Introduction to Probability Theory and its Applications
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Lecture - 27
State Prob First Passage and First Return Prob

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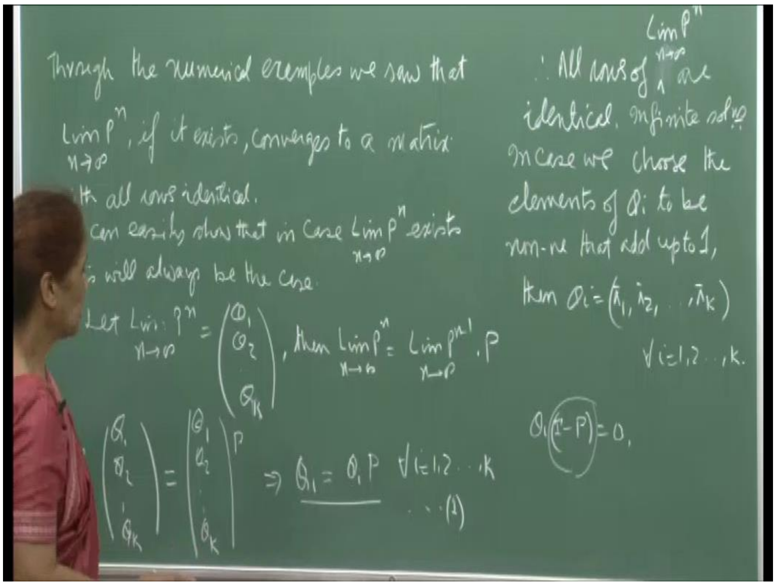

So, through the numerical examples we saw that limit p n - p is a transition matrix - so the nth power, as n goes to infinity, if this exists - if the limit exists - then it converges to a matrix with all rows identical. Whatever two-three examples we considered, we saw that the limit existed, and then, we saw that the rows were becoming identical as we increase the value of n; that means, we continue taking higher and higher powers of p. We can easily show that in case limit p n exists, limit p n then goes to infinity - if this limit exists, this is always be the case; that means, whenever this limit exists, when this will converge to a matrix whose rows are all identical. So, let us show this immediately, very easily.

So, limit p n; suppose, n goes to infinity. So, Q 1 is a row, Q 2, Q k; suppose these are k rows. We are considering the system when it has k states, k possible states. So, then I can write limit p n as n goes to infinity as limit p n minus 1 into p, as n goes to infinity. Then as n goes to infinity, this and this have the same value. So, this same matrix; they will converge to the same matrix. So, this will be Q 1 Q 2 Q k is equal to Q 1 Q 2 Q k p. So, this reduces to the limiting case; this reduces to this system. Therefore, from here, you can say that Q i, the i th row here, would be the i th row multiplied by p, post multiplied by p. So, this is it for all i. And hence you can see that all rows of p are - I should not say of p; what I want to say is, that if it converges to… p is your transition matrix. So, all rows of p raised to n will converge to… So, all rows of limit p n, I should write here limit p n as n goes to infinity are identical.

Now, if you want to solve for this, you can see that immediately, see you know that the rows of p, the rows of p have the property, because it is a transition matrix. So, all rows add up to 1. And therefore, this is not a non-singular matrix. And so, here you will have infinite solutions, to this system you will have infinite solutions; but then, if you also require the elements of Q i to be non-negative and they add up to 1, that means, we are looking for a solution where the Q i (s) - the elements of Q i - form probabilities, then this will be a unique solution and I will denote this solution by Q i is equal to pi 1 pi 2 pi k and these will be known as the steady state probabilities. So, it means when the system has a steady state, it has gone on for a long time, it settles into a steady state, then the probability of being in state 1 is pi 1, probability of being in state 2 is pi 2, and up to pi k. So, this is the whole idea.

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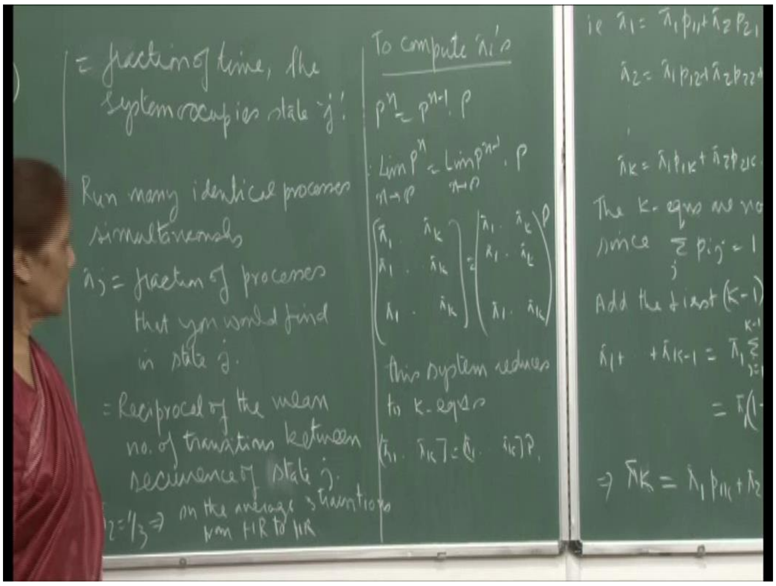

Now we will come up with the method of obtaining these values pi 1, pi 2, pi k - the steady state probabilities. So, now let us evolve the method for computing the pi i(s) - the steady state probabilities. See, p n can be written as p n minus 1 into p, the n th power of the transition matrix. So, then, if I take limit on both sides, then this is the limit p n and n goes to infinity; and this is limit p n minus 1, n goes to infinity into p. Now, as we said since we have assumed that the pi i(s) exist; and so, each row of p n in the limiting value would be pi 1, pi 2, pi k; so, all the rows are identical. Therefore, on this side also you get the matrix pi 1 pi 2 pi k, pi 1 pi 2 pi k, and so on. Similarly, p n minus 1 will also converge to the same matrix and this times p. So, the limiting behavior, I can just break up this in this way and then do it. So, if this is going to the limit, in the limiting value to this matrix, this will also go to the same matrix, and therefore, you have these equations.

Now, this system actually, since the all rows are identical, so actually these k equations reduce to this. So far, I was talking about the three-state processes. So, now, let me just do this much in 3, for the general case, and then we will come back when we want to talk of specific values and examples, we will again revert back to the three-state example that we have been talking about. So, let us just talk about it in general, and therefore, the system reduces to k equations; that means, I can simply just equate the first row here to the first row here. So that means, pi 1 to pi k is equal to pi 1 to pi k times the matrix p. And now let us write out the equations in detail here. So, pi 1, the first component here, will be this multiplied by the first column of p.

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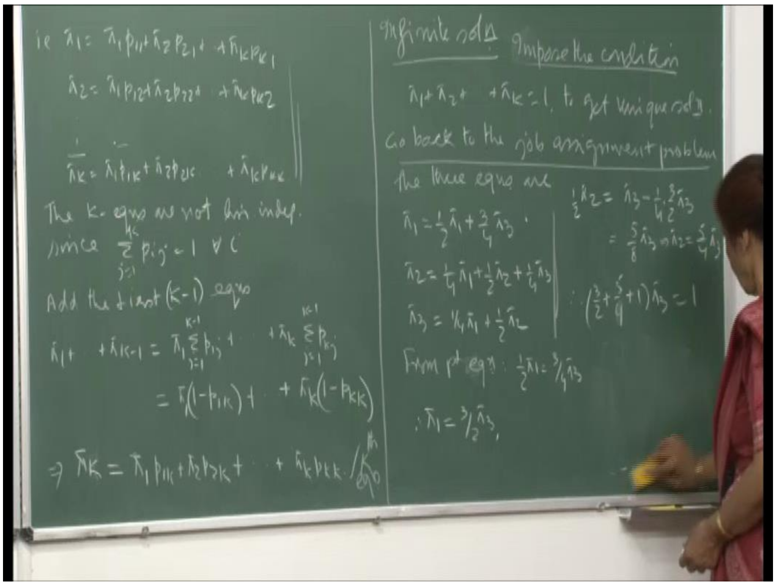

So, the first column p is… so pi 1 p 1 1 1 plus pi 2 p 2 1 and so on plus pi k p k 1. Similarly equate the second, component here, element to the second element; that means, this multiplied by the second column of p; so that gives us this. And finally, the k th equation is pi k is equal to pi 1 p 1 k. So, the k, the k th column we will take when we equate pi k with this multiplied by the k th column. So, I have these k equations, but we can immediately see that these k equations are not linearly independent, since sigma p i j, j varying from 1 to k is equal to 1. So, let us just quickly check this - that is, all these equations. See essentially what I am saying is that your first k minus 1 equations will give you the k th equation. So, therefore - those of you who are familiar with the word rank - so, the rank of this matrix is k minus 1 or you want to show that.

So, let us add the first k minus 1 equations here. So, it will be pi 1 plus pi 2 plus pi k minus 1 and this is equal to... So, when you are adding the first k, so you will be adding p 1 1 plus p 1 2 up to p 1 k minus 1. So, it will be pi 1 into summation p 1 j, j varying from 1 to k minus 1. And similarly, pi k into summation j varying from 1 to k minus 1 p k j.

Now, since the rows add up to the transition matrix we have to put this. I mean we know this that these rows of the transition matrix will always add up to 1. So, therefore, sigma j varying from 1 to k minus 1 p 1 j is actually 1 minus p 1 k. Because this plus p 1 k is 1. So, therefore, this sum is equal to 1 minus p 1 k, and similarly I substitute for all of these sums by. So, this one will be 1 minus p k k.

And now you see that when you open up the bracket, so, pi 1 plus pi 2 plus pi k. So, pi 1 plus pi 2 plus pi k minus 1 cancels out, you are left with pi k, and the other things you transferred to this side. Then you immediately get pi 1 p 1 k plus pi 2 p 2 k and pi k p. So, this is your k th equation. So, because the probabilities of the rows sum up to 1, therefore, these k equations are not linearly independent. So, in fact, the first k or any k minus 1 will lead you to the k th 1 essentially, because here I just choose the first k minus 1, you can choose any k minus 1 and you will be able to obtain the remaining one by adding the k minus 1 equations you have chosen.

So, therefore, infinite solutions - because the matrix is singular; the equation matrix is singular, but when you impose a condition because since we are looking for these steady state probabilities and they must add up to 1, pi 1 plus pi 2 plus pi k has to be 1, because a system will be occupying one of the states - either one, two or k minus 1 or k. So, when you impose the condition that pi 1 plus pi 2 plus pi k is 1, then you get a unique solution. And so, so therefore, we have a very neat way of computing these steady state probabilities and we know that we have a unique solution. So, you cannot say that, you know, that the probability of - the long run probability of - being in a particular state of the system, here you know the probabilities are more than 1; that would not be reasonable solution.

So, now let us go back to your job assignment problem, and let us try to obtain, because I was trying to get you to have a look at the steady state probabilities by taking the powers of p, but now here, this seems to be a quicker way of and a neater way of solving, of trying to get the pi (s). So, because when you are taking the powers you really do not know when to stop; or in fact, you would have to go on doing it till you see that the values are really closing in. So therefore, this would be a better way to get your steady state probabilities. And so, the three equations you see, you can see from pi 1, your this thing p 1 2 is … when you are writing the equation your p 2 1 is zero. So, here you get, yes, the matrix is there in your earlier lectures. So, these are the three equations essentially for solving pi 1 pi 1 pi 2 pi 3.

So, therefore, I can from this equation I immediately get… So, the trick would be that since you know I do not get unique solution to this system, so I will solve for pi 1 and pi 2 in terms of pi 3 and then I will apply the condition that the sum is equal to 1 to get the value of pi 3 and then I will get all the values. So, from here you see - you immediately see - that half of pi 1 is 3 by 4 pi 3. So, that gives you that … where is pi 1? Yes, half of pi 1 is 3 by 4 pi 3. So, pi 1 is 3 by 2 pi 2, 3 by 2 pi 3, and then you can substitute here for pi 1 in terms of pi 3 to get your pi 2. So, pi 2 comes out to be 5 by 4 pi 3. Because this is half pi 2 and this is pi 3 minus 1 by 4 into 3 by 2 pi 3. So which makes it 3 by 8. So, 5 by 8 pi 3; therefore, pi 2 is 5 by 4 pi 3.

So, now I substitute the values pi 1 is 3 by 2 pi 3, this is 5 by… this is 5 by 4, 5 by 4 pi 3 and this is 1. So, therefore, how much is this? I suppose I will have to redo this thing or maybe this was right, I do not know. So, what is it? This will be 6 plus 5 plus 4. So, the value was ok. This was a mistake here, but the value I had computed was ok. So, this is this. So, therefore, your pi 3, pi 3 is 4 by 15. And so, this gives you pi 2 equal to 5 by 4 into 4 by 15, the value of pi 3. So, that makes it 1 by 3. And this pi 1 would be? pi 1 is 3 by 2. So, 3 by 2 into 4 by 15, which is this, and 3 by 5. So, pi 1 is 2 by 5. And now, let us compare these values with what we had obtained by taking powers of p.

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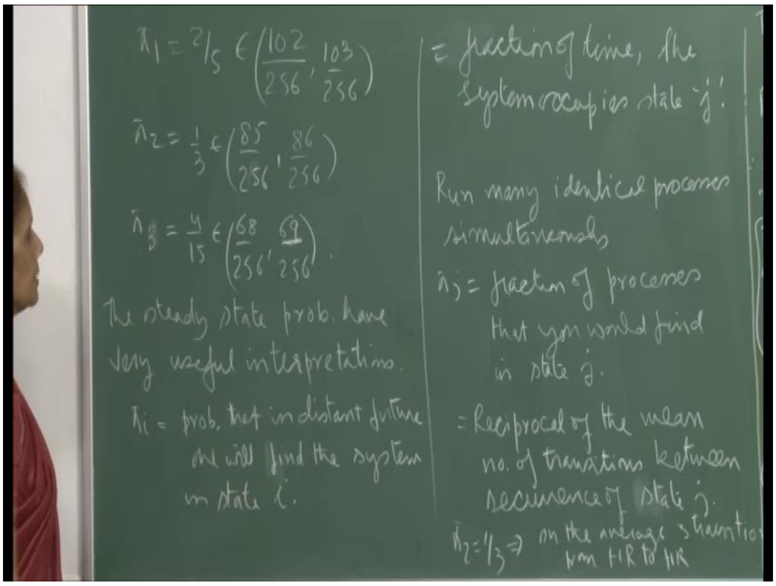

So, our pi 1 had come out to be something because two values of 102 upon 256 and the other one was 103 upon 256. So, you see, 2 by 5, does lie between these two numbers. So, this was up to fourth power. And when you take the fifth and sixth powers, you will see that values will get closer and you will actually reach 2 by 5. Similarly, your pi 2 is 1 by 3, and this is also a number lying between 85 upon 256 and 86 upon 256. You can compare; right this is between. So, 1 by 3 lies between these two. And similarly, 4 by 15 is a number which is between 68 by 256 and 69 by 256. So, the two things matched, but certainly that is a much better, quicker, way of obtaining your steady state probabilities.

Now, these steady state probabilities have very useful interpretations and we will continue seeing through examples, through on, when we analyze the process further. So, essentially what we have said is that pi i is a probability that in distant future one will find the system in state i. So, the probability that your process will be, that means, a particular employee in the automobile manufacturing company, that particular employee, will be in lets say HR, when you know after the process has gone on for lets say for 4 years or 5 years, then we expect the person, the probability, that the employee would be in the HR division is 1 by 3 or the probability that he will be with sales is 4 by 15. So, these are the long term probabilities. And as we said that the initial, that means, the division or the section in which he started his career is irrelevant here. Then you can also interpret this as the fraction of time the system occupies state j; fraction of time the system is occupying the state j. I am writing pi i. So, it should be pi i here. Yes, this is i. So the fraction of time the system occupies state i.

Now, if you run many identical processes simultaneously, then you see, that the pi j would come out to be the fraction of processes that you would find in the state j; that means, if you suppose run hundred identical processes simultaneously, and you find out that maybe 45, 45 of the processes are that particular time, of course, you let the processes run for a long time, and then after a certain particular period of time, you just find out how many of these processes are occupying state j. So that comes out to be 45, then your pi j would be, pi j would be approximately 45 upon 100; that will be the fraction of processes that you would find in state j. And another interesting interpretation of the state pi j is, you know, it is a reciprocal of the mean number of transitions between recurrence of state j. So, this recurrence I will define formulae after some time which is also very important.

So, what we are saying is that, this is the reciprocal of the mean number of transitions. So, on the average how many transitions would be required to go from state j to j? Now recurrence means for the first time; that means, you are in state j to start off, and then for the first time when you revisit j. So that number of transitions, if you take the average of such transitions, then the reciprocal of that is your pi j. And this also we will derive in another way, and of course, this part we will prove later on.

So, for example, what we are saying is that since pi 2 is 1 by 3 and pi 2, 2 is our HR section. So, we are saying on the average three transitions will be required for this particular employee to go from HR to HR. That means, if he starts his career with HR - Human Resource section - then he will after 3, on the average, he would be requiring 3 transitions to get back to HR. So this is the interpretation. So many interpretations that we have we can give, and then, we will see how we make use of these steady state probabilities to analyze these processes further.

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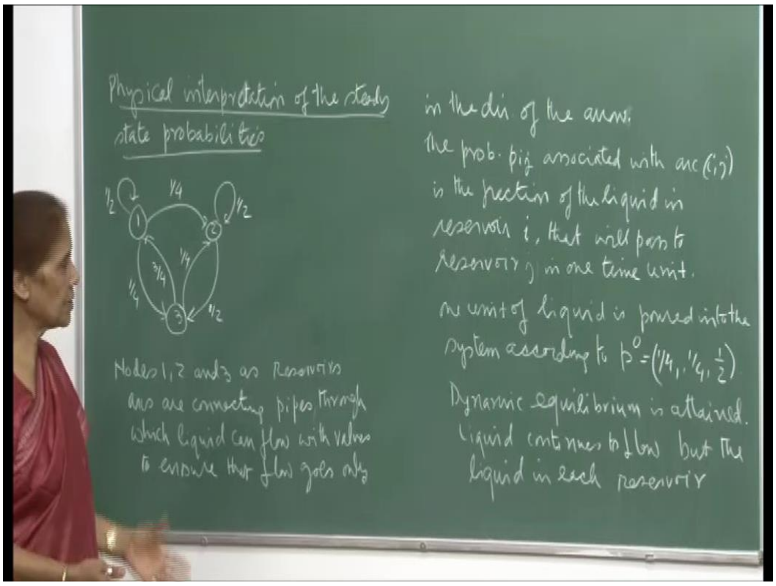

So, let us now… I have taken this example from again Rabindra, Phillips and Solberg. So interesting physical interpretation of state probabilities. So what he saying is that, in the, you know, job assignment problem, you know, you would consider the three states as three reservoirs. So, node 1, and node 2, and node 3 - they are reservoirs, and the arcs connecting the nodes are the pipes through which liquid can flow with valves to ensure that flow goes in the direction in which the arrows are there.

For example, there will be valve here, which will direct the flow from 1 to 3 only, and another valve which will direct the flow from 1 to 2, and then another one which will just direct the flow from 1 to 1. So, this is the idea. So, just think this as a reservoir - representing a reservoir - with these pipes connecting them. The reservoirs and then the valves to ensure that the flow goes in the direction in which the arrows are there. And then the probabilities p i j (s); that means, for example, the probability 1 4 associated 1 2 would be the fraction of the liquid that is there in 1 - in reservoir 1 - which will be sent to 2. Similarly, if you look at 2 to 3, then it is half the liquid, which is there in reservoir 2, will be send from 2 to 3 and so on.

So, these probabilities then can be interpreted as the fraction of the liquid in the reservoirs. So, p i j is the fraction of the liquid in reservoir i, that will pass to reservoir j in 1 unit time. So it will take 1 unit of time for the flow; so that means, half of the flow from here to here in 1 unit of time will go from 2 to 3, because the probability is half.

So, if we think of the system as, you know, made like this, then what you do is, you pour 1 unit of liquid into the system according to these initial probabilities; that means, one-fourth of the liquid is put in reservoir 1; a one-fourth is put in reservoir 2; and half the liquid is put in reservoir 3. And then, the liquid is allow to flow according to this plan. Then what we are saying is that dynamic equilibrium. So what we have discussed - that the probabilities will converge and they will become irrespective of how much liquid was initial poured into the reservoirs. So, finally, dynamic equilibrium will be attained and the liquid will continue to flow, but the liquid in each reservoir will equal the steady state probabilities.

So, our steady state probabilities… I do not have the numbers here, but whatever we had computed pi 1 pi 2 pi 3 as, for example, I remember pi 2 was 1 by 3. So, pi 2 would be the… that means, reservoir 2 will have one-third of the liquid, and then pi 1 would represent the amount of the liquid that is in reservoir 1, and pi 3 will be the amount of the liquid present in reservoir 3.

So, this is the interesting part. So what is being said is that, actually equilibrium will be attained and the liquid will continue to flow according to this plan, but each reservoir will settle down to - even though the starting amounts were this - each reservoir will settle down to the amount of the liquid according to these steady state probabilities; and what do we mean by that? So equilibrium means that for each reservoir the flow out will be equal to the flow in; then only whatever there is - the liquid is there - the steady state liquid that is there in the reservoir will be maintained, which is according to your pi 1 pi 2 pi 3.

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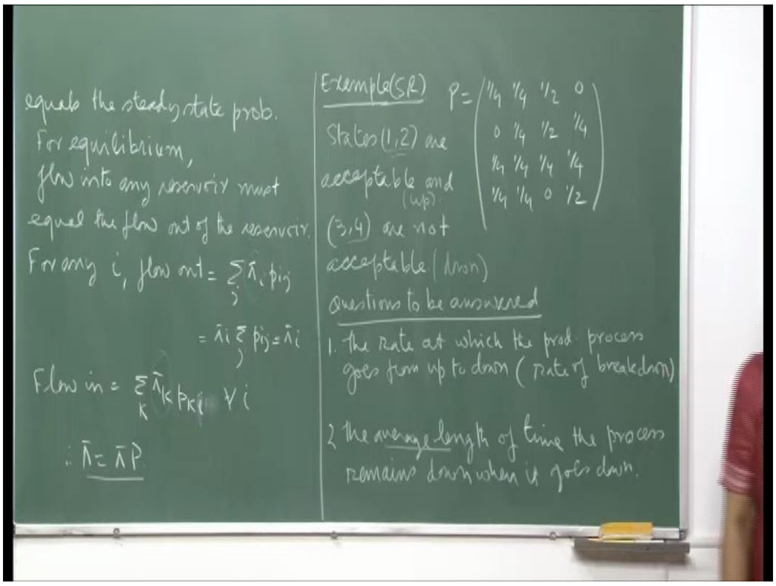

So, flow out from reservoir i would be… So, the probability that you are in a reservoir i into then p i j, summation over j. So, from i it can go to reservoir 1 to 2 to 3. So this is the probability that the amount of liquid flow out from reservoir i… and the amount … This you can write, when you summing up, you are summing up with respect j; so i can, pi i can come out, and this will be sigma j p i j, but sigma j p i j is 1, all these probabilities and the i th row will add up to 1; so this is pi i. And the flow in from other reservoirs that the flow is coming in. So that will be pi k into p k i; the flow is coming from the k th reservoir to the i th reservoir, and so this is the probability of being in the k th reservoir. So this is sigma pi pi k p k; well I am talking of probabilities, but here we are saying this is the amount that is there in the k th reservoir, and so this is the fraction which is going to i.

So, yes, I should actually interpret the whole thing in terms of this particular example. So, here also I should not refer to pi i as the probability of being in i, but this is the amount of liquid that is there in the i th reservoir and the p i j fraction of this liquid is being sent to the j th reservoir. So, therefore, flow out. So, from the i th reservoir this much is the liquid, and from this, these are the fractions of this liquid which are being sent to different reservoir. So this is a flow out and this is the flow in. So, please just interpret it this way; ignore my earlier remark. So this is a sigma pi k p k i, and so, the two must be equal, and therefore, you again get these, if you do it for all i, then you will immediately get this equation pi is equal to pi p. So I thought this was interesting way of looking at these steady state probabilities and somehow these will fixd certain ideas in your mind.

Then another example from Sheldon Ross that I want to … because I really want to spend time on these steady state probabilities, so that you get the ideas, you know, understand them properly.

Now, here this is an example where it is a production system and these are the probabilities of … transition probabilities. So we have four states 1, 2 and 3, 4. Now the states 1 and 2 are considered as acceptable or you can say when the system is up, and 3, 4 are not acceptable which you have to interpret as your system is down - that means, there is a breakdown, the machines are not functioning. So there are four states and this is your transition matrix from state i to state j and questions to be answered.

Now, the question that we want to answer are - the rate at which the production process goes from up to down; that means, the rate of breakdown. So when it is up, that means, the machines are working and then there is a breakdown. So you want to know the rate at which the process - the production process - goes from up to down.

And another question will be the average length of time the process remains down; that is also very important, because you want to know with this kind of transition matrix, you want to know for how long the process will remain. And of course, you always talk in terms of average length of time the process remains down when it goes down. So, when there is a breakdown for how long will it remain in that state before it comes up, but now the new thing is that you have two states which are describing the up situation and two states which are describing the down situation. So, therefore, I took up this example to again show you, of course, we will compute the p i(s) and then I will show you how to answer these and there are many more questions that have to be answered.

So, the third question you want to answer is the average length of time the process remains up when it goes up. So, these are the three questions we will try to answer by computing these steady state probabilities.

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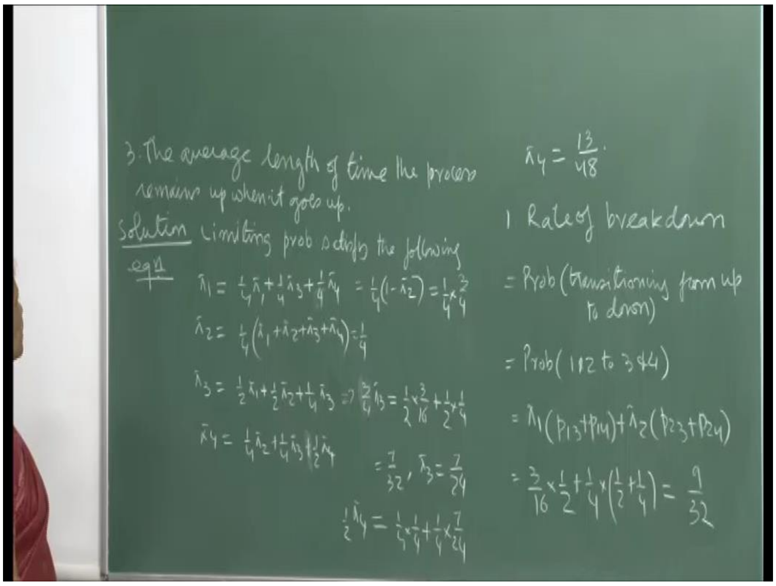

So, we write down the equations for the finding these steady state probabilities. So, pi 1 is equal to this pi 1 pi 2 pi 3 times the first column. So, you get this number, this equation then pi 2. So, you can just by looking at the transition matrix, yes the matrix p, then you can see that these are the four equations that we will obtain.

Now, interestingly the second column here is all 1 by 4, 1 by 4, 1 by 4, 1 by 4. So, when you write this second equation, this, you immediately get the solution for pi 2 because all these add up to 1, the state steady state probabilities have to add up to 1. So this immediately comes out that pi 2 is equal to 1 by 4. So, I have used it already here and now since I have the value of pi 2, I should be able to immediately compute the values of pi 1 pi 2 and pi 3. So, what I do is here, yes… no that is not… I have to now after having got pi 2, yes I can now see from here… yes, no even here… So, I thought that it was immediate that you could compute after you have pi 2, then yes, yes, see here, yes, you see, that is why one has to be a little clever and use inspection. So, here this is pi 1 pi pi 3 and pi 4 and again the coefficients are 1 by 4. So, this I can write as 1 by 4 into 1 minus pi 2. See and I have the value of pi 2 already as 1 by 4. So, this again immediately gives me pi 1 as 1 minus 1 by 4 is 3 by 4 into 1 by 4. So, 3 by 16. So, your pi 1 is 3 by 16.

Now, I have the values of pi 1 and pi 2. So, then from here I can immediately get pi 3 because bring this here. So, this will be 3 by 4 pi 3, 3 by 4 pi 3 and I substitute the values of pi 1 and pi 2. So, this will be 1 by 2 into 3 by 16 plus 1 by 2 into 1 by 4; that gives me

7 by 32. So, pi 3 will be when you multiply by 4 by 3 gives you 7 by 24. And then, once you have, you now have pi 2 pi 3 and pi 4 is on this side, when you bring it will be half pi 4. And so, again substituting for pi 2 and pi 3 you get these values. And so, you get pi 4. So this was quick work. You know this was certainly faster than computing, you know, second, third, fourth powers of p, which is a 4 by 4 matrix. So, lot of multiplications if you start taking the different powers.

Now, let us try to answer the questions - rate of breakdown. So, rate of breakdown is a transition probability of transitioning from up to down; that means, up means when your machines are working or anyone of the machines are working and then any one breakdown will mean there is a breakdown. So, that means, you are transitioning from up 1 and 2 which are up states to their down states which are 3 and 4. So, that means, you want to say that if you are in state 1, so that is the probability into your transitioning from 1 to 3 or 1 to 4. So that is p 1 3 plus p 1 4.

And if you are in state 2, then the transitioning from up to down is 2 3 plus 2 4, p 2 3 plus p 2 4. So, we have all these numbers or the probability of a rate of breakdown will be 9 by 3 2. Because any one of the breakdowns, this means you are going from up to down. So, any one breakdown or two breakdowns, whatever it is, the probability is 9 by 32 and that is your rate of breakdown.

Now, you want to answer the second question, which is the average length of time, average length of time the process remains down when it goes down. And the other one is the average length of time it is up when it goes up. So both the things. So, let us define u bar as the average time the system is up, and d bar as the average time the system is down. Then your rate of breakdown is… now we are, you know, redefining or talking in now. So, then we will make the equations and try to find out u bar and d bar. This is the idea. So, rate of breakdown is 1 upon u bar plus d bar, because you know, see this is the average time it is up and the average time it is down. So, one breakdown at the rate of 1 upon u bar plus d bar, because this is the total time when it is up and then down, average time. So, therefore, 1 upon this will give you the rate of breakdown, because one breakdown for this time, this much period, one breakdown for this much period, and therefore, the rate is 1 upon q bar plus t bar. So, proportion of up time is then u bar upon u bar plus d bar. And similarly, you will define the proportion of a down time as d bar upon u bar plus d bar.

Now, let us try to find out. So, the definition of u bar will be pi 1 plus pi 2; this is because… this is the… your are either in the state 1 or state 2 that is where it is up. So, this is the probability of being in state 1 or state 2, and then so this is a proportion of time and this is your rate of breakdown 1 upon u bar plus d bar which we have already computed as this from here, this is your rate of breakdown; so that is this. And so this is a gives you your u bar, and then we compute d bar, and we will continue with the exercise.

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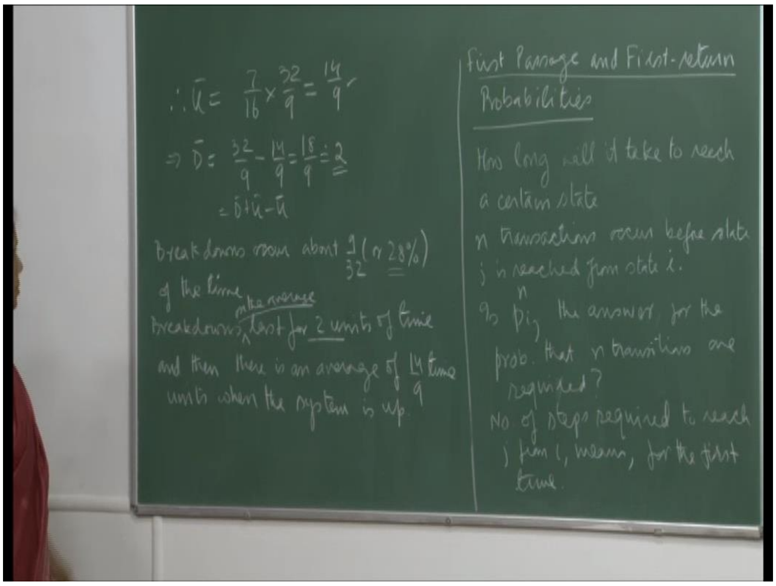

So, therefore, we saw that u bar will turn out to be 7 by 16 which is pi 1 plus pi 2 divided by u bar plus d bar which was your rate of breakdown. So, that was 9 by 32. So, then we multiply and it will be 32 by 9. So, this is u bar comes out to be 14 by 9; that means, this is the average amount of time for which the system will be up. And since u bar plus d bar is a 32 by 9, so to get d bar we will simply say d bar plus u bar minus u bar which is 14 by 9. So, that comes out to be 18 by 9, which is 2 units of time.

So, therefore, what we have been able to answer the three questions that were asked: the first was what is the rate of breakdown? So, this is 9 by 32 or 28 percent of time the breakdowns occur. And then, the breakdowns from the average last for 2 units of time; so, that means, once the system is down then it will remain down for 2 units of time. And the third question was the average amount of time for which it is up. So, then there is average amount of time for 14 by 9. So, 14 by 9 it is up, when the system is up. So, it will remain up for 14 by 9 times.

Now, the thing is certainly the system is not in a very satisfactory situation because your breakdowns on the average the breakdowns have remained for 2 units of time, whereas your actual production time is only 14 by 9 which is less than a 2 units of time. So, certainly the system is not in a very healthy state. And so, this again gives a warning to the manufacturer to do something about it, because the way the transition probabilities are given, this is what your conclusions are.

So, I hope you understand, see the way we computed this because we had to define u bar and d bar, and then of course, the other reason I took this example was that, you know, they were two states which were defining the up system and two states which were defining the down system, and therefore, we had to… these computations were not just straight forward. So, I thought that will be a good example to discuss in this course.

So, once we have talked about these state probabilities pi i(s) which we were answering as to the, you know because this, and of course, we have also computed this. So, for that means, number of transitions required to go from i to j, this was the answer, and then the pi i(s) gave you the… the pi i(s) gave you the long term probabilities of being, of the system occupying a particular state, and then all of course, we also said that this is a fraction of time that the system will be in state i. So, pi i(s). I gave you these interpretations of pi i.

Now, this other kind of questions that are needed which are now the first passage and first return probabilities. So, this is also very important, because you want to know how long it will take to reach a certain state. And so when you say how long will it take to reach a certain state… see the statement it means here that you know that will be the for the first time that you reached the state; so that means, you are starting from a state i and then you are wanting to say that how long will it take for you to reach state j. So obviously, the moment you reach state j you have answered that question. So, therefore, this will be what you mean by this is that for the first time that you reach state j from i. So, that is the understanding.

So, that means… now, for example, if n transactions occur before the state j is reached from state i. So, suppose we want to just, you know, surmise or say that n transactions have taken place before state j is reached from state i, and then we want to know the probability that n transactions will be required for going from i to j, then you might say that would p i j raised to n be the answer, for this probability? Because p i j n also tells you the probability of transitioning from i to j in step one, but see now there is a difference, because you see, when you talked about p i j n then it does not say that you may reach j before… and a number of times before you finally reach j in n steps. The various graphs that I drew for you earlier showed that you may, you know, like you had, you started from state 1 then you stay in state 1 and state 1.

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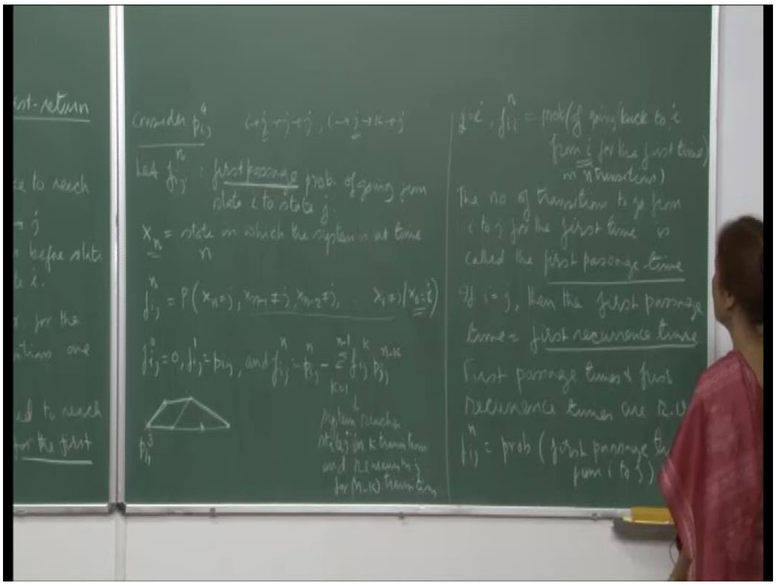

So, this was your, you know, transition probability from 1 to 1 in two steps or in three steps you stayed in state 1. So, here that is fine. So, this was your p 1 1 3, p 1 1 3, of course, also included that you could go from here to here, remain here, and then come here, or you could stay here, then go here, and go here. So, this is fine; this particular path that you are taking, no you are not coming back to state 1 before.

So, starting from state 1, you are reaching here without going to state 1 before hand. So, therefore, this is the kind of path you are looking for; but whereas, when we were computing your p 1 1 3, we had all possible paths to reach from 1 to 1 in three steps; that is what we were doing. Say, for example, if you consider p i j 4, then p i j 4, I could go to 4 in step two also; that means, from i, I could go to j, as I said and then you could again go to j, and then j; this would also be there. Then you will have i to j, and then, you could go to k, and then to j, and this way. So, so many paths, but nobody is stopping you from…

So, when compute the probability p i j 4, remember we said it includes all possible paths of going from i to j. And so you could revisit j in between number of times because you have to, you have to enumerate all the possible paths. So, therefore, this is not the answer what we are looking for. So, we need to make some more definitions and some more terminology has to be introduced to compute these probabilities. So, the first passage and first return probabilities we want to compute.

So, essentially what we were looking for is that for the first time I reach j from i. So, in between I should not have touched state j, and when the first time it occurs I want to compute the probabilities of such. And so, so what we will do is, we will make these definitions here. So, again f i j n, I am defining as the first passage probability. And so, I need to… I have written first passage here, but I will define it here. So, first passage probability of going from state i to state j. And remember x n was the state in which this system is at time n; this is… we have been using this notation when we were describing the transition probabilities on the Markov process.

So, now what are we asking for? We are saying that f i j n is equal to the probability of x n equal to j, but x n minus 1 is not j, x n minus 2 is not j, and x 1 is not j; it is only x naught; x naught is… x naught is i. So, starting from i in between all these n minus 1 transitions that take place you do not ever touch j, but it is only in the n th transition that you reach j. So, probability of that is what we were defining by f i j n; so, that means, probability of reaching j from i in n transitions for the first time. So, if you look at f 1, f i j is 0 then f i j 0 is 0 because we cannot transition; then f i j 1 will be p i j, your … f i j 1 will be simply p i j.

So, now we want to compute f i j n; that is probability of going from i to j for the first time in n steps - n transitions - for the first time. So, we should not have visited j in between, less than n steps. And so, this we will obtain by writing p i j n which is the total probability of going from i to j in n steps - all possible paths - which may imply or revisiting j number of times, and then finally, coming back to j. So, p i j n minus in sigma k varying from 1 to n minus 1, you see, because you want to reach for the first time for y to j and n steps. So, your k can be allow to vary from 1 to n minus 1 only, and then this will be f i j k. So, for the first time you have visited from i to j in k steps, and then once you reach j, and then again you can go to many other states, come back to j or stay in j whatever it is in n minus k steps.

So, for example, if you look at p 1 1 3, then what we are saying is that you can, you know, continue staying here in state 1 for all the three transitions or you can go somewhere here, come back and then again revisit here, and whatever possible paths you can, or you can go this way, this way, this way. So, many other paths you can think of. So, we are ruling out all those paths. So, we are subtracting; so, that means, you have visited from i to j in k steps for the first time, and then, in the remaining n minus k you again from j go to other places or remain in j and then come back to j. So, we subtract all these, then we get the probability that f i j, we subtract these from p i j n. So, we get the probability that probability of visiting j from i for the first time in n steps. And then we will see lot of applications and implications of these probabilities. So, this is what we, how we write down the expression for f i j n.

So, let me… and of course, when j is equal to i, then f i i n would be the probability of going back to i from i for the first time. So, you started from state i, and then when you for the first time you reach a state i again in n steps of going. So, I should say from state in n transitions; so, in n transitions. So, this will be f i i n and that will be the probability of recurrence of i, state i, in n steps. So, f i i n and I used the word recurrence earlier and I said we will define it later on, so that is it. And we will, of course, be talking in detail about these first passage times and first return time.

So, first return probabilities is your recurrence probability, a probability of recurrence; but here, of course, we are saying in terms of n transitions and then you would want to know the probability of ever returning to state i, and so, we will compute that also.

So, now the number of transitions to go from i to j for the first time. So, transitioning from i to j for the first time is called the first passage time. So, the number of transitions required to go from i to j for the first time we will define it as a first passage time. And you can see that this is also a random variable because you do not know how many transitions you will require to go from i to j for the first time. So, first passage time is a random variable, and if i is equal to j, then the first passage time is called the first recurrence time. So, this will be when you want to come back to the same state, starting from your particular state you want to come to it for the first time, that will be your first recurrence time.

So, first passage times and first recurrence times are random variables; and therefore, you can redefine this again f i j n is a probability that first passage time from, probability of first passage time from i to j in n steps; that means, your random variables. So, f i j n is the probability of the first recurrence of the first passage time equal to n. So, f i j n will be because this is the time, first time return, for you go from i to j and when this value is equal to n you want to compute this, the probability of first passage time equal to n is your f i j n; this is how we will define.

We will go through now a very interesting journey when we want to, you know, compute these f i j n and f i i n. In fact, you would finally, want to talk about f i j; so that means, that will be when for the first time you return from i, transition from i to j, so without the n because here this is transitioning the first passage time when it is equal to n. That means the probability of the first passage time equal to n. So, now you would want to then finally, compute your f i j and f i i. So, we will continue with this discussion.