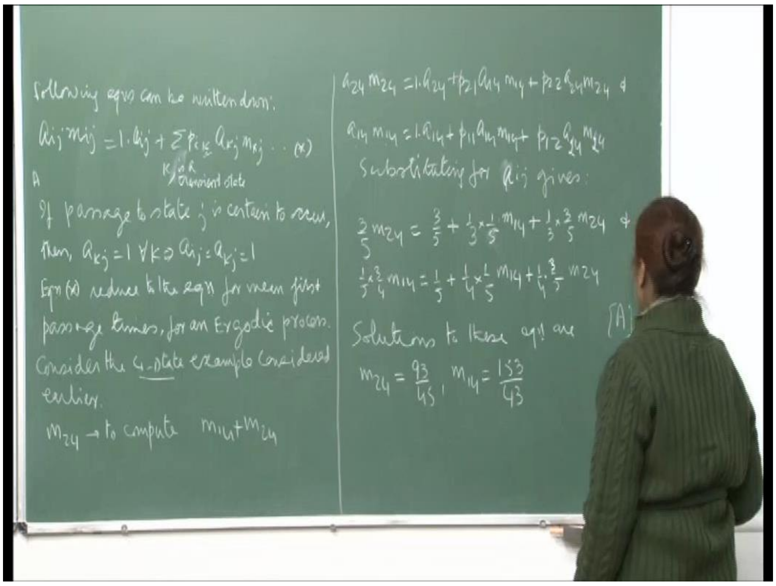


So, now here we will write the following equations. So a i j, remember was your matrix that we had computed to compute the absorbing probabilities. So, then a i j is the probability of transitioning from i to j. i is the transient state, j is absorbing state, then m i j as we have said is conditional mean first passage time from i to j. So, you are occupying state j, then this can occur either in one step. So, 1 into a i j, right, and then here it will be; you may not transition to j right away. So, you will transition to some k and p i k into a k j. So, k will again be a absorbing state. So, you are transitioning to p i k into a k j and k j; right, so therefore, then the transition from k to j, absorbing probability from k to j and then m k j. So, this is, you know, trying to write down the equations for m i j, so that we can have the system of equations and we can solve for these conditional mean first passage times. So, is that ok.

So, here this is i to k; that means, you may transition from i to another absorbing state, but then in that case and then no it has to be, sorry, if k is not; k cannot be in an absorbing state. Because you see, we can immediately see that the argument is not correct because if you are from i to k, then you cannot transition from k to j because once you are in absorbing state, then it is done. So, k is a transient state. So, you are transitioning from i to k. So, i is the transient state, k is a transient state and then you are finding out the absorbing probability of going to j and m k j will be the mean first passage time, right, conditional mean first passage time. That means you want to go to j when you are in k; so m k j into a k j into p i k. So, this you sum up; where k is. So, I should not say that k is not equal to i. So, we are saying that because we want to find the transition probability from i to j, where j is an absorbing state.

And, for the first time; so first passage time, you are computing mean first passage time. So, this will be that you transition from i to another transient state, then from that transient state your absorbing probability is a k j into m k j; so the mean number of transitions that you will require or going from k to j and m k j. So, this gives you an equation for relating the m i j s with the other mean first passage times to the absorbing state j. This is

So, if passage to state j is certain to occur, then of course, a k j will be 1. And, in fact all a i j s; where j is fixed will be 1. So, in that case you see these equations will transform to because this will be 1, this is 1 and this is 1. So, then if you recall your equations for m i j s which you wrote down for the Ergodic process, you will get the same equation; mean first passage time. That means, now you are considering that there is only one absorbing state. And, so the mean first passage time equations will be valid here. Ok.

So the whole idea is, when you have more than one absorbing state, you want to see how to compute these conditional mean first passage times. So, let us say that you; now, in a example that we have been all along following the fourth state example in which 1 and 2 are transient and 3 and 4 are absorbing. So, you consider the four state which is… And, you want to compute m 2 4; that means, you are in 2 and you want to find out the mean first passage time of going to 4; to absorbing state 4.

So here if you look at this equation, let us just write it out. So, it will be a 2 4 m 2 4. So now, 2 cannot be equal to. So, when we will now make use of the conditional mean first passage times. So, I am defining m i j as conditional mean first passage time from i to j, and I am defining the conditional mean first passage time. This may be a general definition. But in particular, now I want to say that suppose j is an absorbing state; I want to talk about that only.

So, in that case I can write this as a i j m i j; right because the transition probability or the absorbing probability from i to j is a i j and that into m i j, the mean; a conditional mean first passage time. Then, either this transition occurs in one step or it will occur…; that means, you will go from i to k where k i. So, I need not write. Here k, essentially I mean that k is a transient state. This is a transient state. That is all I want to say. So, then this will be; you may transition to another state or 2 itself; p i k, then a k j. So, k cannot be j. That is all because j is an absorbing state. So once I transition to j, then it is done. I do not have to. So, it will be p i k; that means, i transition to another transient state. From that transient state, I go to the absorbing state j. So, the probability of that into the mean first passage time from k to j. So, k is a transient state.

Now if passage to state j is certain, then all these absorbing probabilities are 1. Because no matter where you are, since you know that you are going go to j. So, all these probabilities will be 1. And, in that case these equations will reduce to your; you are computing the mean first passage times for an Ergodic process, which we have done already. So, same equation; I mean, this will reduce to the equation for computing the mean first passage time for an Ergodic process.

So, now let us see how we make use of these equations to compute your m i j s. So, let us consist; say for example, I want to compute m 2 4. So, that means the question here that we asked how many transitions will occur before the specified absorbing state is occupied. So, of course I will compute m 2 4, then m 1 4 also, and the sum will tell me so that means your answer to that question. It will be m 1 4 plus m 2 4. That will tell me the mean number of transitions that are required before I occupy state four. This is what we want to compute.

So, let us just write down the equations here. So, a 2 4 into m 2 4. This can be equal to a 2 4 which is actually 1 times a 2 4 plus, then from 2 I can transition to 1 which is the transient state, then a 1 4 into m 1 4 or I can transition from 2 to2. So p 2 2, then a 2 4 and m 2 4. So, this is clear. Similarly if I want to look at m 1 4, then it will be a 1 4 into m 1 4, then 1 into a 1 4. So, I will again transition from 1 to 4 and this will be a ... So, this will be in one step. And then I mean, the mean first passage time will be 1 plus p 1 1 p 1 4 m 1 4 plus p 1 2 a 2 4 and m 2 4. So here again, from 1 you can transition to itself or you can transition to 2. So, if you transition from 1, then again you want to compute a 1 4 and then this will be m 1 4 plus p 1 2 into a 2 4 m 2 4. So, this is the conditional mean first passage times that we are computing. That if I am in 1, and then I can transition to 4. So, what is the conditional probability?

So, now substituting for p i j s. I should not say p i j s. Substituting for a i j s. And, so remember we had, for this thing, we had computed the matrix A. And, if you look back at the matrix A, it is, the numbers are given to you. So, a 1 4; for example, there are a 2 4 is 3 by 5, then this was 3 by 5 plus 1 by 3, your this was 1; this was 1, a 1 4 was 1 by 5, then 1 by 3 was p 2 1. Anyway, you had the matrix this and you had the matrix; the transition matrix p also the transition diagram. So, looking at those if you can just revert back to a few frames earlier, then you have all these numbers. And so by substituting there, I get these two equations and then solving them simply, easily gives me the answer that m 2 4 is 93 by 45 and m 1 4 is 153 by, it should be 45 only, by 43.

So, as I were saying that now if you want to look at the m 1 4 plus m 2 4, then this is equal to 93 plus 153 by 45 and this will be 246 by 45. So, on the average the mean number of conditional; that means if transitioning to four, so the number of transitions that will be required; mean number would be 246 by 45. Provided, you are transitioning to the absorbing state four. So one can, you know, I have tried to look at these processes in a different ways and giving you interpretations.

And so you know given a physical process, there are lot of these kinds of questions because you want to know if one has to plan that one has to know, for example, these are reducible Markov chain; that means you know which the processes which will terminate in short time, then you want to have an idea about these numbers, so that you look at, you can plan accordingly. And, let us hope that in future after having gone through this, you will be coming across such situations or such processes, where then you can look at them in a more meaningful way.

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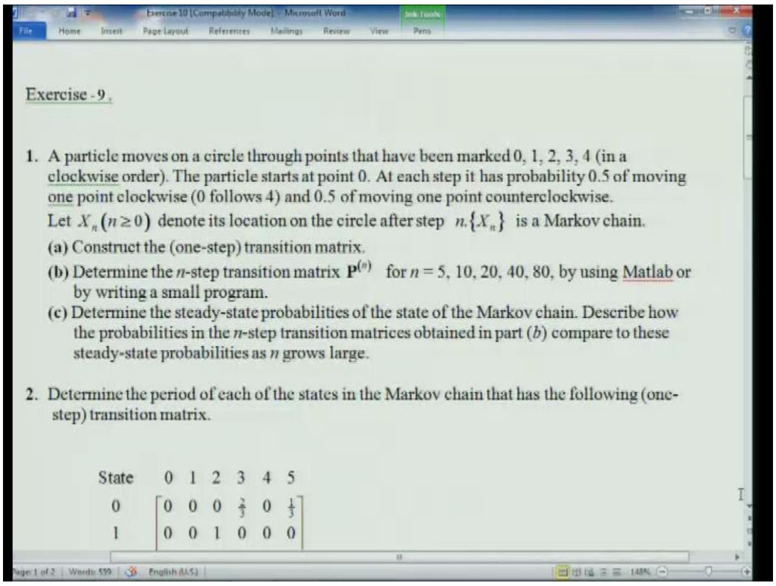


So, I will now discuss exercise. So, let us look at question one. This is, a particle moves. By the way, these questions I have taken from the book Hillier and Lieberman, reference to which will be given to you at the end of the course. So, a particle moves on a circle through points that have been marked 0, 1, 2, 3, 4 in a clockwise order. So, and the particle start at point zero; so let us see. I can just… So, here is a circle and you have 0, 1, 2, 3 and 4; so this. And, the idea is that you can either move from here to here or you can move backwards.

So at each step, the particle starts at point 0. At each step, it has probability 0.5 of moving the point clockwise. So, the clockwise would mean this way or 0.5 of moving one point counter clockwise; means either backwards or forwards. And, both the probabilities are the same. So you remember, this was your, we were looking at the random walk. You were looking at the random walk. And, when we said that probability is half, then we also showed that it will be an Ergodic process; because in that case, all these states will be recurrent. So, the same situation.

Now, let X n denote its location on the circle after step n X n is a Markov chain. So, we have already seen that this will be a Markov chain because it will just depend on where you are, so that you know the probability of transitioning to a next step will just depend on where you are. It will not depend on how you reach there. So, this will be the Markov chain.

Now, construct the one step transition matrix. You can do it. So, it will be a five by five matrix. Then, determine the n-step transition matrix p n for n equal to 5, 10, 20, 40 and 80. No, I have given up to 80, but that is because either you must be either familiar with Matlab or by writing a small program of your own. Then, you can iteratively find out to the multiply to get the power of p raise to 5 p raise to 10 p raise to 20 and so on. So, it is just to familiarize yourself.

Then, determine the steady-state probabilities of state of the Markov chain. Now you, so this is a little; may become tedious because you are solving your five by five; your transition matrix is five by five. So, you will have five variables and five equations. But since the values of p s are half, so therefore it should not be a difficult system to solve. You should get the answer too quickly.

So, determine the steady-state probabilities of the state of the Markov chain. Describe how the probabilities in the n-step transition matrices obtained in part (b), compare to these steady-state probabilities as n grows large. So, want to show. You see that, you should feel the… So, once you find out the steady-state probabilities and you also have; you know p raise to 40 or 80, then you can see that. In fact, at p n equal to 80; that means, when you have p raise to 80, then they should definitely be very close to your steady-state probabilities. And in fact, you can see the pattern even when you compute the 40.

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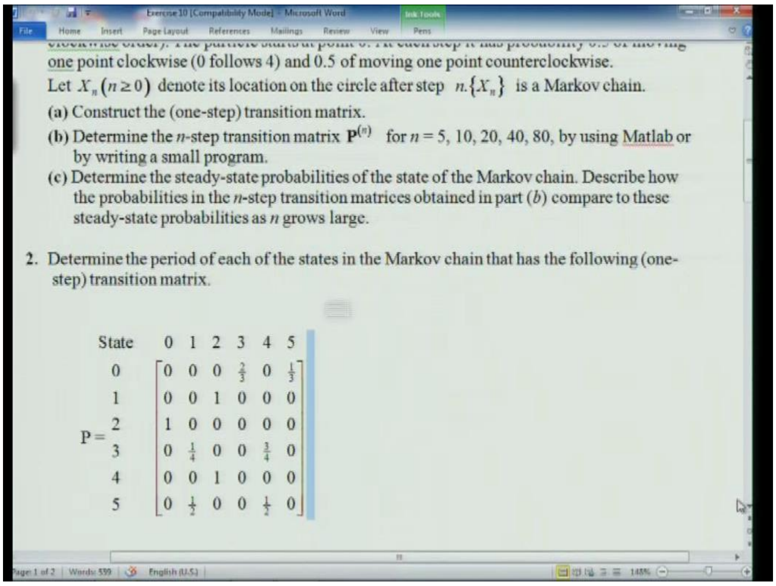


Now, question 2 is just to determine the period of each of the states in the Markov chain that has the… yes. So, here you have again a five by five matrix and I am asking you to determine the period of each of the states in the Markov chain that has the following transition matrix. So, this will be; all the states are periodic. Well, you just find out. So, you have to then determine the period. That means you will have to compute P square, P cube. And, here also you can make use of the same program that you wrote for P and then you can compute P square, P cube and so on to determine the periods of the periodic states. Yes.

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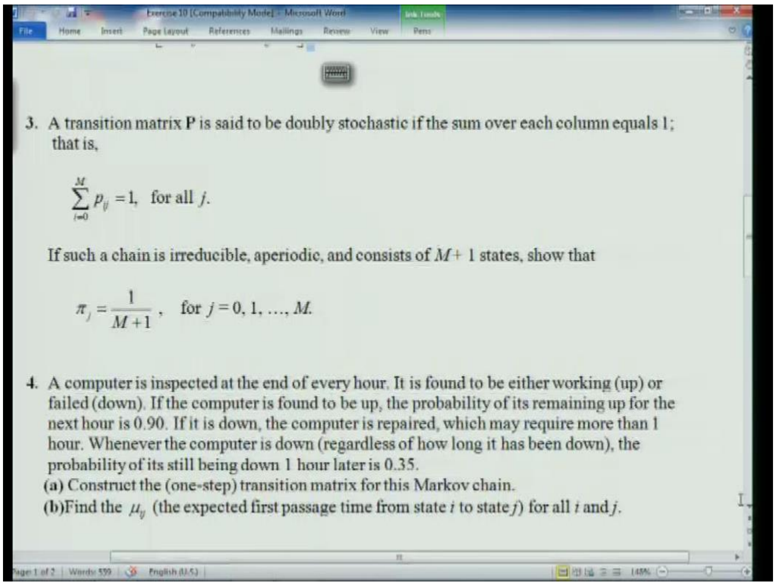

Now question three; a transition matrix P is said to be doubly stochastic if the sum over each column equals 1. So, you know for the transition matrix that we have seen, the row must add up to 1. So, now I am giving you an additional condition. And that is, that the columns also add up to 1. So, therefore P i j as i varies from 0 to M is also 1. So, column sums are 1. In that case, such a chain is irreducible, aperiodic and consists of M plus 1 states. States are already M plus 1; show that your steady-state probabilities. In such a case, you do not have to do any computations. Just right away you will be able to show that your Pi j s are 1 upon M plus 1; where j varying from 0 to M. So, this is simply, you can do it or write down if you think and then make out your… So, this is now, oh, this is ok.

Now question four; a computer is inspected at the end of every hour. It is found to be either working. It means up or failed down. If the computer is found to be up, the probability of its remaining up for the next hour is 0.9; if it is down, the computer is repaired, which may require more than one hour. Whenever the computer is down, regardless of how long it has been down, the probability of its still being down one hour later is 0.35. So, that means your unit of time is one hour and then you have to write down your transition matrix. So, construct the one step transition matrix for this Markov chain. And find the mu i j, the expected first passage time from state i to state j; for all i and j. So this, you will be able to do. So, here I have asked you to; now this sort of question depends on computing the first passage times, which we have also discussed quite thoroughly.

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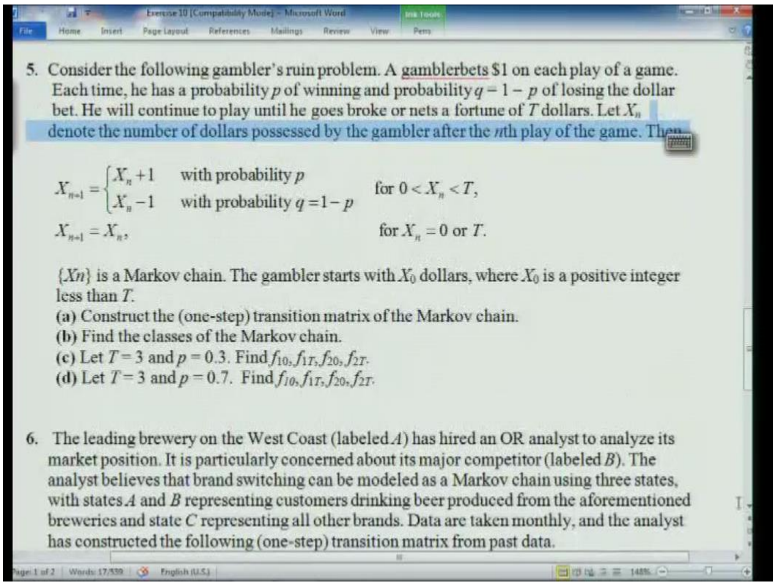

5. Consider the following gambler's ruin problem. A gamblerbets \$1 on each play of a game. Each time, he has a probability $p$ of winning and probability $q = 1 - p$ of losing the dollar bet. He will continue to play until he goes broke or nets a fortune of $T$ dollars. Let $X_n$ denote the number of dollars possessed by the gambler after the $n$th play of the game. Then

$$X_{n+1} = \begin{cases} X_n + 1 & \text{with probability } p \\ X_n - 1 & \text{with probability } q = 1 - p \end{cases} \quad \text{for } 0 < X_n < T,$$

$$X_{n+1} = X_n, \quad \text{for } X_n = 0 \text{ or } T.$$

$\{X_n\}$ is a Markov chain. The gambler starts with $X_0$ dollars, where $X_0$ is a positive integer less than $T$.

(a) Construct the (one-step) transition matrix of the Markov chain.
(b) Find the classes of the Markov chain.
(c) Let $T = 3$ and $p = 0.3$. Find $f_{10}, f_{1T}, f_{20}, f_{2T}$.
(d) Let $T = 3$ and $p = 0.7$. Find $f_{10}, f_{1T}, f_{20}, f_{2T}$.

6. The leading brewery on the West Coast (labeled $A$) has hired an OR analyst to analyze its market position. It is particularly concerned about its major competitor (labeled $B$). The analyst believes that brand switching can be modeled as a Markov chain using three states, with states $A$ and $B$ representing customers drinking beer produced from the aforementioned breweries and state $C$ representing all other brands. Data are taken monthly, and the analyst has constructed the following (one-step) transition matrix from past data.

Now, question five; which I told you in the lecture. This is, you know based on the gambler's ruin problem. And, so I thought that I leave the computations to you. I just explain to you how, what the problem is.

Now, here gambler bets; there should have been a space between. So, now this is a gambler bets dollar here. It is dollar 1 on each play of a game because this is the game from Hillier and Lieberman. Each time he has a probability p of winning and probability q which is 1 minus p of losing the dollar bet. He will continue to play until he goes broke or nets a fortune of T dollars. Now, let X n denote the number of dollars possessed by the gambler after the nth play of the game. Then, you want to find out; X n plus 1 will be X n plus 1 with probability p; right, because he has 1 more rupee if he wins and that is with probability P, otherwise he will have X n minus 1 with probability q which is 1 minus p.

Now, here of course he continues playing, only if x n is less than T. And, X n plus 1 will be X n for X n 0 or T. right because if he has no money, then he cannot play. And therefore, he cannot bet. And, so he continues to be at the same state. That means, he continues to be broke. And if he has T dollars, he has earned T dollars; then again he does not play because he has earned his fortune.

Now X n is a Markov chain. We have already discussed this. The gambler starts with X naught dollars; where X naught is a positive integer less than T because I said; we can say that X naught is i.

Construct the one-step transition matrix of the Markov chain. So this, you will have to write down. And, you could see that it will only be one step forward or one step backward. The other entries will be zeroes. Now, find the classes of the Markov chain. This, I have already told you. Let T equal to 3 and p equal to 0.3. So, if it is a question of earning up to 3 dollars, so that means, your states will be 0, 1, 2 and 3. So, then I have asked you to find out the first passage probabilities f 1 0, f 1 T, f 2 0 and f 2 T; and then again the same things, when p is 0.7. So, it will be not be difficult at all if you write down. See, again the computations, the formulae are given to you for the first passage probabilities. And, so once you write down the transition matrix, you should be able to complete the problem.

So, in the lecture I had told you that I will be asking to you to compute the probability that the gambler will end up with a rupees T. or, in the lecture, it was I think rupees n or dollars n, whatever it is, that does not matter. That means, the gambler with what is the probability that he will earn the fortune that he is wanting to. So, that has not been asked in this question five. But you see, you should be able to set up equations. So, essentially it is just to define the probability p i p i minus one.

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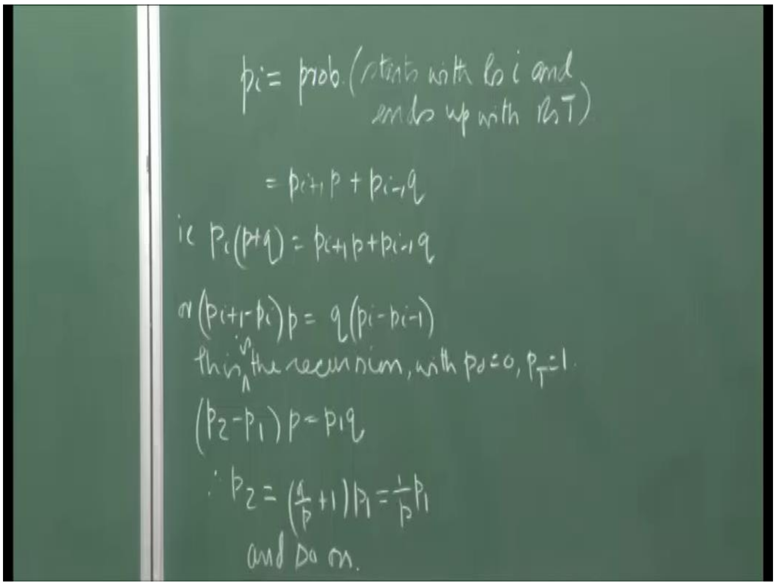


You can interpret the same way. And then it will be p i is equal to p p i plus 1 plus q p i minus 1. And then you multiply p i by p plus q; because p plus q is 1. And then from this equation you have an iterative relationship for different values of i. And then you can find out the iterative relation and then you will be able to find out the probability p i. And there, of course you will use the fact that if you have 0 dollars, then the probability of making your fortune is 0. And, if you have earned T dollars, then your p T is 1. So, using this initial at boundary conditions, you will be able to solve for p i. So, please do that; because in the problem I have not asked you to do it. But, you can certainly do it.

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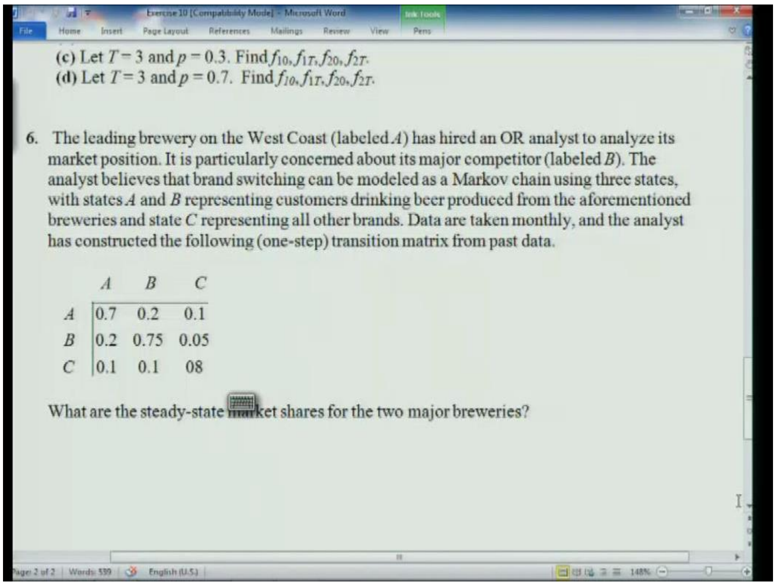

(c) Let $T = 3$ and $p = 0.3$. Find $f_{10}, f_{1T}, f_{20}, f_{2T}$.

(d) Let $T = 3$ and $p = 0.7$. Find $f_{10}, f_{1T}, f_{20}, f_{2T}$.

6. The leading brewery on the West Coast (labeled *A*) has hired an OR analyst to analyze its market position. It is particularly concerned about its major competitor (labeled *B*). The analyst believes that brand switching can be modeled as a Markov chain using three states, with states *A* and *B* representing customers drinking beer produced from the aforementioned breweries and state *C* representing all other brands. Data are taken monthly, and the analyst has constructed the following (one-step) transition matrix from past data.

| | *A* | *B* | *C* |
|---|---|---|---|
| *A* | 0.7 | 0.2 | 0.1 |
| *B* | 0.2 | 0.75 | 0.05 |
| *C* | 0.1 | 0.1 | 08 |

What are the steady-state [illegible]ket shares for the two major breweries?

Now, question six. Question six is which is again a simple one. A leading brewery on the West coast, labeled A, has hired an OR analyst to analyze its market position. If it is particularly concerned about its major competitor, let us say labeled B. So, another brewery which is their competitor for this brewery A and. So, they want to find out how could a competitor or how bad a competitor, this other brewery is. So, analyst believes that brand switching can be modeled as a Markov chain using three states with state A and B representing customers drinking beer produced from the aforementioned breweries and state C representing all other brands. So, you know state A will represent; so that means A to A in this transition matrix that is given below. So, see this probability of A; that means, somebody who is taking, using the, you know, beer from brewery A will continue to do that, use the brewery A only is 0.7, but may switch to B with probability 0.2 or to other brands with probability 0.1.

Similarly, you can explain the row B entries, right, and then C. for other brands, when they switch to A will be 0.1 and 0.2, 0.1 again to B and 0.8; that means, they continue with the same brand that they are already using is 0.8; so that point is missing here. Anyway, just you can make the entry; so 0.8.

Now, what are the steady-state market shares for the two major breweries? So, we want you to find out pi 1, pi 2 and pi 3. So, the answers that they are asking for is pi 1 and pi 2. So, what are the steady-state market shares for the two major; that means, when the process has gone on for some times, you think that the choices have all stabilized. Then, you want to know the steady-state market shares for the two major breweries.