Introduction to Probability Theory and its Applications
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Lecture - 22
Applications of Central Limit Theorem

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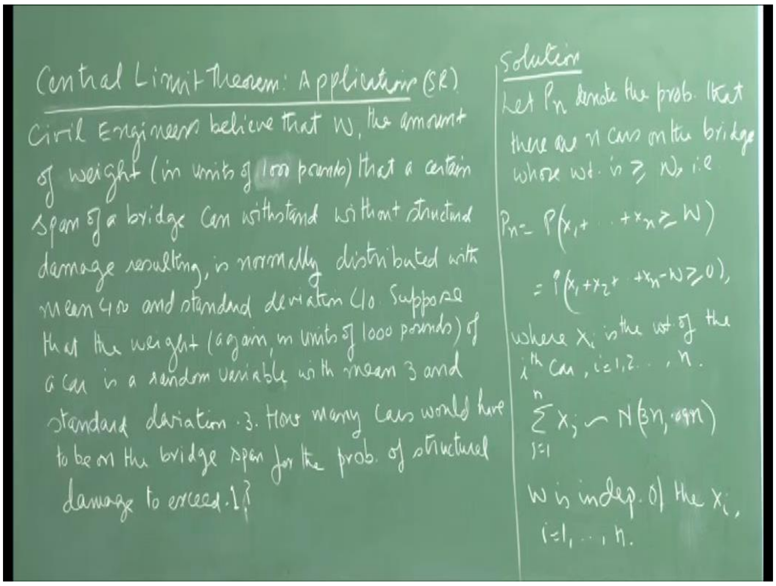

I will continue with the central limit theorem and its applications. This example I have taken from Sheldon Ross's book on probability theory. See the idea here is that, civil engineers believe that W, the amount of weight in units of 1000 pounds at a certain span of a bridge can withstand without structural damage resulting is normally distributed with mean 400 and standard deviation 40. So, the weight, which the bridge can withstand is random variable. And so it is normally distributed with mean 400 and deviation 40. Suppose that the weight again in units of 1000 pounds of a car, is a random variable with mean 3 and standard deviation 0.3. So, the different cars will have different weights. Therefore again we have treated this as a random… I mean this example – the weight of a car is treated as a random variable. And therefore the distribution… And the distribution is normal – approximately normal with mean 3 and standard deviation 0.3.

How many cars would have to be on the bridge span for the probability of structural damage to exceed 0.1. So, at a particular time, how many cars are there, and then the weight of these cars exceeds the weight, which can cause structural damage. And so you want the probability of this whole random of this event to be more than 0.1. So, you want to estimate that. You want to estimate the number of cars that would be on the bridge, so that the structural damage can occur. So, we begin by defining P n as the probability that, there are n cars on the bridge, whose weight exceeds W, because that is… So, the event is this that, when it exceeds w, the structural damage can occur. Therefore, this is same as P n. So, this is X 1 plus X 2 plus X n greater than or equal to W. And, that would be… We can rewrite this as probability X 1 plus X 2 plus X n minus W greater than or equal to 0.

Now, X i's; where, X i's is the weight of the i-th car; X i denotes the weight of the i-th car. So, this is the total weight of the n cars, which are on the bridge at that time. And therefore, by central limit theorem, because for n large, we have said that, when they are identically distributed random variables – independent random variables, because of weight of each car is independent of the other. So, then sigma X j, j varying from 1 to n would be approximately normal with mean 3 n and variance 0.09 n. Standard deviation is 0.3. So, the variance of the weight of a car is 0.09. And therefore, the variance of the n cars is 0.09n. So, this is approximately this.

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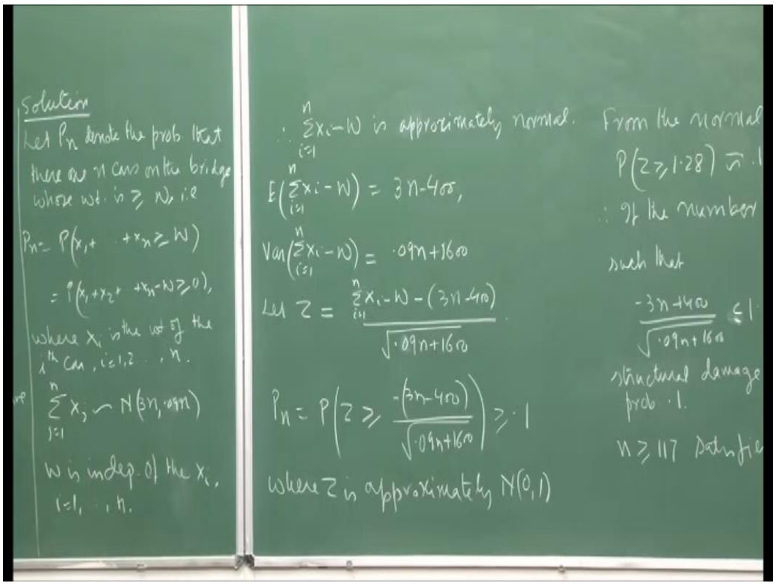

Now, W is independent of the X i's because the weight that the bridge can withstand is independent of the weights of the individual cars. And therefore, where I write sigma X i minus W, this is also approximately normal; yes. And, we will again revisit all these summation of random variables and their distributions. But, right now, we have enough machinery with us to say that, sigma X i minus W, because this is normal – approximately normal. This is normally distributed. So, sigma X i minus W is also approximately normally distributed. And, the expectation or the mean of this normal variate is 3 and minus 400 with minus W. So, mean of sigma X i, i varying from 1 to n is 3 n, and this is 400. And, the variance of course, becomes with the plus sign comes with the plus sign, because they are independent. So, variance of this plus variance of W, which is 1600. So, this is the variance. And therefore, I can standardize.

So, the whole idea is that, this is the variate I am looking at. And, I have said that, this is standard… This is normal distributed with mean 3 and minus 400 and variance this. So, when I standardize, I will say this minus the mean divided by the standard deviation. So, that is standardized. So, now, Z is the standard normal variate and the event. So, when I standardize this, this probability now can be written as probability Z greater than or equal to. So, on this side, it will be minus of 3 and minus 400; I mean bracket minus 400 divided by the standard deviation. So, when I do this operation, I mean this probability is the same as this probability, because this I have standardized the normal variate – standard normal variate by subtracting the mean and dividing by the standard deviation. Therefore, this is equal to this. And so Z is approximately normal.

And now, we want this probability to be greater than or equal to 0.1; yes. And so we look up the tables for the standard normal and we find that, when Z is greater than or equal to 1.28, this is approximately 0.1. So, from the normal tables, I get that, this number should be 1.28 for this to be equal to 0.1. And therefore, greater than or equal to you want. See the whole idea is that, if the number of cars and it is such… So, now, this number – I can say that is equal to 1.28. Therefore, if you take it equal to 1.28, then you get an approximation for n. And, in the sense that, if you write less than or equal to 1.28; then obviously, this probability will be larger. And therefore, the whole thing will still be larger than 0.1; this is the whole idea. So, I get a value of n by equating this to 1.28.

And then n should be greater than or equal to 117. So, here of course, this is a little complex thing to solve, but you can do it or you can start by putting in values of n; and then you can find out for which value of n this is almost equal to this or little less than this. So, one can… There are lot of numerical ways of actually getting the value of n, which satisfies this inequality. So, we can do that. And therefore, it turns out that, n greater than or equal to 117 satisfies the above inequality. And so… that means, if there are more than 117 cars, then these structural damage may occur with probability 0.1. So, there is a chance of 1 in 10 that, the bridge will suffer structural damage. So, this was another interesting example. Actually, you can see the application in the sense that… And then also I chose this example for the reason that, this also is a random variable. And therefore, to convert this event to this event; and then to reduce this to use the central limit theorem and transform this to a standard normal variate; and therefore, get the estimate of the probability that, the bridge may suffer structural damage. So, the interesting example of the central limit theorem.

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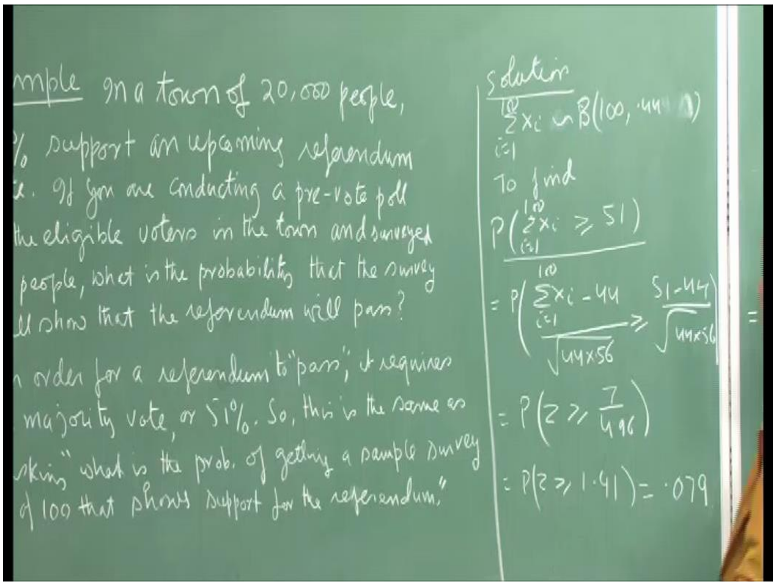

This is in a town of 20,000 people, 44 percent support an upcoming referendum vote. Say for example, currently, the hot thing is Anna Hazare going to form political party or not. So, you might take a referendum; that means you might ask people to vote on this whether he should do it or not. So, let us say… And, it is… That, the feeling is there. So, maybe this is a small town; and, the feeling is that, 44 percent will only support the upcoming referendum, but… So, then what you do is you conduct a pre-vote poll. So, this happens very often; media person do it; lot of magazines – they do it; they conduct their own pre-vote poll to get a feeling or the opinion – and, of the eligible voters in the town and surveys 100 people. Therefore, if for conducting a pre-vote poll of the eligible voters in their town and surveyed 100 people, what is the probability that the, survey will show that, the referendum will pass. So, one needs to understand what we mean by the referendum will pass. In order for a referendum to pass, it requires a majority vote or 51 percent.

See even though the feeling is there that, 44 percent support, but you never know at the time of the voting, more people may vote for the referendum and so on. Therefore, when you conduct a pre-vote poll and you surveyed let us say 100 people, then if in that pre-vote poll, it turns out that, 51 percent or more support the referendum; then you can say that, the pre-vote poll suggest that, the referendum will pass. But, actually, when the voting is done, and then if more than 51 percent people, who have voted; people who have voted – the 51 percent of those people – if they have supported the referendum, the referendum will pass. So, right now, this is just conducting a pre-vote survey of 100 people. So, then you want to know what is the probability that, the referendum will pass. Therefore, the question is… And therefore, that means, if you are taking a referendum – if you are taking a survey of 100 people, then you want 51 people to… Out of those 100 people, 51 should say yes for the referendum or support the referendum. This is what you want to find out.

So, the probability – therefore, one can model the situation using binomial random variables. So, X i is… I mean if the person supports; if the voter or the people you are surveying – they support the referendum, then X i will be counted as a success; otherwise, it is a failure. So, you will say that, sigma X I; i varying from 1 to 100 is binomial 100 with mean as 0.44 into 100, because probability of a success; that means, P is 0.44. So, I am writing here; I should have written only 0.44. This is not… This is only… So, the P is 0.44. And then the mean of the binomial distribution will be np. And, you want to find out the probability that, the people that you are surveying – the 100 people that you have surveyed, how many would support; that means, number of success is here should be greater than or equal to 51, because then the referendum will pass.

And, that is why I chose this, because this is depicting a new situation and we are just trying to model it through this thing here and applying central limit theorem. So, this is a whole idea. And therefore… So, I hope this is clear that, this is sigma X i, i varying from 1 to 100 should be greater than or equal to 51. So, from this 100 people, if they get a feeling that, 51 or more will support the referendum, then they can sort of advertise and they can try to influence people and say that, the pre-vote poll says that, referendum will pass and so on.

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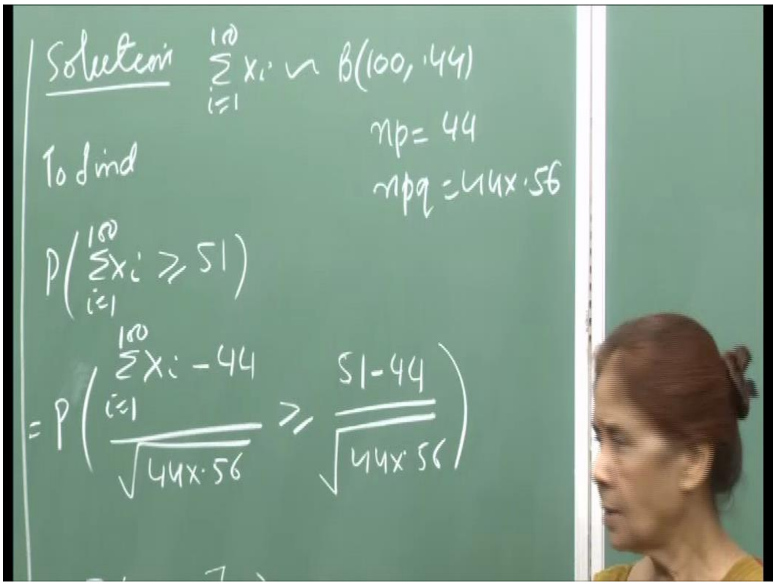

So, standardizing this variate – sigma X i, i varying from 1 to 100; this will be sigma X I; i varying 1 to 100 minus 44 – the mean of this random variable, which is np – 44 divided by the variance, which is npq. So, 44 into 0.56, because p is 0.44. So, q is 0.56. Therefore, this is the variance. And so under root of that – the standard deviation. Therefore, this probability is equal to this probability. So, this is greater than or equal to 51 minus 44 upon under root 44 into 0.56.

Now, as I have been telling you that, wherever you want to approximate a binomial probability by standardizing the random variable and using a standard normal probability, then you should also use the continuity correction factor, which I have not done here. So, anyway. Therefore… So, that would be… If you are saying greater than 51, then it will be 50.5; that would be the right figure. But, anyways, you can do that computation later on. So, right now, the whole idea is just to see that. Therefore... So, to get a feeling for the kind of numbers that you have that, will the referendum pass or not. So, this is this. And therefore, under root of this comes out to be 4.96. So, this probability; and, this is a standard normal variate. Therefore, probability – this is equal to; or, we are approximating this probability by probability Z greater than or equal to 7 upon 4.96, which comes out to be 1.41. So, Z greater than or equal to 1.41. So, this probability, which from the tables gives you the number 0.079. Therefore, this is a very small probability. And hence, the chance of the referendum passing is very slim.

Then, the town is 20,000 and you are only surveying 100 people. And, when you know that, the chances of… There is a 44 percent support the upcoming referendum. So, the probability of 51 percent or more support the referendum is small; and, that is reflected here. So, through the central limit theorem, you have made this approximation to the probability through the required probability and it turns out to be 0.079. So, the chances of when you survey the 100 people and ask for their opinion – whether they support the referendum or not, it shows that, the chances are very small for the people.

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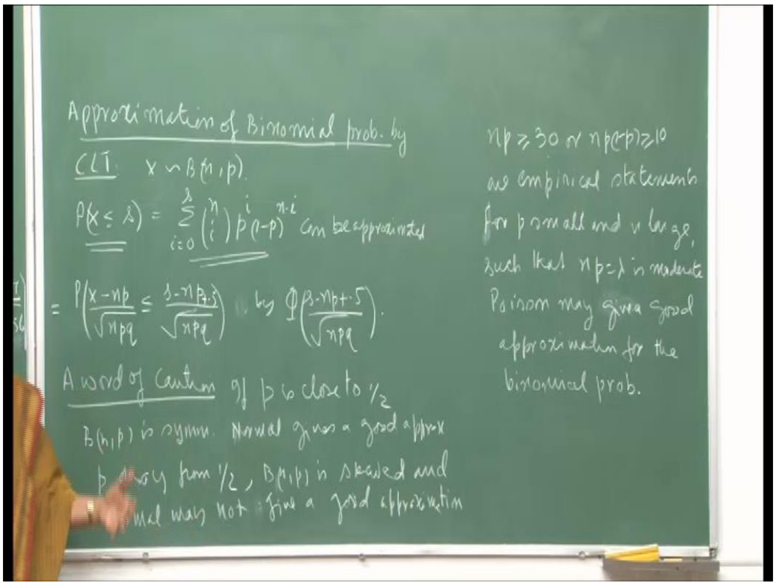

So, again, I mean one can go on and on about the applications of central limit theorem and how to various different situations you can apply it. When I want to get back to… And, other thing that, we had also sort of use… We had used the central limit theorem, but probably did not… And, I have said that, we will prove it later on. But, I just want to add word of caution also to it. So, here this is that, we had X equal to… X is a binomial n comma p; and then we said that, if you want to compute this probability – X less than or equal to s; then you will have to compute these numbers; and, this can be quite messy; i varying from 0 to s. And, see i p raise to i; 1 minus p raise to n minus 1. So, this can be quite too cumbersome to compute. But, then we said that, we can approximate it by a standardizing this thing. And so here this is X minus np divided by under root of npq – the standard deviation. And then this is less than or equal to s minus np.

Now, you add 0.5. Remember I had talked about the correction factor when the binomial is a discrete random variable and we are approximating it by a continuous distribution. Therefore, this continuity correction factor is also added. So, you have 0.5 and this. Therefore, this probability – this cumbersome thing can be approximated by the normal probability, which is s minus np plus 0.5 upon under root npq. And, we look up the normal tables and we can compute this number. Now, the thing is that, of course, when you are approximating, the question does arise – how good an approximation it is?

And, see what happens is that, when for a binomial distribution, if p is close to half, then the binomial distribution is symmetric; in the sense that, the values keep on increasing and decreasing in a symmetric manner. And then because normal itself is also symmetric distribution about its mean; therefore, a normal distribution will give a good approximation as long as p is close to half, because then you are approximating a symmetric distribution – a discrete symmetric distribution by a continuous symmetric distribution. And so… But, when the p is away from half, then the binomial will be skewed may be to the right or to the left. And, in that case, it is not necessary that, the normal distribution will give you a good approximation of the binomial probabilities.

Now, it is said often that, if np is greater than or equal to 30 or np into 1 minus p is greater than or equal to 10, then the central limit theorem will always give you a good approximation of the binomial probabilities, but… And, these are empirical statements. And, in some cases, it may turn out that, when you have np greater than or equal to 30 or np into 1 minus p greater than or equal to 10, you may get good approximations, but it cannot be said that, this will happen all the time, because certainly, symmetry plays a role. And, for p small and n large such that np equal to lambda is moderate; so then in that case, Poisson approximation may be a good approximation. And, I had… When we were discussing discrete random variables, I had shown you that, how a Poisson probabilities can approximate the binomial probabilities. But, then of course, the condition was that, p is small and n is large, and np is moderately small, is reasonable number; then Poisson may give a good approximation for the binomial probabilities.

So, with this word of caution, of course, these approximations can be used and they are very helpful. And so I just thought that, once we have talked about the central limit theorem, we have proved it and shown its applications. I will just revisit what we had done earlier when we talked about approximating the binomial probabilities by standardizing the variate – normal variate and reducing it to a standard normal variate, and then computing the probabilities.

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1. A random sample of size n=81 is taken from a distribution with μ=128 and standard deviation σ=6.3. With what probability can we assert that the value we obtain for will not fall between 126.6 and 129.4 if we use:
   (i) Chebychev's
   (ii) The central limit theorem.

2. Let the random variable Y_n have a distribution that is binomial (n,p). Prove that Y_n/n converges to p in probability.

3. Consider the sequence $\{X_n\}$ of random variables
   $$P(X_n = x) = 1 \quad \text{if } x=4+2/n$$
   $$= 0 \quad \text{Otherwise.}$$
   Does it converge in distribution to some random variable X ? If so, find the distribution function of X.
   Show that $\{X_n\}$ converges in probability to X.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. r.v with density function
   $$f(x) = 1/\theta \quad 0 < x < \theta \qquad 0 < \theta < \infty$$
   $$= 0 \quad \text{Otherwise.}$$
   Let $M_n = \max\{X_1, X_2, \ldots, X_n\}$. Find the distribution function F_n of M_n. Does F_n

Now, which has problems for you to try on Chebychev's inequalities, central limit theorem and law of large numbers – weak law of large numbers. Now, the first problem is straightforward; it says that a random sample of size and equal to 81 is taken from a distribution with mu equal to 128 and standard deviation sigma equal to 6.3. With what probability can we assert that, the value we obtain for X bar will not fall between 126.6 and 129. Chebychev's inequality. So, you can see that, it will be… You will have the absolute value. So, X bar minus… When you essentially… I was saying that would be greater than 129.4 and less than 126.6. So, I have given it specifically, because I want you to then convert it to the form of the… when you are saying that, it is… when you apply Chebychev's inequality or the central limit theorem. So, we have already tried given exam… I have discussed examples, where you can compute the probabilities given that, n is 81 and the standard deviation and mean are given to you.

Now, you just want to make a comment here is that, as we have seen through examples in the lectures that, the number n… For example, the probability that you get – the bound that you get by using Chebychev's inequality on the required probability would be loose bound; and, the central limit theorem will give you a tighter bound – a tighter this thing on the probability. Now, the thing is… And, of course, you can also say that… But, one point that is important is that, the probability when you compute it by the central limit theorem, may sometimes depend on the distribution that you are handling; whereas, Chebychev's is a universal inequality. And therefore, it may give you a loose bound; but, then the number does not change with respect to different distributions. So, Chebychev's is the general statement – a universal statement. And, later on when I have occasion, I will again point out the difference between the… Even though we say that, Chebychev's is a looser bound, there are other advantages of using the Chebychev's inequality.

Question 2 – that the random variables Y n have a distribution that is binomial n, p; prove that, Y n by n converges to p in probability. So, this is the use of weak law of large numbers. I may have already done it for you in the lectures; but, anyway go through it and try to prove it by yourself. Then, the third problem is consider the sequence X n of random variables, where p n is X; probability of X n equal to X is 1, if x is 4 plus 2 by n and 0 otherwise. So, now, here the probability… So, X n is equal to X – the probability of that is equal to 1, if X is 4 plus 2 by n. Does it converge in distribution to some random variable X? So, that means, find out the… You will define the cumulative distribution function. As n goes to infinity, can you find distribution bridge. If so find the distribution function of X; show that the sequence X n converges in probability to X also. So, should be interesting thing, but we go by the basic definitions and then try to solve the problem.

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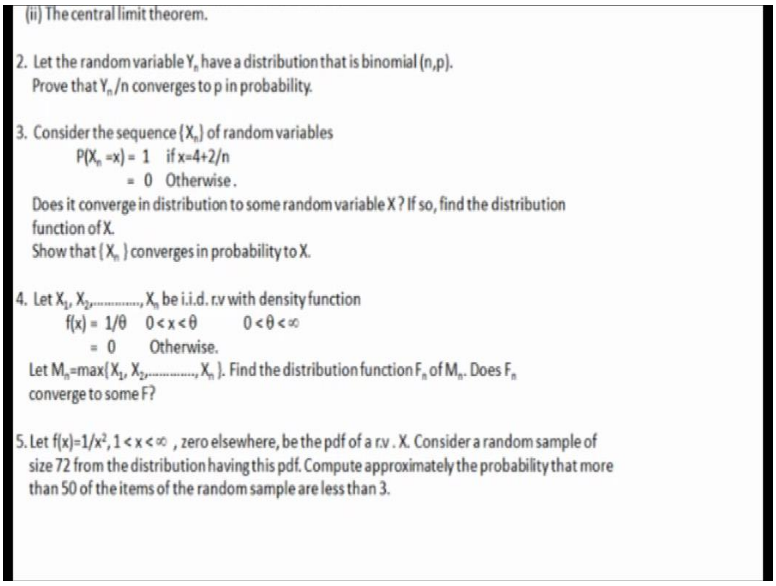

Upon X to X n are identically independently distributed random variables with density function F X equal to 1 by theta and 0 otherwise. It should be equal to this – 0 less than theta less than infinity. Let M n be max of X 1, X 2, X n. So, M n is the random variable, which is the maximum of these n sample values; find the distribution function F n of M n. Does F n converge to sum F's? Yes, it will. And, see but, we will not talk much about it because… The second part is a little difficult part, but you can certainly see that, F n will converge to some F. So, find the distribution function F n of M n. So, that part is okay; that you can do through the tools that you have already learnt, because when you find out the… To find the distribution function, you have to say probability M n less than or equal to t.

Now, since M n is the max of X 1, X 2, X n, this will reduce to probability that, each X 1 is less than t; X 2 is less than t; X n is less than or equal to t. And, since they are independent, this will reduce to probability X 1 less than or equal to t raise to n. Therefore, you can sort of do it in the regular way, and then see if you can get a feeling for convergence of F n; that is all; we will not talk in detail about it, because this becomes a little complex. If you have given that F X is 1 upon X square and X varies from 1 to infinity, 0 elsewhere.

So, this is how you are defining this pdf; and, this is the pdf of a random variable X. Consider a random sample of size 72 from the distribution having this pdf. So, that means, the sample – identically independently distributed random variables – they are 72 of them; compute approximately the probability that more than 50 of the items of the random sample are less than 3. See the thing is now – that the problem… I have include this problem, because these two steps. See first is that, you want the probability that, more than 50 of the items of the random sample are less than 3. So, there is a probability I will use this here.

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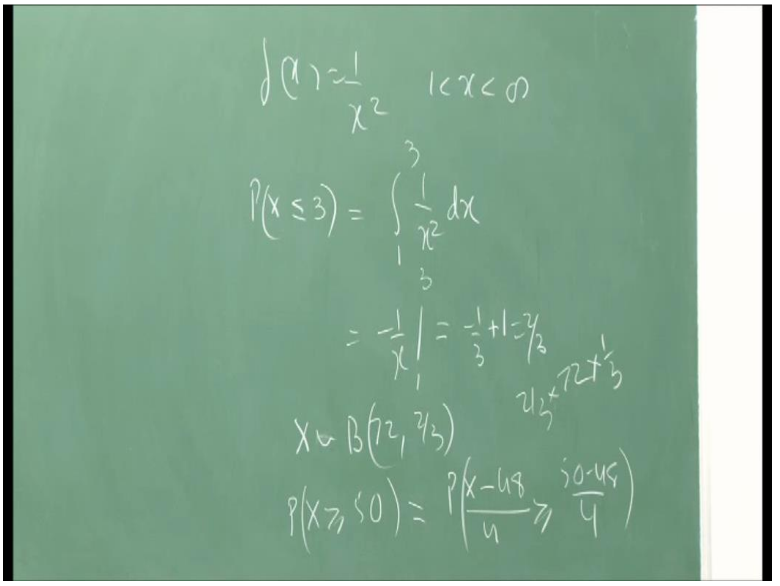

See you are given that, f x is 1 by x square; 1 less than x less than infinity. So, you are wanting to find probability x less than or equal to 3; this is the problem – that more than 50 of the items of a random sample are less than 3. So, this is x less than or equal to 3; this will be 1 to 3 of 1 by x square dx – the probability that random variable, which has this pdf. So, then the probability of x less than or equal to 3 will be given by this, which is minus 1 by x from 1 to 3. So, this comes out to be minus 1 by 3 plus 1, which is 2 by 3. So, now, what I will do is you are selecting a sample of size 72 and we will say that, if a sample has a value less than 3, then that is a success. Therefore, the probability of a success would be 2 by 3. So, now, this gets converted to a binomial situation; where we are selecting a sample of size 72 and we say that, if a sample value is less than 3, then it is a success. So, that means…

Now, the question is that, from a binomial 72 comma p – 2 by 3, I want a sample of the items. So, more than 50; that means, you want that, if you are writing sigma X i; so random variable X coming from binomial 72… Maybe I can write it here. So, essentially, what I am treating is that, X is binomial 72 and this is this. So, I am wanting that, probability X is greater than or equal to 50. And so when you standardize, this will be X minus… The mean is 2 by 3 into 72, which is… This is 24. So, 48 – 48. And, that will be 1 by 3. So, minus 9 – 16 – 4. So, this is greater than or equal to 50 minus 48; that is, 4; this comes out to be 9 ((Refer Slide Time: 29:49)). So, this is the whole thing. So, that is why I chose this example. Therefore, you have converted this to a binomial situation and then you are computing the approximate probability that, more than 50. So, here again, I am now using the central limit theorem; I am standardizing the variate there and then… Therefore, you are computing the approximate probability; because to compute the actual probability, would be – you will have to sum up those 72, 50 and beyond the binomial probabilities of 50, 51, 52 and 72. So, this is this problem.

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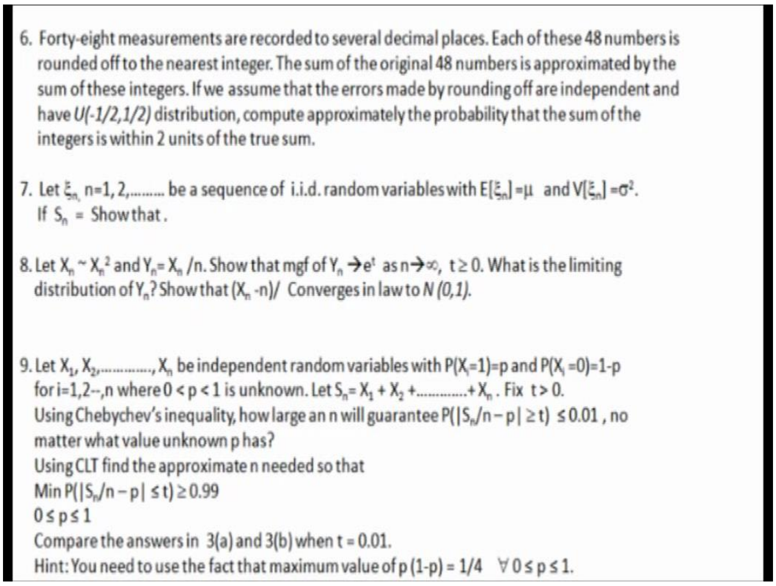
6. Forty-eight measurements are recorded to several decimal places. Each of these 48 numbers is rounded off to the nearest integer. The sum of the original 48 numbers is approximated by the sum of these integers. If we assume that the errors made by rounding off are independent and have *U(-1/2,1/2)* distribution, compute approximately the probability that the sum of the integers is within 2 units of the true sum.

7. Let ξ_n, n=1, 2,……… be a sequence of i.i.d. random variables with $E[\xi_n]=\mu$ and $V[\xi_n]=\sigma^2$. If S_n = Show that .

8. Let $X_n \sim X_n^2$ and $Y_n = X_n/n$. Show that mgf of $Y_n \rightarrow e^t$ as $n \rightarrow \infty$, $t \geq 0$. What is the limiting distribution of Y_n? Show that $(X_n - n)/$ Converges in law to *N (0,1)*.

9. Let $X_1, X_2, \ldots, X_n$ be independent random variables with $P(X_i=1)=p$ and $P(X_i=0)=1-p$ for i=1,2--,n where $0 < p < 1$ is unknown. Let $S_n = X_1 + X_2 + \ldots + X_n$. Fix $t > 0$.
Using Chebychev's inequality, how large an n will guarantee $P(|S_n/n - p| \geq t) \leq 0.01$, no matter what value unknown p has?
Using CLT find the approximate n needed so that
Min $P(|S_n/n - p| \leq t) \geq 0.99$
$0 \leq p \leq 1$
Compare the answers in 3(a) and 3(b) when t = 0.01.
Hint: You need to use the fact that maximum value of $p(1-p) = 1/4 \quad \forall\, 0 \leq p \leq 1$.

Now, let us go to measurements are recorded to several decimal places. Each of these 48 numbers is rounded off to the nearest sum of these integers. So, when you say rounding off; that means, if the sum of the original 48 numbers; if the decimal is below 5, then you drop the decimal number point. And, if it is 0.6, 0.7, then you take it to the next integer. This is how we say that, when you round off the numbers is approximated by sum of these integers. If we assume that, the errors made by rounding off are independent and have a uniform minus 1 by 2 comma 1 by 2 distribution, compute approximately the probability that the sum of the integers is within 2 units of the true sum.

So, now, here we are assuming that, the errors made by the rounding off are independent; surely, that you can expect because the errors that occur are not dependent on each other. And then this… Therefore, the rounding off that, you are doing is between minus 0.5and 0.5; thus, I said if the number is something like 10.4, then you will round it off to 10. If the number is 9.7, you will round it off to 10 – an integer. Therefore, you are assuming that, the error part; that means, the actual number minus the rounding – that difference is uniformly distributed between minus 1 by 2 and 1 by 2. The approximate probability that, the sum of the integers is within 2 units of the true sum. Therefore, what we are doing is… So, you have 48 errors – 48 numbers that you are rounding off. So, sigma… And, each is…

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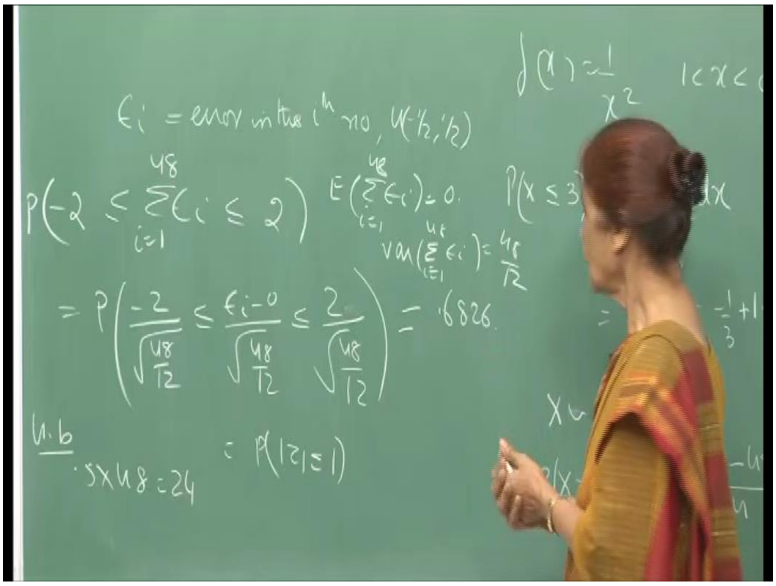

Yes, I can again write here that… See epsilon i is the error in the i-th number. So, we are wanting that, summation epsilon… And, each epsilon i is… And, this is uniform minus 1 by 2, 1 by 2; each error is uniformly distributed. Now, you are wanting the probability that, this thing should be less than or equal to 2; I think this is the question that, the sum of the integers is within 2 units of the true sum; which means that, total error that occurs should be within 2 of the original; so that means, sigma epsilon i, i varying from 1 to 48 – this should be within minus 2 and 2; the error can occur either on the… when you round down or you round up.

Therefore, this total error we are saying, what is the probability that this error is within two of the original sum of numbers. So, I have added up the errors. And so this sum should be greater than or equal to minus 2 and less than or equal to 2. This is what we want to approximate – this probability. And, that again by the use of central limit theorem, we will say because, now, epsilon i's are all uniform. Therefore, sigma epsilon i – expectation of this i varying from 1 to 48, is because the mean is 0. So, this is 0. They are all independent; the errors we have assumed are independent.

And similarly, the variance of sigma epsilon i, i varying from 1 to 48 will be sum of the variances and which will come out to be… So, the variance here is remember it is b minus a whole square raise to… b minus a whole square divided by 12; b minus a whole square by 12. So, this is… The variance here is 1 by 12 and so variance… This will be 48 by 12. So, the variance will be 48 by 12. And therefore, standard deviation will be under root of 48 by 12. So, I standardize. And, here this is what we get; and then by the normal this thing, it says that probability. So, this is actually equal to probability mod z is less than or equal to… This is 48 by 12. So, this is 1. And, that comes out to be 0.6826 from the normal tables.

Of course, you have to do some more competitions here and this will be… Therefore; that means the error can be kept within 2; the total errors of rounding up and rounding down can be kept within 2 with probability 0.6826. So, that is a very… There is high probability. But, if you look at a loose upper bound; that means if you are suppose rounding up all the numbers, then this will be 0.5 into 48, which will be 24. So, that means, an upper bound on the number of total errors – that can occur – can go up to 24. But, here the central limit theorem gives you the idea that, the probability that, the errors will be within 2 is reasonably high.

So, this is something about the problem I wanted to talk to you about. Varying from 1, 2 and so on is a sequence of identically independently distributed random variables with expected value of psi and is mu, and variance psi and is sigma square. Now, if S n is the sum of the first n sample values, show that S n upon n goes to mu with probability p. This is again just reiteration of the weak law of large numbers. Then, I want you to sit down and work out the proof by yourself. X n… Show that mgf of… as n goes to infinity; t greater than distribution of Y n… And, square. See the notation because we could not get it.

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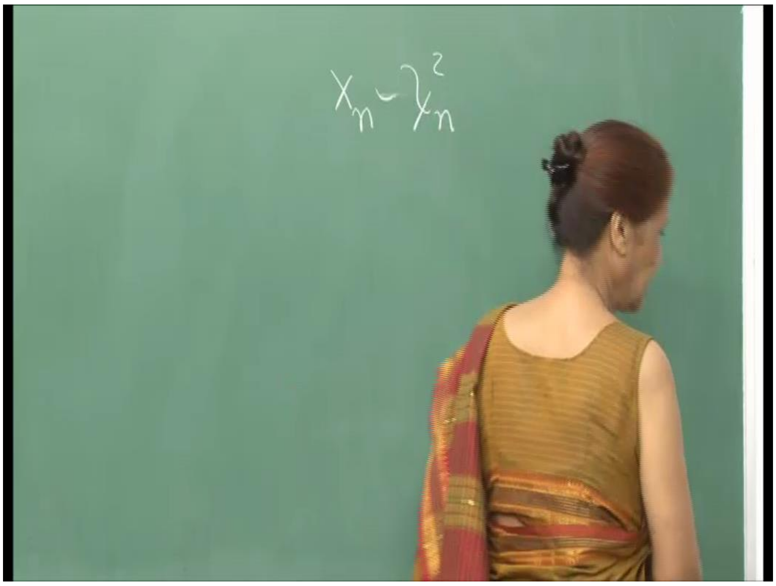

So, X n I am saying is chi square n. So, we have talked about the chi square n distribution also. So, here this is the notation; it looks like… In the print, it looks like X n square, but it is actually chi square. So X n is chi square n and then Y n is X n upon n, because again we wrote X n instead of chi, because chi was not coming out nicely. So, Y n is X n upon n and show that, moment generating function of Y n will go to e raise to t as n goes to infinity, for t greater than 0. So, it is defined for t greater than 0. So, this you can work out. And then what is the limiting distribution of Y n.

So, once you get the limiting mgf of Y n, then you will be able to say what is a distribution of Y n – limiting distribution of Y n. This is the whole idea through this exercise. And then show that X n minus n; where so X n is actually chi square n. So, chi square n has mean n and variance 2 n. Therefore, now we are standardizing this. So, this is actually the use of central limit theorem, because remember – central limit theorem is convergence in law. So, X n minus n upon under root 2 n for n large will converge to a standard normal variate. So, this is again the central limit theorem.

That X 1, X 2, X n are independent random variables with probability X i equal to 1 – p and probability X i equal to 0 – 1 minus p for i varying from 1 to 2n; that means, each X i’s… So, X i's are identically independently distributed Bernoulli random variables; p is of course, between 0 and 1 and it is unknown. So, this is what we have to estimate. I will get back to this thing. So, now, if you define S n as X 1 plus X 2 plus X n and you fix the t, then the problem says using Chebychev's inequality, how large an n will guarantee that, the probability of S n upon n minus p is greater than or equal to t? So, the probability of this event is less than or equal to 0.01 no matter what value unknown p has. So, obviously, we are trying to say that, we want to find out how many sample values we should take – X 1, X 2, X n, so that this ratio S n upon n or the average of the sample values is different from p by… So, greater than or… t we have fixed. So, this difference greater than t – probability of that is less than 0.01. So, you want to use Chebychev's inequality.

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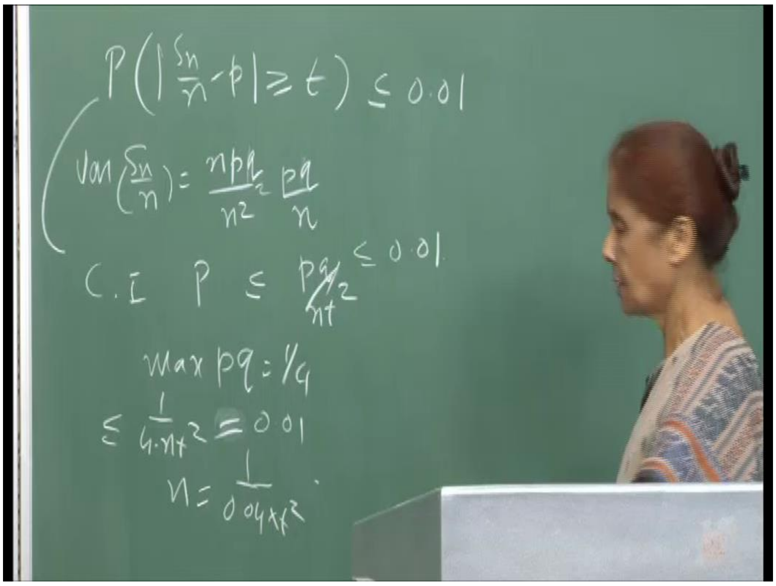

So, here by Chebychev's inequality, as we said, this is S n by n minus p. So, this you want greater than t – probability of this; and, this you want less than or equal to 0.01. Now, by Chebychev's inequality, because this is the variance of S n by n is because remember – now, S n is what? Each X i is a Bernoulli. Therefore, this is binomial. And so this is the variance of S n, is npq. And so 1 by n; this will be n square. So, this is pq by n. Therefore, by Chebychev's inequality, this probability is less than or equal to… This is pq by n into t square. And, this you want to be less than or equal to 0.01. So, now, what it says is p's are known. So, q is also unknown. And therefore, no matter what the value of p is…

Now, since maximum of pq… We have already gone through this in the lecture also; maximum pq is 1 by 4. So, if I take the… If I write the maximum value here since n is in the denominator; so this will… I will get the value of n, which is smaller. See what I am saying is that, this probability is less than or equal to 1 by pq by… This 1 by 4 into n t square. And, this we want less than or equal to 0.01. So, suppose I put this equal to 0.01. And, this tells me that, n should be equal to… From here n should be equal to… If you take n to this side, it will be 0.04 into t square. And, since I have written, see… So, this value has become 1 by 4, is the maximum value of pq.

So, now… that means, for n greater than or equal to this, this will always be satisfied – less than or equal to 0.01; can you see that? See here I am writing the maximum value; this upon nt square is less than this. So, n would be greater than or equal to this. So, I am taking it… So, if I put the maximum value here, then obviously, I get a value of n, which will meet this inequality, because n will be greater than… Otherwise, if I write the actual value of pq, then what I get – the value of n would be smaller than what I am getting here. Therefore, this will always satisfy this inequality; this is the idea. Therefore, by Chebychev's inequality, this is the thing.

Now, part 2 says that, using CLT, find the approximate n needed, so that… Now, here you see it has put the word minimum of this probability. And, the probability here is the compliment of the event that you had in the part a. Therefore, it is a same thing, because here the probability of less than t is greater than 0.99. So, exactly… But, the minimum part I will explain again here, because this is now…

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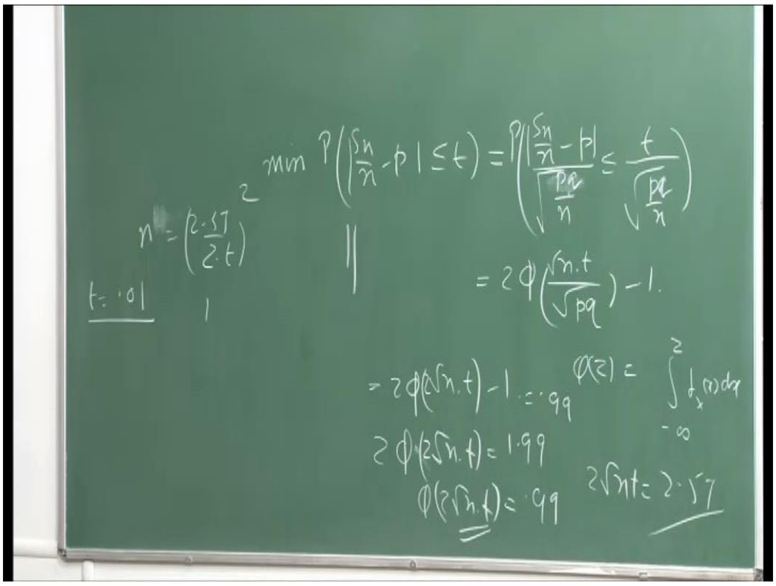

By central limit theorem, minimum this… So, minimum this probability will be attained when I put the maximum value of this. And therefore, the minimum… When you write this here, this will be twice 5 and root n into t; and, this is 1 upon 4. So, 1 by 2, so that the 2 also comes here. So, this minus 1. So, that satisfies this. And now, you want to compute again. This you want to say is equal to 0.99; which means that, 2 of 5… 2 root n into t is equal to 1.99. And, now, you can continue. And, in fact, you will find out the value of t, because I think… Maybe we will complete the problem. So, this is 2 divided by this – 0.99; and, the corresponding Z value here; I think from the tables if you look up, it says that 2 root n t is 2.57; I think that is the thing. And so you can compute root n from here. And now, what it says is again… Since you have the numbers… In this case, n comes out to be equal to 2.57 divided by 2 into t whole square.

And, in the third part, it asks you to… When you fix the value of t, I think the value of t is given – 0.01. If you do this, then it wants you to compare. So, for example, from here when t is this, for t equal to 0.01, n comes out to be equal to 250000. And then when you compare it with the central limit thing, I think this comes out to be… It is computed somewhere; I have done it here – 16. So, n will be greater than or equal to 16500. So, this is the idea, because the Chebychev's inequality gives you a loose upper bound. And therefore, the numbers will be different. So, this is the idea behind this thing. And now, you can sit down and work it out yourself to get a better feeling. Is equal to 1.99 and now you can continue. And in fact, you will find out the value of t, because I think…

Maybe we will complete the problem. So, this is 2 divided by this – 0.99. And, the corresponding Z value here – I think from the tables if you look up, it says that, 2 root nt is 2.57; I think that is the thing. And so you can compute root n from here. And now, what it says is again since you have the numbers… I mean in this case, n comes out to be equal to 2.57 divided by 2 into t whole square. And, in the third part, it asks you to… When you fix the value of t; I think the value of t is given 0.01 if you do this. Then, it wants you to compare.

So, for example, from here when t is this; for t equal to 0.01, n comes out to be equal to 250000. And then when you compare it with the central limit thing, I think this comes out to be… It is computed somewhere; I have done it here – 16… So, n will be greater than or equal to 16500. So, this is the idea, because the Chebychev's inequality gives you a loose upper bound. And therefore, the numbers will be different. So, this is the idea behind this thing. And now you can sit down and work it out yourself to get a better feeling.