Introduction to Probability Theory and its Applications
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Lecture - 28
First Passage and First Return Prob. Classification of States

So, we will continue our discussion with first passage and first return probabilities; how to compute them and so on.

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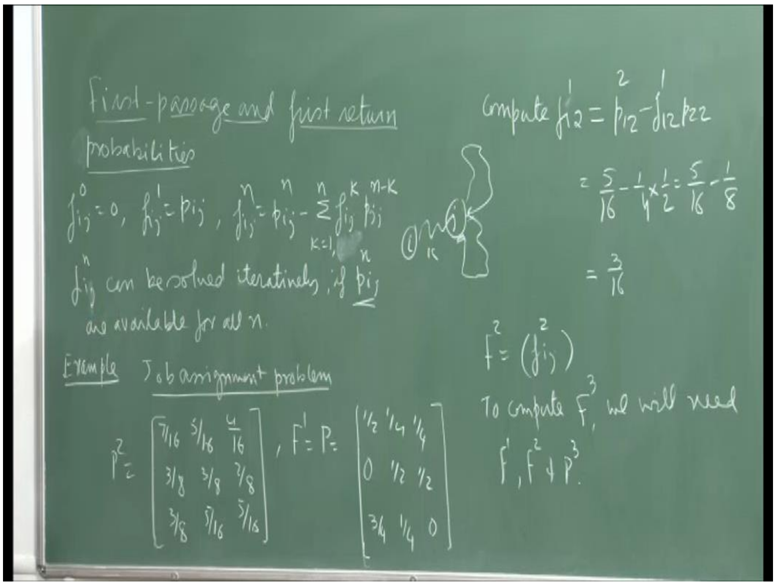

So, I just recall quickly that f i j 0 will be 0, because there cannot be any transition in 0 time; then, f i j 1 that mean the first time, j is reached from i in one step, so that will simply be p i j. So, the first step transition probabilities are the first-time transition probabilities also, and then for f i j n, we wrote down this formula through which we can recursively compute the f i j’s. So, an f i j n will be p i j n minus sigma k varying from 1 to n; f i j k into p j j n minus k. And I had explained last time also that, we are… because p i j n simply gives you the probability of transitioning from i to j in n steps. And in between you could number of times have visited state j. So, all that is included here, because we were computing all possible paths from i to j. So, that may include going to j and staying over j and so on; and then, going from j to somewhere and then coming back to j and all that.

So, now, we will want to subtract those probabilities from p i j n to compute f i j n. And so, this says that, here this is the first time in k steps you transition from i to j for the first time and then from j to j in n minus k. So, this again will include… No, going from j and coming back to j, but in between also you can visit j any number times.

So, this p j j and minus k; and here this is… You have already reached j in k steps and then all this is happening, because you are computing re-transitioning probabilities up to n. Therefore, we have to subtract. So, these particular paths we are subtracting; that means from i to j, suppose you have this; and then, you go back here; you come back to j; then, again you go back and come back to j and so on. So, n minus k steps you will be doing this; but, in k steps, you reach j for the first time from i. So, we have to subtract the possibility of traversing all such paths from this one to be able to compute the first passage probabilities. And so, that is what we are doing. So, I thought that, I will revisit this and explain it much better.

Now, we can solve these iteratively from these equations given the initial conditions here. But, then we need the p i j n’s are available for all n to compute this. So, just look at this example. Again a job assignment problem; we have p square is this; and F 1. So, here I we are using the notation that, now, just as a P is a matrix of one-step transition probabilities; P square is the matrix of two-step transition probabilities; we will also denote the first time first passage transition probabilities by the matrix F 1. Then, it will be F 2, F 3, and so on. So, if you want to compute for example, f 1 2 1; no, what am I doing here; this is f 2 1 2. So you want to compute the probability that a transition from 1 to 2 in two steps. So, this will be by the formula p 1 2 2 minus f 1 2 1 p 2 2.

See here the formula is… Therefore, you reach… that means in one step, you will go from 1 to 2 and then you have to stay with 2, because there is only 1 step here. And therefore, if you compute this, you get 3 by 16. So, now, what you see is that, you have to... Therefore, now using this formula, you can compute… you get the matrix F 2. So, you have to compute all two-step transition probabilities before you can compute other the three-step first passage transition probabilities. And for computing F 3, you would need F 1, F 2 and P 3 also; that means P, P 2, P 3, F 1, F 2 you will need to compute F 3 – elements of F 3. And so, this is lot of work. And so, I will give you now an alternative method of computing these first passage probabilities without wanting to… also, without wanting the higher powers of P; just the first transition matrix would be enough.

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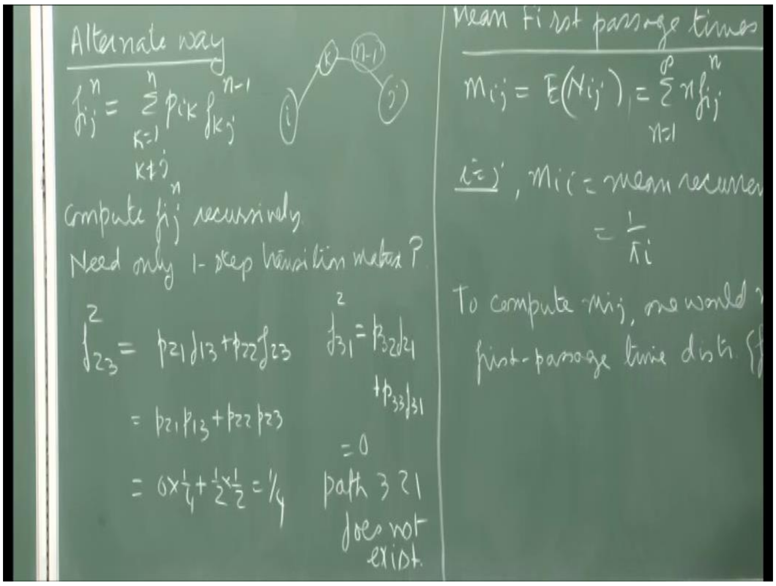

So, the alternate way. Now, the alternate way is see the way we explain; this is now, look at it as a one-step. So, f i j n – I am writing as sigma k varying from 1 to n p i k. So, one-step you transition right in the beginning; you are starting from i. So, suppose you go to k. So, that probability is p i k. And then, from k, the first-step transition probability… I mean first passage transition probability k to j in n minus 1; is it okay? that means, k is varying from 1 to n. So, for all possible states, you are starting from i and you may go to some other k; and then, from k, you want to go to j for the first time. So, here you will require n minus 1. So, if you have already computed f k j n minus 1, then you can compute f i j n by using the first-step transition probabilities. And so, this is a neater and more efficient way of computing f i j n's recursively. So, the argument is okay because I have to move out from i to some this thing. So, k should not be j obviously, because I have to visit j for the first time; but, then in… that means I will definitely visit some other state from i to k in one-step since I want to visit j from i in n steps.

So, in the first step, I will definitely go somewhere, which is a state, which is different from j. And then, i need f k j to n minus 1. So, from k to j, I should visit j from k for the first time in n minus 1 steps. So, this is a neater way of computing the first passage probabilities. And now, here we just need the one-step transition matrix P. And of course, we are computing these f k j… – all of these before I compute f i j n. And so, here again, I am just doing the same exercise. So, if you want to compute f 2 3 2; so, then this will be simply p 2 1. So, remember the k does not have to be… So, k cannot be 3.

So, k can take the value 1 and 2. Therefore, p 2 1 f 1 3 plus p 2 2 f 2 3. So, this is all. And so, by taking the values because this is only p 1 3; so, this is p 2 1 p 1 3 plus p 2 2 p 2 3. And you can looking up the values in the matrix P, you get this.

Now, look at f 3 1 2; f 3 1 2 – so, here j is 1. So, k can be 2 and 3. So, this will be p 3 2 into f 2 1 plus p 3 3 into f 3 1. And what is happening is that, what is p 3 2? p 3 2 is 1 by 4, but then what is f 2 1? f 2 1 is p 2 1. So, p 2 1 is 0. So, this is 0. Then, p 3 3 is 0 into f 3 1. So, this is 0. But, then you remember if I have not drawn the diagram here; but, if you remember, the path 3 2 1 does not exist, because you did not have the arrow from 2 to 1 in the job transition – this matrix, because this is 0. So, you do not have the arc from 2 to 1. Therefore, this path does not exist. So, anyway this is you cannot transition from 3 to 1 in two steps first time. First time, from 3 to… You cannot reach from 3 to 1 for the first time in two steps in two time periods. So, this is… So, this is definitely better way of computing f i j n.

So, now, once we know this, then let us talk about the mean first passage times. So, remember N i j we have denoted as the number of transitions that you require for going from i to j for the first time. So, then the mean first passage time will be expectation of the random variable N i j, which you will write as n varying from 1 to infinity and f i j n. And when you put i equal to j, then it will be M i i, which will be the mean recurrence time. And this we have said is equal to 1 by pi; 1 by pi i. So, the steady state probability of being in… So, this is a proportion of time being in state i over the long run. So, 1 by pi i would be the mean recurrence time. So, that will be the time required for reaching. On the average, this is a time that you will require for reaching from i to i for the first time.

Now, to compute M i j's, you would need the first passage time distributions. So, you need to compute… You cannot apply this formula, because you have to then have all f i j n. So, again, we will look at a nicer way of computing mean recurrence time and mean first passage times.

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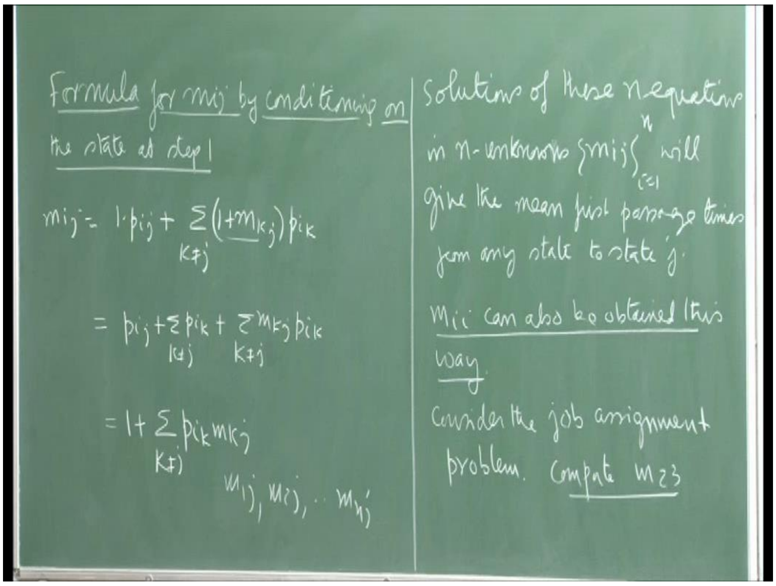

So, as we said that, to compute m i j by the formula, where you need all f i j n up to infinity, this is not practical. So, let us now come out with another method for computing these first passage times. So, you see we will condition on the state at step 1. So, see what happens is if you are computing the mean first passage time, then either the transition from i to j takes in one step. So, then this is 1 into p i j; remember you are computing the expected value of f i j n. So, m i j – so, if it is one step, then probability of transitioning from i to j in one step, that is p i j.

So, 1 into p i j. So, either we will transition from i to j in one step, and therefore, 1 into p i j or we go from i to k in the first step, since it has to be… Either we transition from i to j in one step or from i, I go to some other state. And then, from k, I will transition back to j. And so, then it will be… Once I do this, then the passage time becomes 1 plus m k j, because m k j is the mean time of going from k to j. And so, 1 plus, because one transition has already taken place. So, the mean time of transitioning from k to j for the first time will be 1 plus m k j and into the probability of transitioning from i to k, where k is not equal to j. So, this should be clear.

Now, we just rewrite this expression, because here you are summing with respect to k and k is not equal to j. So, p i j is missing, which is available from here. So, when you add up p i j plus sigma p i k, k not equal to j. So, this whole thing is the summing up the components of the i-th row of the first transition matrix. This must be equal to 1, because from i, you have to transition to one of the states. And therefore, this is 1 plus sigma m k j into p i k, k not equal to j. And we can rewrite this as summation p i k into m k j, where k is not equal to j. So, this is now again, as I said, the way we computed the formula for f i j n, this is also a simple way of computing m i j without really requiring to have the whole distribution for the first passage probabilities f i j n. So, these are n equations and n unknowns; that means, n unknowns means in the sense that, you are asking for this – m 2 j and so on, m n j. So, you are asking for a first transition passage to j from anyone of the states 1, 2 and n. So, these are n variables, because i is varying from 1 to n in n unknowns; n equations in n unknowns. Now… And similarly, we can also compute i i i; can also be obtained this way. So, the mean first recurrence times can be also obtained by solving corresponding set of equations here. And let us just work out the formula. This will be…

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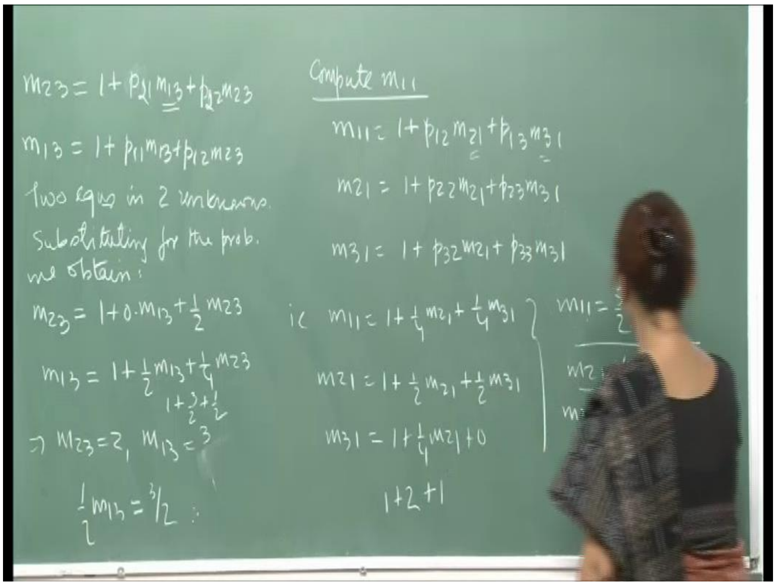

So, if you want to compute m 2 3, then m 2 3 will be 1 plus… And remember k is not equal to 3. So, k can take the value 1 and 2. So, it will be p 2 1 into m 1 3 plus p 2 2 into m 2 3. So, now, in order to compute m 2 3, I need m 1 3. So, I will write down the formula for m 1 3. So, m 1 3 will be… So, here again k is not equal to 3. So, this will be 1 plus p 1 1 m 1 3 plus p 1 2 into m 2 3; so, then two equations in two unknowns. And we can… Substituting the probabilities p 2 1, p 2 2, p 1 1 and p 1 2, I get these equations. And so, it is not difficult, because from here you immediately get like half m 2 3 is 1. So, m 2 3 is 2; immediately you get it from here. And then, once m 2 3 is 2, this is half. So, this is 3 by 2. And when you bring this to this side, let us see; m 2 3 is 2. So, this is 1 by 2. So, this is 1 by 2 and this is 3 by 2 and this you bring here. So, this will be half m 1 3 is equal to 3 by 2. Therefore, m 1 3 is 3; it is not just 3 by 2; m 1 3 is 3; yes, from here. You can again substitute and make sure. So, m 1 3 is 3; this is 3 by 2 and m 2 3 is 2. So, this is… What is the mistake? [FL] This is equal to 1 plus m 1 3 – we are saying is coming out. So, this is 3 by 2 plus [FL] and this is half. So, this is 2 plus 1 3. So, m 1 3. Therefore, that was the right solution. So, this is m 1 3 is 3. So, working out always helps you, because you can find out…

Now, similarly, let us just look at the way we compute m 1 1. So, the first mean recurrence time for going from state 1 to 1. And here again k cannot be equal to 1. So, it will be 1 plus p 1 2 m 2 1 plus p 1 3 m 3 1. And now, you need m 2 1 and m 3 1. So, three unknowns are there. Therefore, three equations: m 2 1 will be 1 plus… Again, k is not equal to 1. So, p 2 2 m 2 1 plus p 2 3 m 3 1. And then, finally, when you write it down for m 3 1, it will be 1 plus p 3 2 m 2 1 plus p 3 3 m 3. And so, substituting for probabilities, I get these equations. And again, in a very simple way, you can solve and you can check… Let us see m 1 1 comes out to be 5 by 2. And if you remember the calculations for pi 1, pi 1 was 2 by 5. So, the steady state probability of being in state 1 was 2 by 5. So, the mean of first passage time will be 5 by 2.

Now, m 2 1 comes out to be 4 and m 3 1 is 2. So, I hope this is a right calculation, because let us see if you want to do it here, just verify. See if m 2 1 is 4, then this side is 1 plus 2 and m 3 1 is 2. So, that is also 1. So, that is 4. This is coming out to be high, because remember there is no arc from 2 to 1; there is no direct arc from 2 to 1. So, you can of course, go from 2 to 3, then 3 to 1 for the first time, but… So, that is 2. But, then other paths when you go around, it will be... So, the mean first passage time is coming out to be 4 and m 3 1 is 2. So, now, we will see… Of course, see the thing is that, as I said when I have concluding the steady state probabilities and the mean first passage times, there are certain conditions under which these are valid. And so, we now have to look at the situations, where these things may not be valid. And in fact, I had said that, these may not even exist and so on. So, we will now start looking at… So, right now, under certain conditions, which we have to specify and we will do it soon. So, want to show you that, this will be true – all these ways of calculating the steady state probabilities and mean first passage time and mean recurrence times; you will be doing it under certain conditions; they are valid. And when they are not valid, then we have other quantities to define those states. So, we will talk about it.

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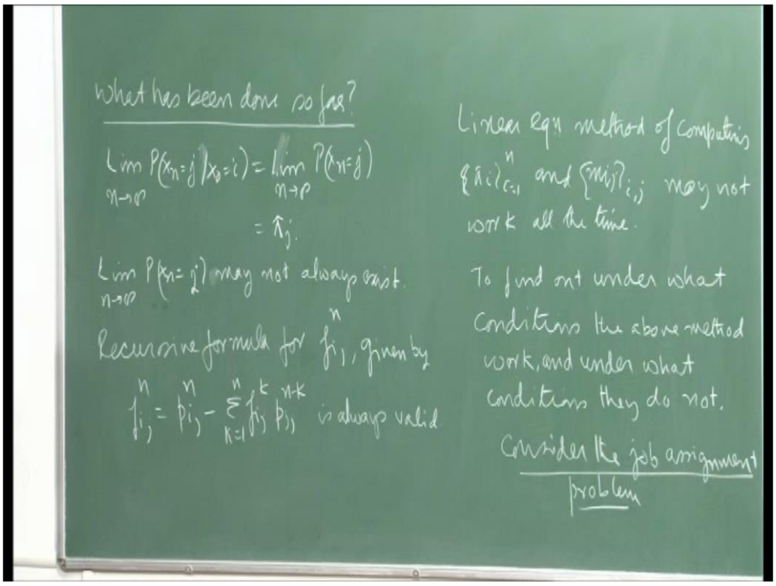

So, let us just recall what has been done so far. See we said that, limit probability X n equal to j given that X naught is i is actually limit P X n equal to j. And then, we defined this as pi j. So, this was the tacit assumption that, this limit exists and that it is independent of the initial state. And therefore, we said that, you can solve these pi j's through the system of linear equations. And that is what we did. But, this was under the assumption that, this limit exists. But, now, it is not really true that, this will always exist. And this is what we need to now talk about and find out. Therefore, that means the linear equation method of solving the pi i's depends on whether this limit exists. And we said that, this limit is also independent of the starting state. And therefore, we could write down the system of linear equations and say that, the solution exist under the condition that, sigma pi j is add up 1. Then therefore, things were simple, life was easy. Therefore, we will now look at the conditions under which this limit exists.

Second is that… And the recursive formula for f i j n that we wrote down – this formula is valid. Here we are not asking for any limits or anything, because the powers of p can be computed. And then, therefore, through that, you can recursively compute your first passage probabilities. So, that is valid. But, here… And m i j again – because remember I am saying that, the m i i will be equal to 1 upon pi i. So, there again the existence of pi i is assumed and even otherwise. So, the linear equations for solving the m i j's also needs to examined under what conditions this method will be valid. So, let us start looking at examples and then we start talking about the conditions. So, first suppose… So, now, you again consider the job assignment problem and you see that, the matrix p 2 had got all entries positive.

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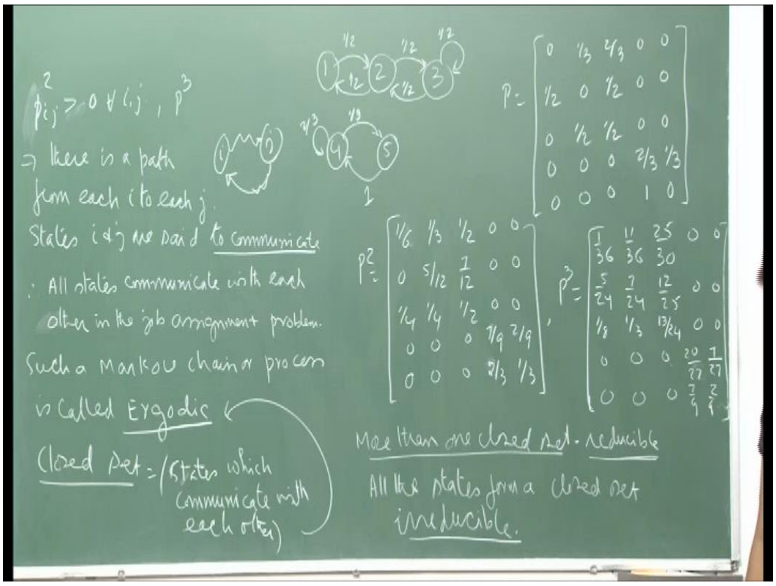

In the first transition matrix, there were some zeros in the sense that, you were not able to go from 2 to 1 and you did not have a loop from 3 to 3. So, you had 0 entries. But, when you took the square of the matrix, then all entries became positive. And after that, P 3 was also… All entries were positive. So, this implies that, there is a path from… Remember because this is simply a transition probability in two steps from i to j. So, if it is positive; that means there is a path from each i to each j. And in such a case, when you have a path from i to j and then you have also have a path from j to i, because p i j 2 is positive and p j i 2 is also positive.

Since all entries are positive, you have this; you have a path here again from j to i with positive probability; after such a pair of states is said to communicate with each other. So, they said to communicate. And if all states communicate with each other; and so, now, because p i j 2 is positive for all i j. So, we conclude that all states for the job assignment problem communicate with each other. And such a chain or such a Markov chain or process is called ergodic. And we will talk about this some more. But, first let me say that...

So, now, what we will say is that, we will define a closed set as a set of states, which communicate with each other. And for the job assignment problem, it turns out that, the closed set actually consists of all the states – 1, 2 and 3. But, it is possible that you may have more than two closed sets or three closed sets; whatever it is, all states may not form one closed set; that means all states may not communicate with each other. And now, let us look at this example here. See this is the transition diagram. So, you see that, 1, 2 and 3 – you can see that, they all communicate with each other; there is a path from 1 to 3, from 3 to 2, 3 to 1 and so on; and there is a path from 4 to 5 and 5 to 4. But, there is no path from 1, 2 and 3 to 4 and 5. And this you can… Of course, see the thing is that, I have drawn this diagram with five states; but, when the number of states is large, the transition diagram will be very big and it may not be again… Same problem as I was talking about earlier.

Let you know you cannot possibly enumerate all possible paths when you have a large number of states… So, the recourse is to taking higher powers of p. So, let us start. And of course, here it is evident that, these two... Therefore, the two closed sets are 1, 2 and 3 and 4 and 5. So, this chain is definitely not… All the states not form a closed set. And… So, let us see. Let us start with P – the transition matrix. So, you see there is no arc from 1 to 4 or 1 to 5. Similarly, there is no arc from 2 to 4, 2 to 5 and from 3 to this. And same way, there is no arc from 4 to 1, 4 to 2, 4 to 3 and 5 to 1, 5 to 2 and 5 to 3. There is no arc. So, this is the zeroes. And let us see when we take the second power; that means P square, then these zeroes remain intact. These probabilities change; these zeroes may… like this 0 goes away. But, this…

And then, again when you take P 3, these zeroes do not go away and here it becomes… Therefore, now, you can say that, among the states 1, 2 and 3 – the closed set 1, 2, 3, now, you have path from each node to the other two nodes, because these are all positive. So, they all communicate; that means you need at least three transition steps to see that, there is a path from each of the nodes 1 to 3 to the other two. And similarly, here this is of course, quite clear. So, this is also… All the probabilities are positive here. And therefore… And you see that, this structure will continue no matter what higher powers you take of P. You will continue to have zeroes here and zeroes here. Therefore, this is a situation, where all the states do not form a single closed set. And in fact, if you have more than one closed set, we call such a chain or a process reducible. And if all states form a closed set – one single closed set, then it will be irreducible. So, what we have been talking about so far has been irreducible chain; and irreducible means that, all states will communicate with each other. And therefore, it will also be ergodic. But, again even this classification is not enough; we need to do it further. And so, continue the discussion.

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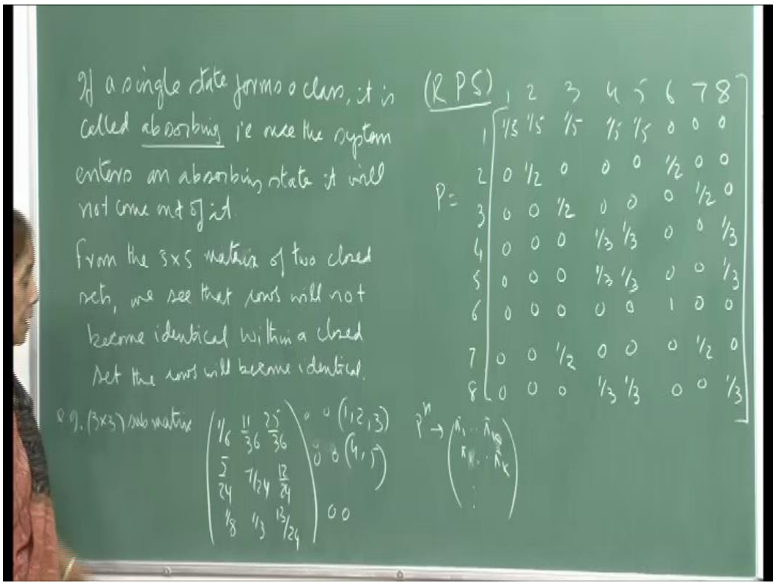

So, if a single state forms a class; that means there is only one state in a class. It is called absorbing, because obviously, the state is not communicating with any other states; it is just communicating with itself. So, then it is called absorbing; that is… So, once the system enters in an absorbing state, it will not come out of it. So, we will just look at the examples for all these – the kinds of states that we are talking about. Now, from the 5 by 5 matrix of the two closed sets, we just looked at this example in which you had two closed sets has formed by the states 1 2 3 and the states 4 and 5. Remember there was no communication between the states 4 and 5 and 1, 2 and 3, and vice versa. So, these were two closed sets. So, if you look at the submatrix – 3 by 3 submatrix; see because otherwise it is all zeroes; this is 3… It was 0 and 0, because there was no communication from 4 to 5.

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So, in the submatrix, the entry 1 3 was by mistake written as 25 by 36; it should be 19 by 36, because all the rows – the elements of any row must add up for a transition matrix; must add up to 1. And so, with 19 by 36, the numbers 1 by 6 plus 11 by 36 plus 19 by 36 add up to 1. So, make that correction. And in P 3 also, the same correction has to be made, because this mistake got carried over from P 3, where it was by mistake written as 25 by 36 and not 19 by 36. So, you see that, the submatrix itself…

Now, the case that, we were talking about that, the matrix P n in the limit will converge to where the rows are all identical – whatever these numbers states k 1. So, this was a pi 1 to pi k and so on. So, all rows were identical. Remember this is the case we had talked about. But, now, you see it has no meaning here, because the rows… These zeroes will remain forever. So, we cannot say that, the rows will become identical in the sense that… These three rows will become identical maybe you can say. But, then below here you have zeroes, because again 1, 2, 3, 4 and 5 are not communicating with 1 and 2 and 3. So, you will have zeroes here and then you will have some positive entries here. Therefore, you cannot say that, the rows will become identical.

These three rows will become identical; that is what I want to show you that, if you… This is P 3. So, P 3; this is 1 by 6, 5 by 24 and 1 by 8. So, supposedly, this number will go up a little and this number will also go up, because you need 6 by 24 is 1 by… So, you need 1 by 4. So, supposedly, you can see that, the numbers are close. Similarly, 11 by 36, 7 by 24 and 1 by 3. So, 1 by 3 is 8 by 24. Therefore, again the numbers are getting close. And similarly, here. So, you see… that means these three closed states – they themselves will satisfy the condition that, the steady state probabilities… You can compute the steady state probabilities; and here it will not matter in which the system started. But, to say that, all the rows of the 5 by 5 transition matrix will converge to the same values would not be valid.

Now, again this is a big example I have taken from Ravindran and Phillips and Solberg. And so, this is 8 by 8 transition matrix; and the diagram I have drawn on the other board. And see I have written down this matrix though I am not making use of it right now. But, it will be a good exercise for you people also to sit down and try to… If you can write it a small ((Refer Slide Time: 31:16)) I am sure from matlab and so on, if you feed this matrix, you can get a different powers of the matrix. So, it is not too much of a problem getting higher powers. And then, whatever I am saying, you can also conclude the same things by looking at higher powers of P.

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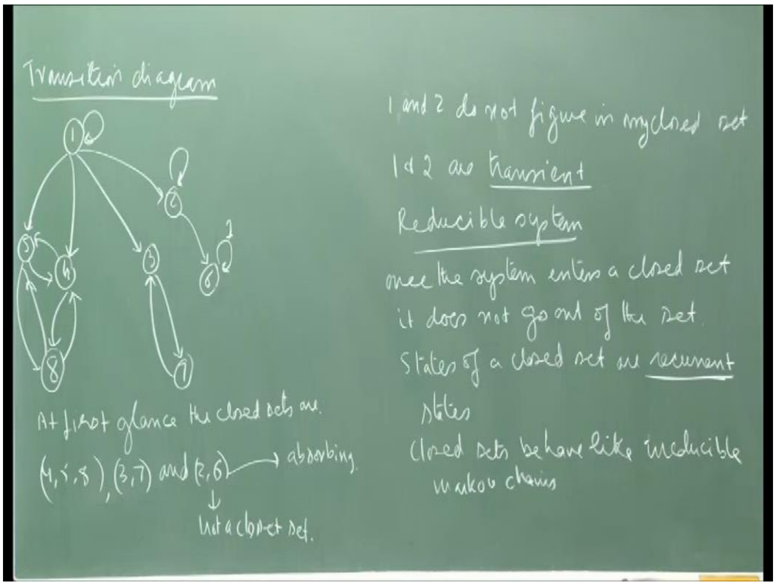

But, in any case, if we draw the transition diagram, it will look something like this. And I am not writing down the probabilities, because they are here and anyway it will clutter up the diagram; we just need to look at the connections and whatever i am concluding from that is enough; I do not need the probabilities really, except that this P 6 6 is 1. So, fine. So, now, if you just glance at this transition diagram, then you will see that, 4, 5, 8 are connected; 3, 7 are connected; and you may say that, 2 and 6 are connected. But, then when you go further, you see that, there is no arc from 6 to 2. And when you reach 6, then the probability of coming back to 6 is 1. So, this is a certain event; so, that means once the system comes to state 6, it just stays there. And so, 6 is an absorbing state. And this is not a closed set, because closed set 2 and 6 should communicate, and 6 and 2 communicates. So, there is an arc from 2 to 6, but there is no arc from 6 to 2. So, this is not a closed set. Therefore, you have two closed sets, which are 4, 5, 8 and 3 and 7; that for sure, because you have an arc from 3 to 7 and 7 to 3.

Now, 1 and 2 do not figure. Therefore, this is not a closed set. So, states 1 and 2 do not figure in any of the closed sets. And such states we define as transient, because see what will happen is that… So, first of all, they are transient. And since you have more than one state – closed state in the system… Therefore, this is a reducible system. So, that is another thing. Now, thirdly, once the system enters a closed set. So, for example, it enters state 5 or 4, then you see it will keep hopping amongst these states only; there is no way of going out; there is no arc, which is going out of 4, 5 or 8. So, the system will then… For infinite number of times, go on hopping among the states 4, 5 and 8. And such states we will call as recurrent, because they will keep occurring again and again. So, the moment a system enters the closed set, then there is no way of going out, because if this communicated with any other states, then that will also form part of the… For example, if 4 communicated with 1, then this will become a closed set. So, that is not true. So, once you enter a closed set, you cannot get out of it.

And similarly, here if you come to 3, then you will go on going from 3 to 7 or 7 to 3; that is all. You will just go round and round. Or, because there is no loop here either, there is no loop here. So, you will continue doing it. And from here if you come to 2 and then you go to 6, then you just stay put in state 6; you do not get out of it. So, first of all, now, after defining the recurrent state from transient state and an absorbing state, you have the closed sets. What we want to say is that, each closed set if you look at, this itself behaves like now a reducible Markov chain. So, a subchain of the whole system, which is irreducible – ergodic in the sense that, if you just consider this. And similarly, if you consider the chain consisting of sets 3 and 7, then they together form a irreducible chain. And this is both of them – 3 and 7 communicate with each other. And so, talk about… Further classification is needed.

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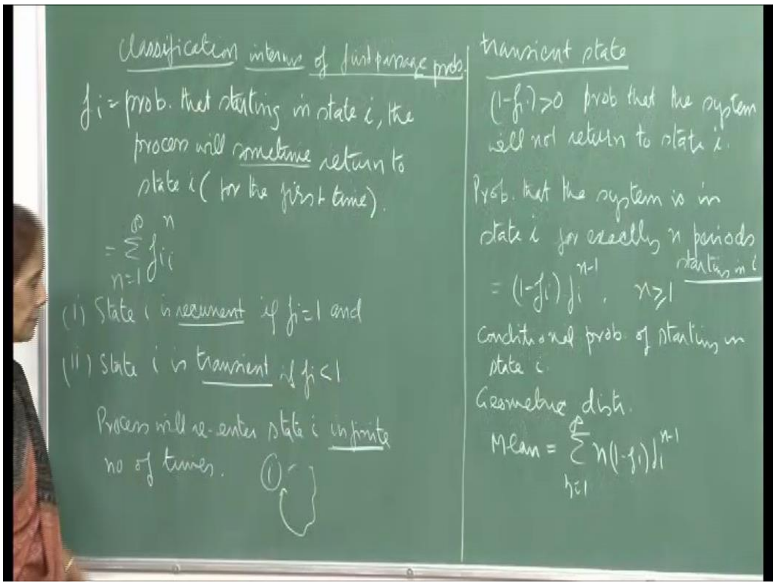

So, now, I will give you another classification of these states in terms of first passage probabilities. And that will also give you a good understanding. And of course, another alternate way of determining whether the state is recurrent or transient or whatever it is. So, now, let us define f i as the probability that, starting in state i, the process will sometimes return to state i for the first time, because f i i n was the probability that, it returns to state i from i for the first time in n steps – n transitions. So, now, when we add up this from 1 to infinity, this will give us the probability of the system returning to – starting from i returning to i for the first time in any number of steps. So, all possible. Therefore, this is will sometimes return to state i. So, that is important. So, this is a probability.

Now, we say that, state i is recurrent if f i is 1; so, that means there is a positive probability or it is a certain event that, the system will sometimes return to the state i from which it started. So, if f i is 1, then it is a certain event. But, if f i is less than 1, then it is a transient state. And the… So, now, you can interpret this as saying that, if f i is 1; that means the system will return. So, starting from i, it will return to itself after sometime. And then, because it is a Markov process, memory less; that means the past history is not to be considered; then, you will again… So, the whole process starts fresh; and then, you will again go to other states and so on, and then come back again to i. So, every time you come back, the system starts fresh. And since f i is 1; therefore, you will keep coming back to i infinite number of times. Therefore, the way to characterize this is recurrent state; that means is that, once you come to i, then you will keep coming back to i an infinite number of times. But, for a transient state, see what is happening is since f i is less than 1, 1 minus f i is positive. So, this is the probability that the system will not return to state i. And so, if this is a positive probability, then we will say that, this event will also occur. Therefore, how you want to interpret this saying that, a transient state will be visited only a finite number of times. And so, that will be the difference between a recurrent state and a transient state. So, let us just interpret this.

Now, you see if the system is in state i for exactly n periods; starting in i and exactly for n periods, visit the state i again for n periods – n times; then, the probability of that is f i raise to n minus 1, because f i is the probability of x returning to state i. So, that into n minus 1. And again, these are independent probabilities. So, that is why I am raising it to f i raise to n minus 1, because of the Markov process, the Markov property. And then, it does not come back to itself. So, for what… So, I am computing the probability that, exactly for n periods, it has visited state i starting from i; I mean this is I am writing down the conditional probability of starting in state i and then visiting it for exactly n times – being in state for n times. So, it has already started in state i. So, now, it needs to revisit state i n minus 1 times. So, this is the probability of revisiting state i n minus 1 – n times starting. And then, 1 minus f i is the probability that it will never come back after that.

Now, this is… And for n greater than or equal to 1, this is now… This has the probabilities from a geometric distribution if you recall. So, what we are saying is that, coming to itself is a failure and then not coming to itself is a success, remember. So, you are asking for the probability in n trials; you are asking for the… Here this is the… The best way to say it is that, this is not coming to itself. So, that happens once. And then, this is coming back to itself n minus n times. Or, coming back to itself – it is n minus 1, because it is starting in i, but it has been in state i for exactly n periods, that is… So, there are two different things I am saying. Therefore, here you are saying revisiting itself n minus 1 times. Therefore, f i raise to n minus 1. Now, when you want to compute the mean for this distribution, then it will be… that means n now varies from 1 to infinity and you compute the mean. And it will be n times 1 minus f i into f i raise to n minus 1.

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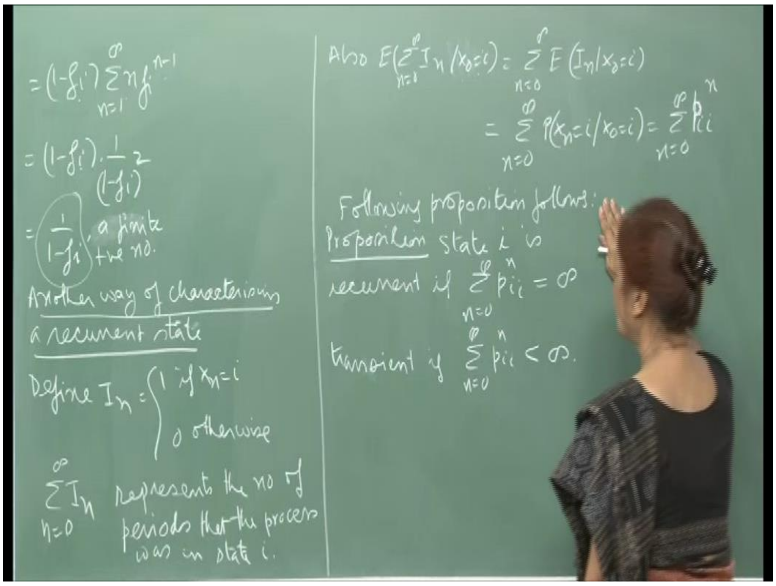

And, here we can take 1 minus f i outside. So, sigma n f i n minus 1. And this is a geometric distribution; f i is less than 1. So, this is convergent; and this is an arithmetico-geometric series. And you should all know how to evaluate it; we have done it right in the beginning of this course. And so, the sum is 1 upon 1 minus f i whole square; and so, multiplied by 1 minus f i. So, this is this. So, there is a finite; this is the finite positive number; that means the average of a… You see if it was to visit itself, infinite number of times, the mean will not be finite. But, here it is the mean is finite; so, that means, you can again interpret this as saying that, the state will not be revisited infinite number times; only a finite number of times it will be revisited.

Now, another way of characterizing a recurrent state; so, you see that, we have been trying to talk about the same thing in different ways. And that certainly helps you to understand other things better. So, now, let us say define I n as 1 if X n is i; that means if the n-th time the system is in state i and 0 otherwise. So, this is an indicator variable. And then, if you add up sigma i n from and to infinity, this will represent the number of periods that the process was in state i. So, starting from time period 0 – the initial time, then this will count the number of times the system was occupying state i. And now, if you want to compute this conditional expectation, that is, given X naught is I; you want to compute sigma I n, n varying from 0 to infinity – the expectation. So, I will exchange this summation sign; that means if this thing is finite; that means if this exists, then obviously, I can exchange the expectation. So, this is also convergent; that is all I need to interchange. Therefore, this is expected value of I n given X naught is i, which… But, I n is equal to 1 if X n is i. Therefore, this is probability X n equal to i into 1 when you… Conditional probability of X n given, i given that X naught is i. So, 1 times; that you will write down.

And, this by our definition, is P i i raise to n. Therefore, summation n varying from 0 to infinity P i i n. So, you want to say that, sigma I n, n varying from 0 to infinity represents the number of periods that the process is in state i. So, now, if… So, once you have computed this, then a proposition immediately follows. A proposition says that, if state i is recurrent, that is, state i is recurrent if this is infinity, because this is the expected value. And if the definition is that, the recurrent state will keep revisiting itself infinite number of times; then, the expected value will also be infinite.

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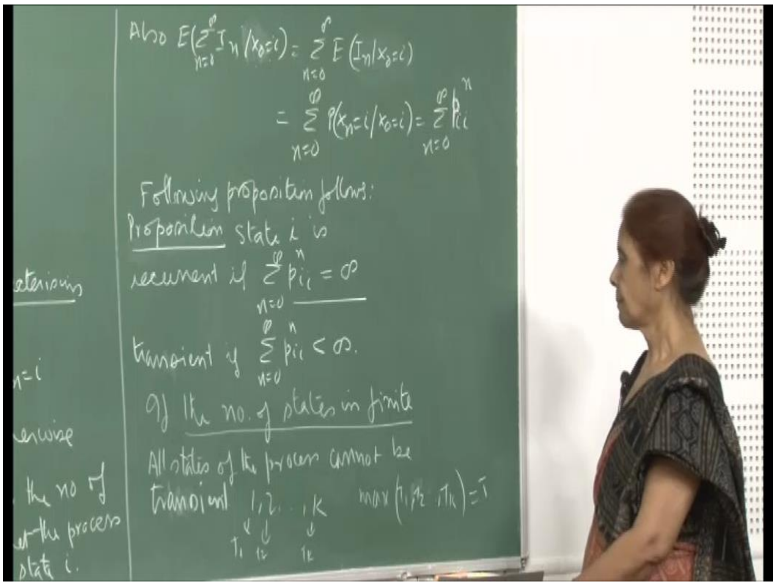

Therefore, this will be infinity. And for transient, this has to be less than infinity; so, another way of looking at it. Now, see interesting outcome from here is that, if the number of states is finite; see that is important, is finite. So, the number of states is finite; then, all states of the process cannot be transient – all states cannot be transient. And why? Because if a transient state can be visited only a finite number of times and you have finite states; let us say the number of states are 1, 2 and k; you have k states. Now, this is a… If all are transient; so, this can only be visited let us say T 1 number of times; this can be visited only T 2 number of times, and this can be revisited only T k number of times. And now, you take max of T 1, T 2 and T k; take the maximum of this; define this as capital T. And then, now, when you consider the time T plus 1, T plus 2, what happens? Because all the transient states have already been visited; you cannot revisit, the process has to go on; the process has to be in some state at time T plus 1, T plus 2. So, this is a contradiction, because if you have only finite number of states and all are transient, they will all get visited the finite number of times.

And after that, beyond that time period, the process is going on. So, where does it go? It has to transit to one of the states. Therefore, in a finite process, a finite process we mean finite number of states – all states cannot be transient. Therefore, immediately, the question is asked. If the number of states is not finite, then is this possible that, all states may be transient and yes. So, we will also look at an example. And we have already done, looked at such a process, but we did not really talk about this aspect of the process at that time. So, yes, if there are number of states are infinite, then all the states maybe transient.