Introduction to Probability Theory and its Applications

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Lecture - 3
Conditional Probability Independence of Events

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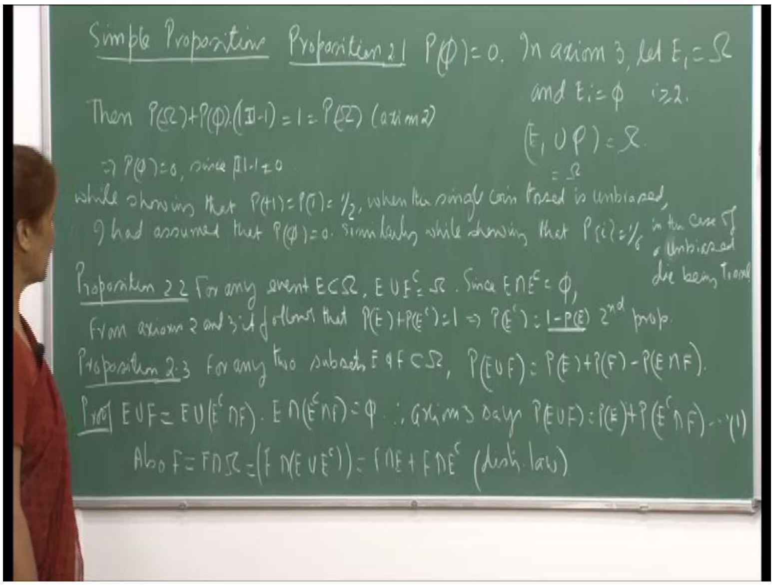

So, after having defined the axioms of probability I am trying to, I try to show you how probability function can be obtained. Let me now derive a few propositions using these axioms. So, the topic that I am going to talk about is simple propositions. So, proving this that proposition that P phi is 0, so the axiom 3 we have taken we take E 1 to be omega and then E i s are all, all remaining E i s are empty.

And here we will take the index set be a finite index set. So, it does not matter; I have to prove it for, I have to just show that P phi is 0. So, whether I take; so here it is convenient to take i to be a finite index set. So, I take i to be a finite index set. Then all other E i s are empty sets. So, therefore, the condition for axiom 3 is satisfied, because certainly E 1 is disjoint with an empty set, and all other sets are also because they all are empty. So, they are disjoint.

So, therefore, the I write down applying the axiom 3; I say that P omega plus, p phi, and phi gets added i minus 1 times, because 1 has gone here; so 1 less than the index of i, to the cardinality of i. So, therefore, this is P phi into, cardinality of i minus 1; this has to be equal to 1 because this is P omega, right. Since you add up all the E i s then this all add up to omega; remaining sets are all empty sets.

So, this is by axiom 2 this is 1, right; and so and therefore, since i is a finite number which is greater than or equal to 2 because we are saying that at least 1 set which is equal to omega, and the other there is at least 1 set which is the empty set. So, therefore, the number of, the cardinality of i has 2 or more than 2; and so this number is not 0.

Therefore, here, you see, because P omega, P omega cancels out; then you get that, and also P omega is 1. So, anyway, this cancels out; and we are left with this; this left is equal to 0 which means that either this is 0 or this is 0, but then this number is not 0 since cardinality of i is 2 or more than 2, so P phi must be 0; so this is how we obtain the preposition that the P phi will always be 0.

Now, proposition 2 is this actually; that probability of the, complement of the event E is 1 minus probability of the event E; and this can be very simply derived. And see here the attempt is to show you that once you define the axioms and then using the axioms logically, we can derive at so many results and build up a good structure. So, that then you can estimate probabilities of more complex events and so on.

So, now for any event E, E union E compliment is omega, right; either the element of omega is in E or is in E compliment, so this holds. Then, since E and E complement are disjoint sets, they are mutually, this thing, exclusive. From axiom 2 and 3 it follows that P (E) plus, P (E c) is 1. So, from axiom 3 it say that probability of E into, probability of E union E complement is P (E) plus, probability of E compliment. And then axiom 2 says that probability of omega is 1; so this is what holds.

And therefore, from here it immediately follows that probability of E compliment is 1 minus P (E). So, that is your second proposition. So, therefore, now using the basic 3 axioms I have been able to arrive at the result that P phi is 0. And then now I have shown you that P of the, probability of the compliment of an event is 1 minus the probability of that event.

Proposition 2.3 says that now, we can see that the results are getting more and more complex. So, if the 2 events, E and F, subsets of omega, then probability E union F is equal to, probability E plus, probability F minus, probability E intersection F. And you see that, in case, E intersection F is empty then this will be 0, because we have just derived the result that P phi is 0, so in that case P E n F will be this; and then this is again is valid for axiom 3 because in case E intersection F is empty then E and F are disjoint, and then we said that probability of the union is the sum of the probability, right. So, this is in complement with you axioms, and so on.

Now, let us start proving this result. So, I first say that E union F is E union E compliment intersection F, right; because this set E union F, so either whatever the elements of F are E that cover, that is covered here, and the remaining elements of F are not in E, but they are in E compliment. And therefore, you can write E union F as E union E complement intersection F.

Now, by doing this, you see, I have broken up this union into, the union of 2 disjoin sets because E intersection that E complement of F is empty; so there cannot be any element which are in E compliment. So, therefore, this is empty. So, now axiom 3 again says because these 2 sets are disjoint; therefore, probability of this union, so probability of E union F is equal to probability E plus, probability E complement intersection F because I could decompose this union into the union of 2 disjoint sets. And so I can immediately apply axiom 3, and I get this, right.

Now, we will try to write, again decompose up in a way so I can make use of the axiom 3. So, now, F is F intersection omega, but then omega as we said we can write as E union E complement; I just used it here. So, then F intersection of E union E complement which I can write as F intersection E union, F intersection E complement; this is by the distributive law which I defined sometime ago for you; so therefore, in the earlier lecture. So, therefore, this is equal to this; and here again you see that these 2 sets are, ok; it should be union; be careful because we have no meaning of putting a plus sign here; these are subsets, fine. So, this is equal to this; now, here again both the sets are disjoint.

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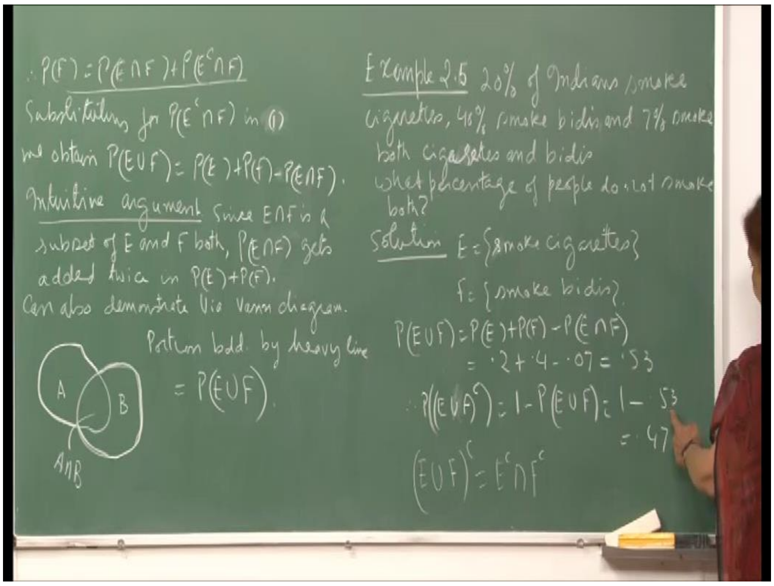

So, therefore, when I write probability F, I can simply write it as probability E intersection F, this probability E complement intersection F, by using our axiom 3. Now, from here, from this relationship I can now compute probability E compliment intersection F as equal to probability F minus, probability E intersection F, right. And this, I will then substitute in 1. So, I will get P (E) plus, P (F) minus, probability E intersection F.

So, substituting this in 1, we obtain the desired result, that probability for any 2 sets probability of the union of those 2 sets is sum of the probabilities of the 2 sets minus, the probability of the intersection of the 2 sets. And it is very intuitive argument for this, for the validity of this result. Now, since you see P intersection F is a subset of E and F, both, right; E intersection F is at a common element, so they occur in E as well as in F.

And therefore, when you are adding up the probabilities P (E) plus P (F), probability of E intersection F, gets added twice. So, I have to subtract it once to make the equation balance. And therefore, this is the result, right. And even through Venn diagram you could explain that when you; see this is A, this is B. So, probability A is this area, probability B is this whole area; when you put them together, you have added this area twice because this is your A intersection B. You have added this area twice, and therefore, you need to subtract it once, so that the 2 equations are balanced, ok.

So, you may get corollary of this result P (E) union F is P (E) plus, P (F) minus, P (E) intersection F, is that if A is a subset of E then the probability of A is less than or equal to probability of E. Now, we can write E as, because A is a subset of E, therefore I can write E as, this subset E I can write as A union A complement intersection E, right. Because if this is E and this is subset of A, so then this is A and this portion is your A complement intersection E, right.

So, therefore, and then these 2 will be disjoint sets. So, therefore, P will be P (A) plus, P (A) complement intersection E; again we are applying axiom 3 because I have decomposed the event E into sum of 2 disjoint sets, 2 disjoint events. And therefore, the probability of P (E) would be P (A) plus P (A) complement intersection E. And since P (A) complement intersection of A is greater than or equal to 0, therefore, this implies that from here that your P (E) is greater than or equal to P (A).

So, we get consequence, and I will just put it down so that for complete in a set, otherwise one can immediately; once you have proved this result, you can conclude this result immediately. So, once you have these 3 propositions now you can see, you can get some more results; you can compute probabilities of more interesting and complex events.

So, consider this example. Example, this says that 20 percent of Indians smoke cigarettes, 40 percent smoke bidis, and 7 percent smoke both cigarettes and bidis. So, what percentage of people do not smoke both? So, you are asking what percentage of people do not smoke cigarette as well as bidi. So, now here if I write the event E as set of people who smoke cigarettes, and F is my set of people who smoke bidis, right.

Then, I first compute E union F. So, probability of E union F by adding the proposition I just proved is P (E) plus P (F) minus, P E intersection F, right, which is 2.2 plus 4; this is 20 percent, 40 percent, and then minus 0.07; so therefore this comes out to be 0.53. And I am looking for the, what percentage of people do not smoke both. So, if this is E union F, right; so E union F, remember, again I was talking of de morgan laws, this compliment was E compliment intersection F complement, right.

So, this is people who do not smoke cigarettes, people who do not smoke bidis, so the intersection will give me the sort of people who do not smoke either cigarette or bidi, right. So, therefore, I am computing E union F complement. Now, here I have used proposition 2, I think 2.2. We are also; I have computed probability of E union F, so now, I want to compute probability of E union F complement which is 1 minus probability E union F; and so therefore, this is 1 minus 0.53 which is 0.47.

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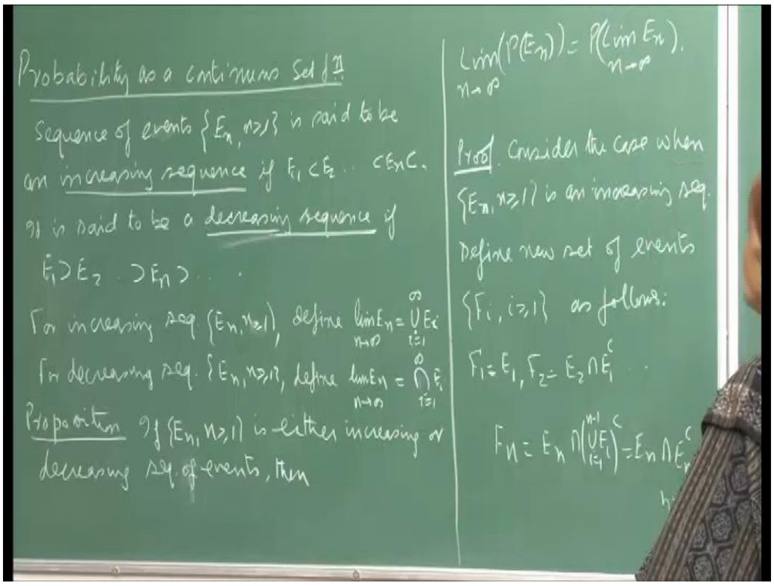

See, we have already, you have come across continuity of a function on the real line, now we want to introduce this concept of probability function as a continuous set function. And this is very useful because often when we have to take limits of sequences, of your sequences of events, and then we need to have this notion of probability as a continuous function on the sets, so that I can inter change the limit on the, in the process of taking the limit and probability, in taking the probability. So, these are what we are going to formulize.

See, sequence of events E n is greater than or equal to 1 is said to be an increasing sequence if E 1 is contained in E 2 contained in E n, and so on, right. So, this is the notion of increasing sequence of events; and then similarly a decreasing sequence would be when E 1 contains E 2 contains E n, and so on. So, for increasing sequence E n, we define the limit of E n, n going to infinity as union i varying from 1 to infinity E i, right.

Because, anyway, it is a increasing sequence of this in fact is E n; and therefore, there is nothing new in the definition, because these subsets are all, events are part of E n. So, therefore, union actually is, because you want to take the concept of inter changing the probability in the limit, so this is why we are doing it. For decreasing sequence E n, we defined the limit of E n as if intersection i varying from 1 to infinity of E i.

Now, the proposition is that if E n is either increasing or decreasing sequence of events then limit P E n; so here this will be mean result that limit probability E n, as n goes to infinity is probability limit of E n as n goes to infinity. So, what we are saying is that you can interchange the operation of taking the limit and taking the probability because of continuity of P on the sets.

So, let us just quickly look at the proof. Consider the case when E n is in increasing sequence; and then we can also go with the proof when E n in decreasing sequence. So, define new set of events F i as follows: F 1 is E 1, then F 2 is E 2 intersection E 1 complement. So, therefore, you see whatever common components of E 1 are there, and E 2 they are there anymore; so F1 and F 2; F 1 is E 1; F 2 is E 2 intersection E 1 complement.

So, therefore, you can immediately say that F 1 and F 2 are disjoint, right. Then finally, this way you want to find F 1 as E n intersection union, i varying from 1 to n minus E i n complement. So, all the previous sets, E i s, you take the union and take its complement, then you take the intersection in the end so that is E F n. So, therefore, here you can see the systematically the way we have defined F 1, F 2, F n, and so on, that these are all disjoint subsets or disjoint events, right.

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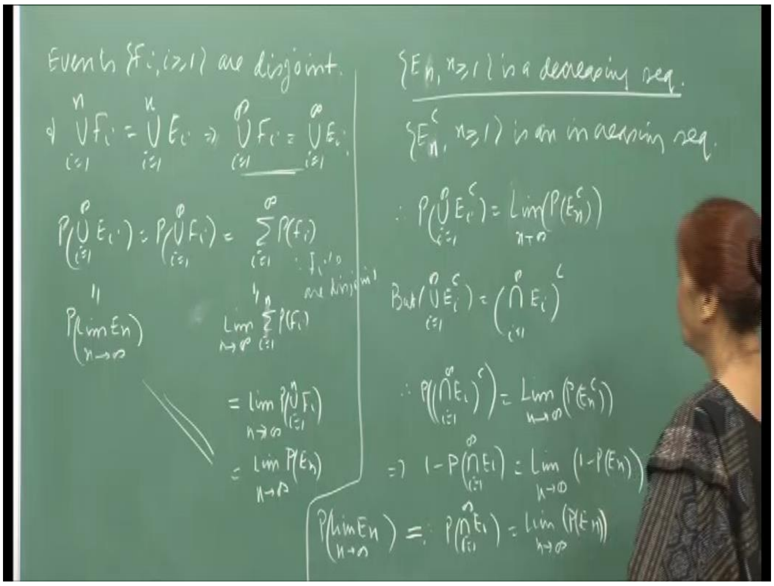

And so that is what we are saying that F i, i is greater than or equal to 1 are disjoint. Now, union i varying from 1 to n F i, is union E i, right, because I am not taking away anything; it is just the common parts I am removing, and so making these disjoint. But, otherwise all the elements of, all the components of E 1, E 2, E n are all there. So, union F i, i varying from 1 to n is union i varying from 1 to n E i. And therefore, you know, the limiting case also, this will be true, right.

Now, probability union E i, i varying from 1 to infinity is probability this, because this wholes; so therefore this is also probability of E i union i varying from 1 to infinity, F i. And since, F i s are disjoint I can write this as summation i varying from 1 to infinity P F i, since, F i s are disjoint, right; I can write this. And now, this side you see, this is nothing but limit E n by our definition, right; we defined that union E i, i varying from 1 to infinity is limit E n and goes to infinity. So, this is this.

And from this side you see this is sigma P F i, i varying from 1 to infinity which can be written as limit n going to infinity of summation i varying from 1 to n P F i which; again here this you will write as union i varying from 1 to n F i, and then this will be limit; so this is E n, right. This part is also union F i, yes; you can see it immediately that union F i is E n because E n is an increasing sequence; and therefore, this is this. So, now this is what you wanted to show, right. Probability limit E n, n goes to infinity is equal to limit probability E n is n goes to infinity, for an increasing sequence.

Now, when the sequence is decreasing then you see E n complements will be increasing, right; because E n s are decreasing, so E n complements would be increasing; and so you apply what we have just proved to this sequence of events E and c; and so probability union i varying from 1 to infinity E i complement is limit probability E n complement, and n going to infinity, right.

But, this union is nothing but intersection of E i, i varying from 1 to infinity complement that is what your de morgan's law is that we have done in lecture 2 or lecture 1, I think the previous lecture. So, then this is this. Therefore, intersection E i this thing, from here, this is, because this is equal to this number because we have taken this as increasing sequence, and this is equal to this; so it follows from here, and this is equal to this, right, this thing.

And this you can write as 1 minus probability intersection E i, i varying from 1 to infinity because this is complement, probability of the complement event; so 1 minus probability intersection i varying from 1 to infinity E i. And this is same thing; you are writing probability of E n complement; so this will be 1 minus P E n. So, this comes to this side. So, here you have this. And this cancels out 1, 1.

And therefore, this now is by your definition because it is a decreasing sequence, so by definition i varying from 1 to infinity E i is limit E n as n goes to infinity. So, therefore, you have proved the result for both. And now there we will be many occasions where we will be using, I mean referring to because P is a continuous function, continuous set function; therefore, I would be very often exchanging the process of taking the limit and the probability.

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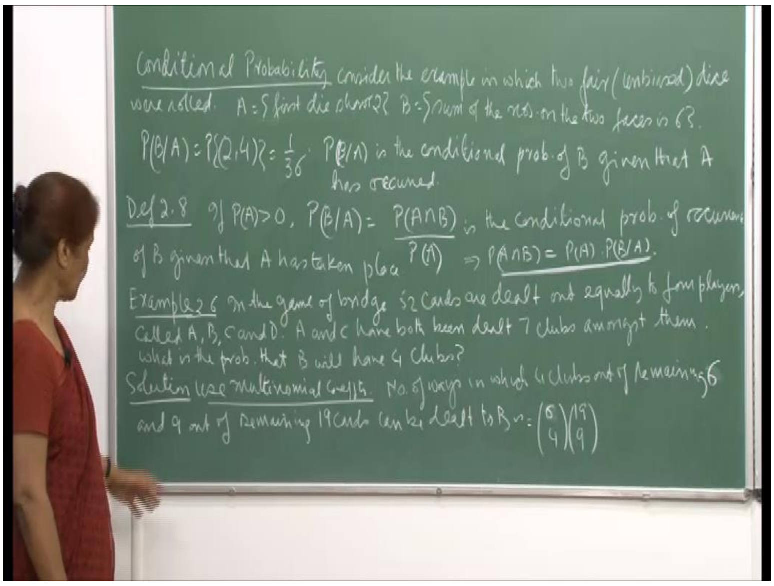

So, now, I will define another new concept which is the conditional probability. And we will just see that how starting from the 3 axioms I am able to develop more and more theory about the probability. So, conditional probability again I will take an example first. So, suppose you have 2 fair dice, the dice we can also call phial; all these words are synonyms.

So, when you have the example we had considered of rolling or tossing 2 dice; then if my event A is that the first dice shows 2, and B is the event that sum of the numbers on the 2 phases is, they add up to 6, right. So, the first dice is already showing a number 2. Now, in the conditional probability, that means knowing that the first dice has shown 2, then if I want to sum up the 2 numbers on the 2 phases to be 6; and now you want to find the conditional probability of the given A.

So, therefore, see, what will be the B intersection A; that means, the first dice showing 2, and then you want the sum of the 2 numbers is 6, that means the other dice must show the number 4; so this gives you the intersection of A and B, 2, 4, right. And here, given that your first dice; so the probability that the first dice has shown 2, then the second dice can show any of the 6 numbers.

And therefore, the probability of A, because this probability is probability of P intersection A divided by probability A. So, probability A will be the first dice showing the number 2, and the second dice is showing any of the 6 numbers; so therefore this add up to here; so the probability of 2, 4; just one of the 36 equally likely elements because we are assuming that the dice is unbiased, both the dice are fair.

So, the probability of this pair occurring is 1 by 36; and each of these is again equally and each of them has probability 1 by 36; that is the 36 upon 36 which will be 1 by 6; and therefore the outcome is 1 by 6. So, this is how you compute the conditional probability. And so we say that probability of B, given A is the conditional probability of B given that A has occurred.

So, once you know that this event has taken place, now you are wanting to find out the probability that B will occur. So, therefore, this will be conditional probability of B, given A; that means, you want to compute the probability of B, once you know that certain event A has already occurred.

Now, so the formal definition would be that if P (A) is positive, that means A is an event which actually may occur. Then, if probability P (A) is greater than 0, we say that probability B conditional A is equal to probability of A intersection B divided by, probability A. So, this is the definition; and this is the valid one because we have already assumed that P (A) is positive.

So, here we can immediately say that this implies, that probability, you can also write, the way of writing probability A intersection B is, probability A into probability B conditional A; so this is a very impotent formula which we keep using. So, let me illustrate this concept through this example. Now, in the game of bridge 52 cards are dealt are equal to the 4 players; that means, each player gets 13 cards.

So, the 4 players are A, B, C, and D. A and C have both been dealt 7 clubs amongst them. What is the probability that B will have 4 clubs? This is the question. Now, the event that has already occurred is that A and C both together have 7 clubs amongst them. Now, we want to find out the probability that B will get 4 clubs. So, here again I am using multinomial coefficients, and I am saying that the number of ways in which 4 clubs out of; see, because they have been, there are 7 clubs amongst them; number of ways in which 4 clubs out of remaining 6; this should be 6, sorry; this should be 6 because 7 clubs have been dealt to A and C, so you are left with 6 more clubs; and therefore, this will also be number 6.

So, now, you want 4 of the, out of remaining 6 clubs you want 4 of them to go to player C, player B, right. And so if he gets 4 clubs, then that means, since he is getting 13 cards, so after the remaining 19, so this will be then 20. So, in the remaining 20, A should get 9, right. So, I have to collect the thing here also. This is 6 and this is 20. So, I am make the, I have computed this probability by using the multinomial coefficients that, you know, the 2 sets of cards; and so the choices are from 6, 4 out of 6, and 9 out of 20.

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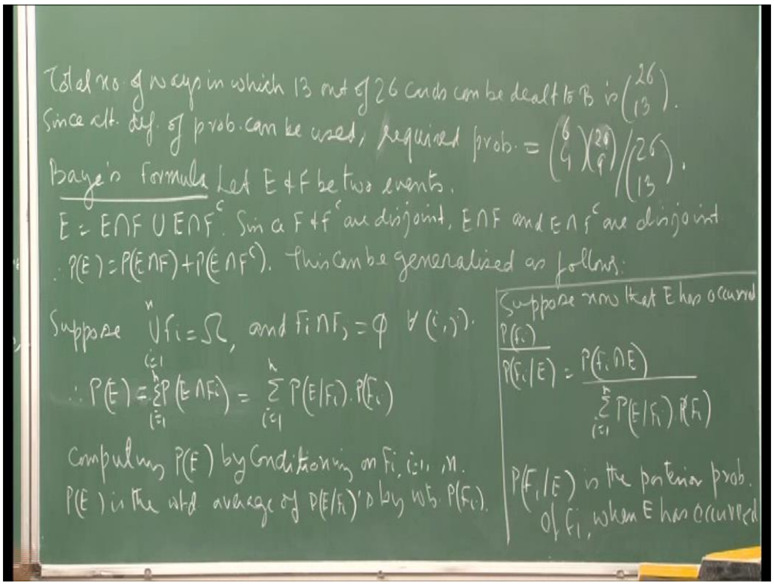

And the total ways in which it can be dealt 13 cards out of the 26 cards because 26 have already been given to A and C. So, out of the remaining 26 it is given. So, total number of ways in which that can happen is 26 choose 13; and this is this. Now, you have to explain this in terms of; so I am saying that the conditional probability of B getting 4 clubs, when A and C are already been given 7 clubs, is given by that number.

And then I would like you to know because; so just bringing it out how this will fit into this definition. And I will come back and then discus with you again a few and figure it out how this fix into that definition. Now, let me give you another extension of this concept of conditional probability; and then so just want to point out here that, you know, this is a name. So, therefore, the positive will occur, we put after s. This is the name of the, such a station.

And so those formula which we will, as we go on we will show that it is very useful way of computing certain probabilities, conditional probabilities of course; and so we are consulting over the, telling over the formula is; and then taking some examples to illustrate how it is. So, suppose E and F are 2 events, then as I have already shown to you, I can write E as E intersection F union, E intersection F complement, right.

Since, F and F complement are disjoint, and so E intersection F, and E intersection F complement are disjoint. And therefore, in exactly we have concluded that P E, can be written as P E intersection F plus, P E intersection F complement. Now, we can generalize this formula; because if I have n mutually disjoint P of E intersection F i, right. If you remember this then that means, I am giving you a formula for computing probability of E, by conditioning E on the F i s.

So, if I have a set of these mutually exclusive F i s such that they add up to, the union is equal to my whole sample space omega; then I can decompose P E in this way; and then using the conditional probability formula for E intersection F i I can write this as this; and now this is a way of computing P E by conditioning E on the occurrings of F i s, into F i. Or, in other words, you can also look upon this as a; because your P F i s, summation this will be equal to 1, remember; because this is equal to this; they are disjoint.

So, probability of union F i s will be sum of the probabilities of P F i s; and therefore that will be equal to 1. So, this is equal to 1. So, I can treat probability F i s as weights. And therefore, this formula saying that P E is the weighted average of probabilities E conditional F i by the weights P F i. So, whatever conditioning event you take here you multiply it by the corresponding weight. So, there are many ways of looking at it.

Now, why I am discussing this is because suppose that E has occurred then you want to find out the probability of occurrence of F i. See, actually the role of the 2 events saying that suppose E, here I was computing E dependent conditional occurrence of the events F i s. Now, I am saying that if I already know that E has occurred then what can be the probability of the occurrence of a particular F i.

Because, see, the thing is that F i intersection F j is empty; and union F i is omega. So, therefore only one of the events can occur here exactly, because they are mutually exclusive, so 2 events cannot occur at the same time; so this is occurred. So, in that case what we are saying is that when E has occurred I want to compute the probability of F i occurring, right; so that means, I want to compute the probability F i, given E, given that E has occurred.

So, again using this formula that I just wrote it, this will be P F i intersection E divided by this; because probability E, I have just computed for you, is given by this. And this is F i intersection E by my definition of the conditional probability. So, this is this. Now, P F i, given E is the posterior probability of F i when E has occurred. So, I am going to call this as the posterior probability of F i, given that E has occurred. And so now, I take up an example to show you, what we mean by this.

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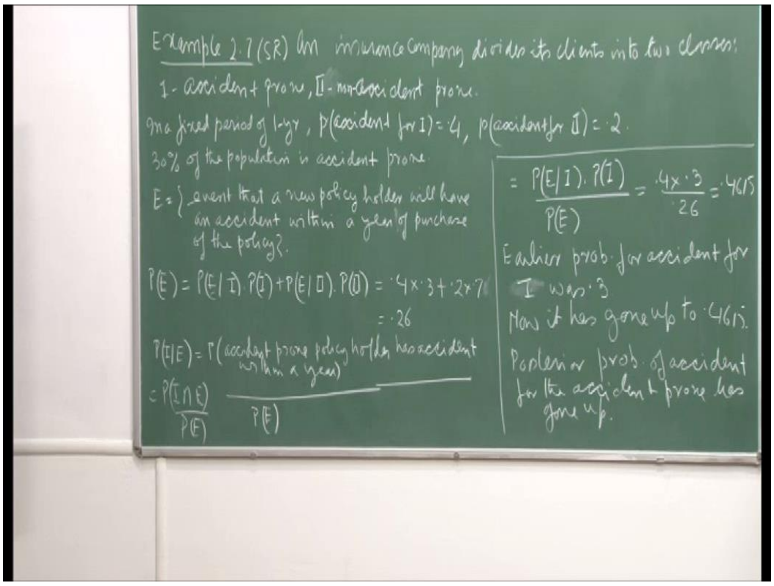

So, I will give you now an example of computing the base probability. And here, I am considering this example; and i you know, even in some other previous examples in the other lectures I have mentioned F i, which means that I am taking these examples from the book by Shelton Ross. And I will give you a proper, the 2 books by Shelton Ross which book I referred to while preparing this lecture; so I will give you the proper references at the end of the this section.

So, now, let us understand this problem very well. There is an insurance company. It divides its clients into 2 classes. See, there is, somehow, there is this information that a sort of people, are both accident prone, then some others; there is some ways of computing this. And this may be because of the past history or something; you know, if somebody has had lot of accidents and somebody has not had, then you know, because it shows companies they depend on the this kind of data.

So, let us see that class 1 is the set of people who are accident prone, and class 2 is the set of people who are non-accident prone, that means they had very few accidents in the past. Now, in the period of 1 year, the whole thing that we are looking at is during given period of 1 year, fixed period. The probability of accidents for class 1 of people is 0.4; that means, the probability of accident prone, having an accident is 0.4, and the probability of non-accident prone person having an accident is 0.2, and in that fixed period of time that you have.

And it is also known that 30 percent of the population is accident prone. That means the number of people being in set 1, the probability is 0.3, ok. Now, suppose the event we are looking at is that a new policy holder will have an accident within a year of purchase of the policy. So, there is a person who has just bought a policy, insurance policy. And now, you want to compute the probability this person will have an accident within the year, ok; within the year of the purchase of the policy.

So, again using this decomposition principle I say that probability E can be written as probability E condition on i into, probability i plus, probability E condition on 2; so that means this is the accident prone, and this is non-accident prone. So, probability E condition on 2, into probability 2; that means the people having accident, right; when I said that P 2, I mean that probability of a person who is non-accident prone having an accident.

And here, this is the probability that an accident prone person has an accident. And I have this probabilities, right. I have this probability. So, the whole thing is accident for set of people in i, which I am writing in short form as P i, and for this I am writing P 2. So, this we can easily compute because this conditional probability is given to me which is 0.4, right; given that the person policy holder is accident prone; if he has an accident that probability is 0.4. And the probability that the person in accident prone, 0.3.

So, therefore, this number is 0.4 into 0.3 plus, the probability that the person policy holder having accident when he is non-accident prone; that probability is given to me as 0.2. And then the probability that he is non-accident prone is 0.7. So, when you compute this number, this comes out to be 0.26; that means, a new policy holder having an accident within a year of purchase of the policy is 0.26.

Now, I want to compute this conditional probability i, given E; that means, an accident; so when you look at the intersection; so this will actually be by our definition i intersection E divided by P E, right; which means that; see, if you just read this, this intersection of 2 event says that a new policy holder. That means, a person who has purchased a policy currently and is accident prone will have an accident within a year of purchase of the policy, right. If you read carefully, the event i intersection E, E simply says that a new policy holder that was a person who has just purchased the policy will have an accident within a year; I says that the person is accident prone.

So, therefore the intersection will be that if accident prone person, the policy holder will have an accident within a year of purchase of the policy. So, that is this. And this is divided by P E, right. Now, what I will do is because, so i, given E conditional E, I do not have; but I have this probability. So, therefore, using this definition of, remember I gave you the alternative definition of intersection of the set.

So, this is P i intersection E; I can write this as P E, given i; that means, accident prone and then the person is having an accident within a year into, P i divided by P E. P E, I have already computed as 0.26. And so here, this is 0.4 into 0.3, this I already know, right, remember, from here the first part, and so this is; so now this goes up to 0.4615.

So, earlier probability for accident, for accident prone, right; earlier probability for an accident prone person to have an accident was for; earlier probability for accident for i was, sorry; this is 0.4, right; probability of accident for an accident prone was 0.4. But now, the posterior probability; that means, after knowing that the person had an accident, remember, I am computing now that policy holder has had an accident.

So, the probability accident prone person having an accident mean that he has had an accident has now gone up to 0.4615. So, the posterior probability of accident for the accident prone has gone up from 0.4 to 0.4615. So, actually this is a little complex concept and I will keep coming back to it through examples to make sure that you, you sort of get a better feeling for these computation of base probability, ok.

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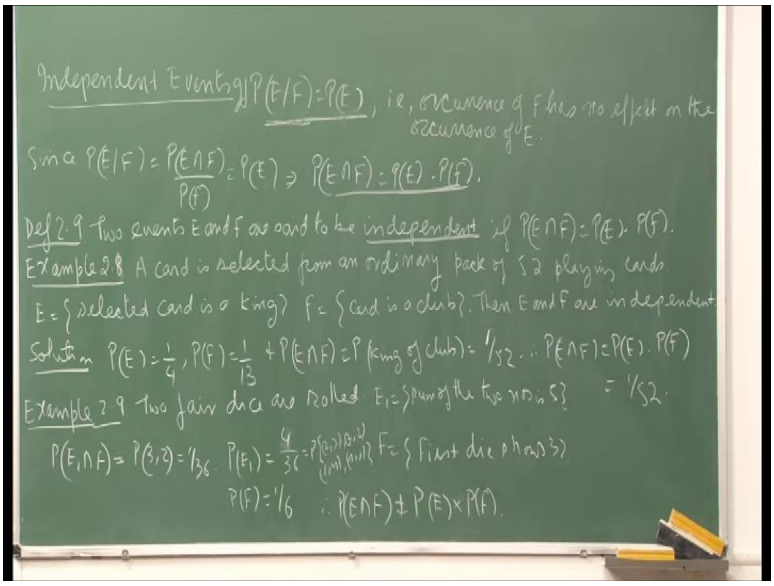

Now, the moment you define conditional probability, you then come to the concept of independent events. So, let me first motivate the reason, motivate the definition. And so here you see what is being said is that, if this conditional probability of E given that F is occurred, is equal to P E; that means, that the occurrence of F has no effect on the occurrence of E, right; because this probability has remained unchanged, eventhough I know that event F has occurred.

So, these are the kinds of things because you want to know what kind of events can have effect on the occurrence of some events and so on. So, this is also very important concept and you need to know how compute it. So therefore, if this is so then you see, by definition of the conditional probability of E given F is P E intersection F divided by P F, but then we are saying that this is equal to E, probability of E, right.

That means, I am just trying to formulize this concept that if occurrence of an event is not dependent on some sense on the occurrence of another event, then we would be defining the concept of 2 events being independent. And so this is, essentially this is what you will say; that if this is equal to this, then occurrence of F has no effect on the occurrence of E.

And now, by our definition of conditional probability I will write it as this; that since this is equal to P E, this implies that probability of E intersection F is P E into P F. And so this is our concept of 2 events being independent. And so I will just write down the definition 2.9 which says that 2 events E and F are said to be independent, if P of E intersection F, that means, probability of E intersection F, is equal to P E into P F, right.

So, let us look at this example. A card is selected from an ordinary pack of 52 playing cards. Now, E is the event that selected card is a king, fine; and then F is the event that the card is a club. Then, we can immediately show that the events E and F are independent which you can also argue intuitively also, that the king need not be a club, and so on. So, let us now show that; this gets validity by definition also.

See, to compute the probability of E that the selected card is a king, so since there are 4 kings in the pack, so therefore, the probability of P E is 4 by 52. And selected card is a club, so 13 cards belong to the club. So therefore, the probability of F will be 13 by 52. And P of E intersection F is probability king of club, so that is only 1 card, and therefore the probability of that is 1 by 52. So, therefore, P of E intersection F is P E into P F which is 13 by 52 is 1 by 4; and so 4 into 4 gets cancelled. And so P E into P F is also 1 by 52. So, the 2 probabilities are equal; and i therefore, E and F are independent.

So, if you look at this other example; 2 fair dices are rolled; and i if E 1 is the event of sum of 2 numbers is 5; and F is the event that the first dice shows 3. Then you see E 1 intersection F is simply with the event 3, 2; that means I must have the pair 3, 2, then only, you all know that this is already 3, so then this number must be 2 inorder for the sum to be 5.

And this again the probability is 1 by 36; but P E 1 is what? That means, E 1 is the sum of the 2 numbers being 5; this is (2, 3), (3, 2), (1, 4) and (4, 1). So, these are the 4 possible pairs which will give me the sum as 5; and so this probability will be 4 by 36; each being equally likely. And the first phase showing you 3 is 1 by 6 because here again each phase is equally likely, each number is equally likely. So, then this probability of E intersection F is not equal to P E into P F. So, we will say that P and F are not independent events.

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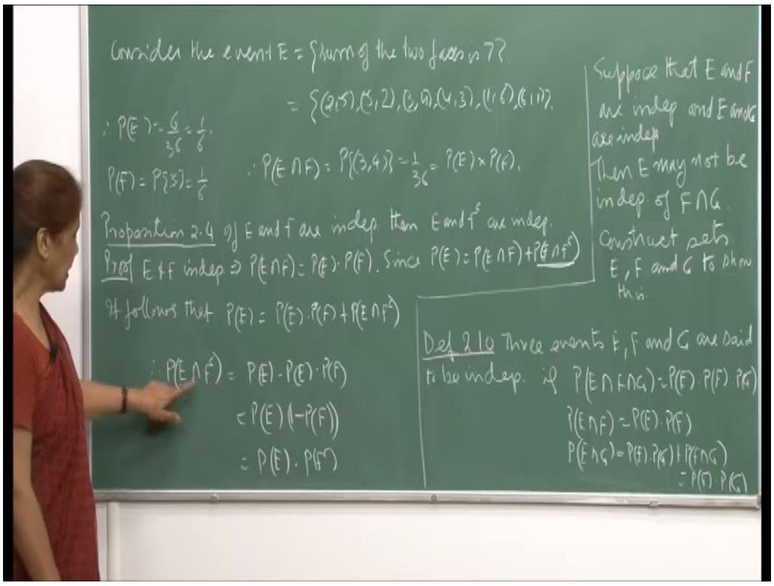

Now, consider the event E which is the sum of 2 phases is 7, right. So, if I change the event from E 1 to E which requires that the sum of numbers on the 2 phases is 7; then you can see that these are the pairs which are valid, so this event. And therefore, they are 6 in number. So, probability E will be 6 upon 36 which is 1 by 6; and probability F already we have seen which says that the first dice must show number 3, so that probability is again 1 by 6.

And so now you see that, if you look at the event E intersection F, since F says that the first dice has to show the number 3. So then for the sum to be 7, the second dice must show the number 4, so this is the only possible element of omega which is favorable to yield this section F. So, therefore, here the probability of E intersection F is 1 upon 36 which is equal to P E into P F.

So, therefore you see, you can see how one can define events which given the event F; for example, what are the event which are independent of F, and which are not. In the earlier case when we took the event E 1 to be, the event of the sum of the 2 phases is 5; then you saw that P E n f where E 1 and F are not independent; then if I take the sum to be 7 then it says the 2 events are independent.

And now, if you try to define another event E 2 phase which says that the sum of 2 phases is 8. Then again you can show that F and E 2 will not be independent. So, you should now play around and get a feeling for the definition by constructing different;

take the same experiment that is throwing up of 2 dices; and then try to construct a set of events which are independent, which are not, and so on. So, you can get better feeling.

Now, continuing with the, again as you see you define a concept; then using the axioms we come up to new propositions; and you see the theory keeps developing. So, if 2 events are independent then you will say that E and F complement are also independent which is very intuitive and if you able to rationalize it, but in any case, we will prove it analytically also. So, the proof is simple.

If E and F are independent then by definition probability of E intersection F is P E into P F; and i since we have already seen that you can write P E as probability of E intersection F, plus probability of E intersection of F complement. So, then we want to compute this; so this will be, from here probability of E intersection F complement will be P E minus, from here P intersection F, but then because E and F are independent.

So, P E intersection F, I can write as P E into P F; and so you see this comes out to be P E multiplied by 1 minus P F. I can take P E common outside. 1 minus P F is probability F of complement. And therefore, probability of E intersection F complement has been shown to be equal to P E into P F compliment; and therefore, by our definition E and F complement are also independent. And which says that if occurrence of F has no effect on the occurrence of P; and when I say occurrence that means the probability we are talking of.

So, E and F being independent shows that occurrence of F is not equal to the occurrence of E. Therefore, occurrence of, non occurrence of F shows somewhat have an effect on E. So, that is what we have concluded, that if E and F are independent, then E and F complement are also independent.

Then, now here, I am again just make it a statement and I want you to construct examples for yourself, so that we are saying is that, if suppose E and F are independent, and E and G are independent, then E may not be independent of F intersection G. Now this may not sound very intuitive. But, you can construct sets E, F and G to show this, right; so that means, now, I have already told you to explain this the example that I took earlier.

And now I want you to also construct sets E, F and G, construct an experiment; and then construct the corresponding events E, F, and G to show that if E and F are independent, and E and G are independent then E may not be independent of F intersection G. And infact, you can, you know, that the throwing of 2 dices that I have taken up to experiment. And there also you can construct such E, F and G to show this is valid, this statement is valid. But then I can come back and discuss some examples with you.

Now, if you want to extend the concept of independence of 2 events or more than 2, then I will just show how things start becoming difficult, even if you just take the, want to extent the concept to 3 sets, that means if 3 events E, F and G are independent then we require that not only should the probability of intersection of E with F with G. And intersection with G should be equal to the product of the individual probabilities.

But, when you take 2 at a time, these 3 sets, from these 3 sets, then P of E intersection; that means, any 2 subsets here should also be independent. It is not just that the product of the, probability of the intersection of the 3 sets is equal to the product of the individual probabilities, that when you take 2 at a time then also that should be, the condition of independent should be satisfied. So, all 4 conditions have to be met before we can say that the 3 sets, E, F and G are independent.

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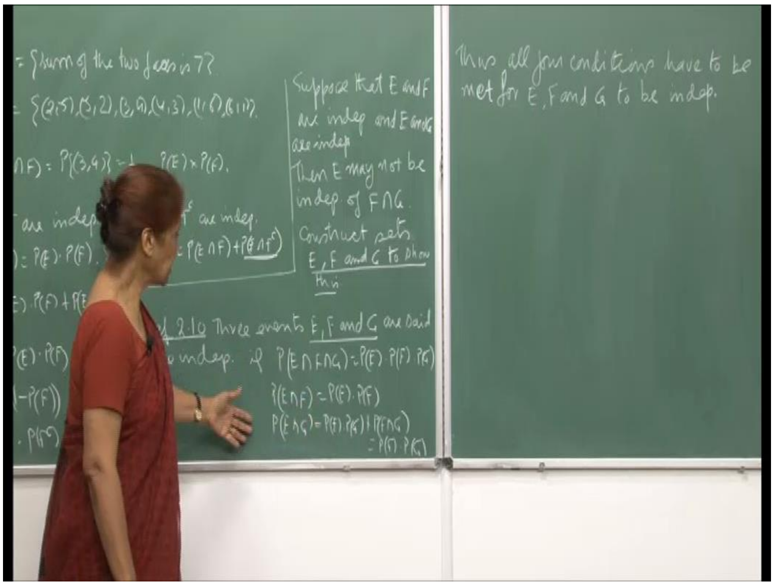

So now, here again we need to construct; and we can immediately see that the moment you have increased the number of sets, and you want to talk about their independence, then this will get more complex; and the number of conditions will go on, becoming larger and larger. So, therefore, I just leave it at this point for this thing. And then one can always look up at textbooks where they will give you conditions for any independent sets, I mean, any number of subsets.

So, here again a interesting way to try to understand this concept would be to construct subsets E, F and G which may satisfy not all the conditions, but only some, to show that all 4 conditions are necessary for independence of the 3 events, E, F, and G. So, I will now take up set of exercises too, and then we will come back to constructing examples for these situations.