Introduction to Probability Theory and its Applications
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Lecture - 15
Conditional Disti Joint Distr of Functions of R. V. Order Statistics

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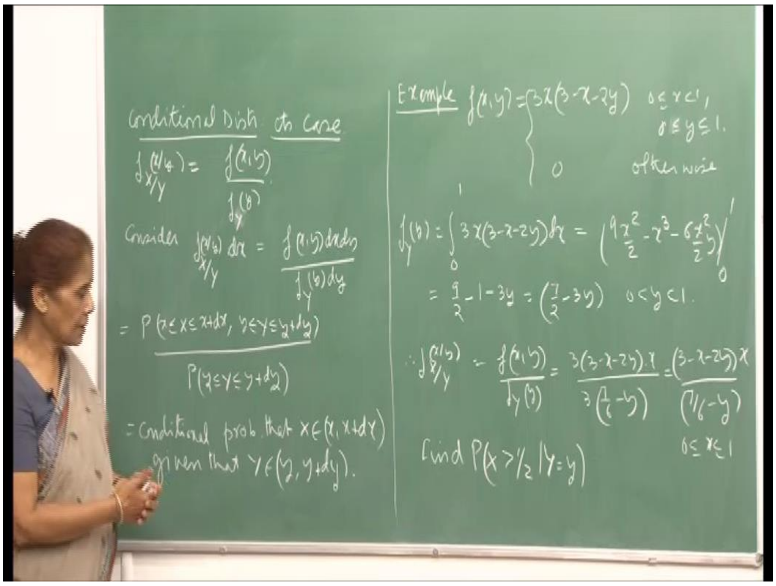

So, let me now talk about the conditional distribution when the random variables are continuous. So, in that case, it will be probability density function x given y. So, the notation will be this. So, you will write it as the joint of f x y at x, y divided by the marginal of y at small y, right. And see the way to explain this is because since we know that the continuous case, the probability at a fixed point is 0.

So, therefore the way to look at it is you know if I multiply both sides by d x and then multiply and divide by here, then you see this represents, this will be the conditional probability of x given that capital wise Y and X is between x comma x plus d x and here on the right hand side you can it will be you can interpret f x comma y d x d y as probability x less than or equal to x plus d x comma y less than or equal to y. That means capital Y between y and y plus d y, you can look at this way, d y d x are small divided by, so this will be probability capital Y between small y and y plus d y.

So, therefore you can say that this ratio represents the conditional probability of x lying between small x and x plus d x, when you are given that capital y is between y and y plus d y. So, this is what I have expressed here that capital X will belong to x, x plus d x given that y belongs to y comma y plus d y.

Now, let us just look at an example. So, suppose this is a joint density function x lying between 0 and 1, y is between 0 and 1 and you can verify that this is joint p d f. That means, double integral of this expression should be equal to 1 when your x and y are between 0 and 1, but to find out the conditional. So, if I wanted to write down the conditional p d f of x given y and I need to compute the marginal of y which I hope the arithmetic is, so this will be when you integrate right 3 x into 3 minus x minus 2 y. This will be the expression between 0 and 1. So, this will be 9 by 2 minus 1 minus 3 y minus 3.

Yeah. So, remember that because y, capital Y is fixed at small y, yes. So, this is the marginal, yeah this is the marginal y. So, obviously it will be a function of y only, but I had something else in mind which I will tell you right now. So, now, if you want to find out the conditional of x given y, then that by definition is this ratio f x, y divided by f y of a small y and this will be because I have computed f y y for you. So, this is the ratio and therefore, this comes out to be this. So, that means, for a fixed x and y, this will be the conditional p d f of x given y as. So, here your x will vary between 0 and 1. Now, if you have to find out this probability x greater than half given that capital Y is equal to y.

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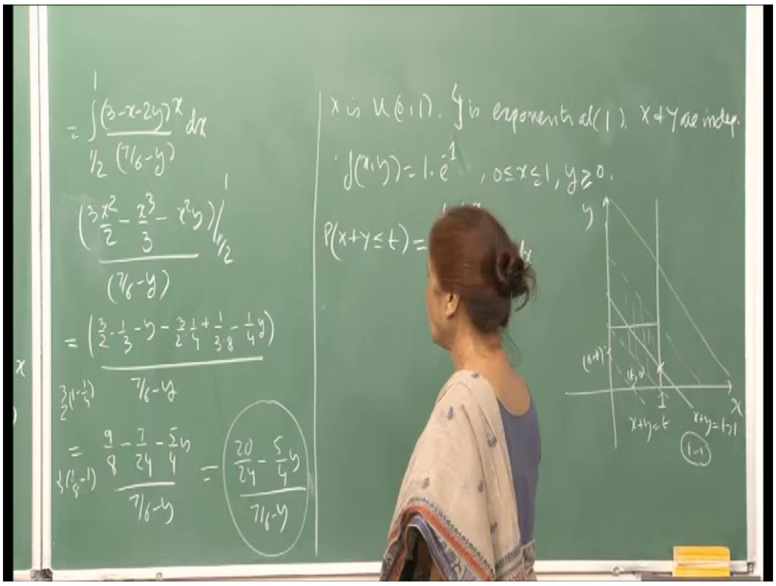

So, this will be the integration half to 1 of this conditional p d f, where y is being treated as constant, right. So, you integrate respect to x and here again, this is arithmetic 3 x square by 2 minus x cube by 3 minus x square y because a 2 cancel from half to 1 and then, this is the denominator which I do not have to do anything. So, I do the computations here. Please verify the arithmetic thing should be add up. So, the final expression is this which will be the function of y because here you are given a value to x, all values of x greater than or equal to half.

So, therefore, this conditional probability x greater than or equal to greater than half or does not matter for a continuous case, it does not matter given that y is y. It turns out to be this expression. So, once we have this and I think for the continuous case for the discrete case also, we wrote down the distributed cumulative distribution function or cumulative, these things and then, you can write down the probability conditional probability mass function also exactly in the same way. So, therefore, this is nothing new. Maybe I did not actually write down the expression, but that does not matter.

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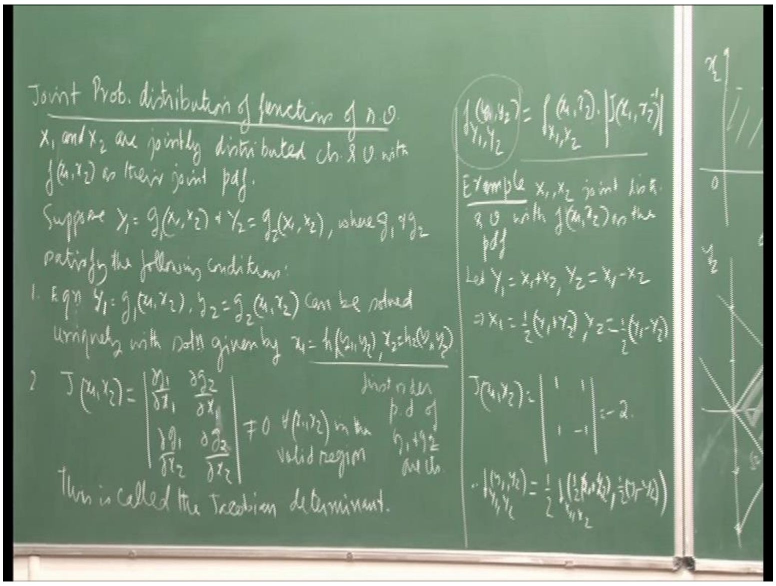

So, let me now begin the topic Joint Probability Distributions of Function of Random Variables. So, just before this I talked about joint distribution of random variables. Now, let us take the joint distribution probability distribution of function of random variables because this also we need often to compute certain probabilities and so on.

So, now x 1 and x 2 are jointly distributed continuous random variables with x 1 and x 2 is there joint p d f. So, then suppose y 1 is a function of x 1 and x 2 and y 2 is a function of x 1 and x 2. So, g 1 represents a function of x 1 and x 2 which is represented by y 1 and g 2 is a function which represents y 2, where g 1 and g 2 have to satisfy certain conditions and this is what we were saying that if you look at these two equations, then they must be unique solution. In fact, if they are more than 1 or you should be able to fix the values of y 1 and the values of x 1 and x 2, that you will take corresponding to values of y 1 and y 2. So, in other words, what you are saying that we should have the solutions in a deterministic way. We should be known liquidity about it. So, x 1 should be h 1 of y 1, y 2 and x 2 is h 2 or y 1, y 2.

So, I can solve this set of equations to get the values of x 1 and x 2 for given values of y 1 and y 2 and then, this second condition is that this Jacobian as we call it, they should be set of partial derivatives which are continuous. So, the first order partial derivatives I have not written here or maybe that also has to mentioned first order partial derivatives of g 1 and g 2 are continuous. So, therefore, they exist and continuous.

So, first order partial derivative exist are continuous. Then, we define this determinant which is delta g 1 by delta x 1, delta g 2 by delta x 1, then delta g 1 by delta x 2 and delta g 2 delta x 2. So, the notation is sometime some people also use a notation. This will be y 1 y 2 upon x 1 x 2 and so on, because whatever you have in a denominator, there would not to other notation for the Jacobian also, and this should not be a 0 for all x 1 x 2 in the valid region. So, this should be a non singular matrix and therefore, its determinant is not 0. Now, this quantity we called a Jacobian determinant here and then, the transformation is that means, when you wanted to find out the p d f of y 1 y 2 respect to p d f of y 1 y 2, then that can be obtained in terms of the p d f of x 1 and x 2 as this f x 1 x 2.

Now, here of course you will substitute for x 1 in terms of because you are able to solve x 1 and x 2 in terms of y 1 y 2. So, you can substitute that here this will be the absolute value. These two lines indicate an inverse of the Jacobian. So, you have this matrix, compute this determinant and then, take in inverse and the absolute value, ok.

Now, I will try to give you feeling about see what this says is that the fact that this is none. You can see that if it was 0 p d f of f y 1 y 2 at small y 1 y 2 is defined, since the division by 0 is not permissible, but otherwise there is no point in talking of such transformations, where the Jacobian is 0 and there are. So, many ways you can interpret this concept of Jacobian, but I will just try to show you now one aspect here, and that says that absolute value of the Jacobian determinant at a point p gives us the factor by which the function expands or swings the area and bracket volume if you're talking in three-dimension near the point p.

So, it will swing if the absolute value of Jacobian is Jacobian determinant is less than 1 and it will expand if the determinant of the Jacobian is greater than 1 and near the point p. So, if you are the coordinates that we are considering are x 1 x 2 and y 1 y 2 in the transformed plane, then near the point p in the transform space. So, the area in the x y plane in the x 1 x 2 plane will get transformed to the area element of around the point p. That means we are talking in terms of element of area around p in y 1 y 2 plane by the value determinant of the Jacobian.

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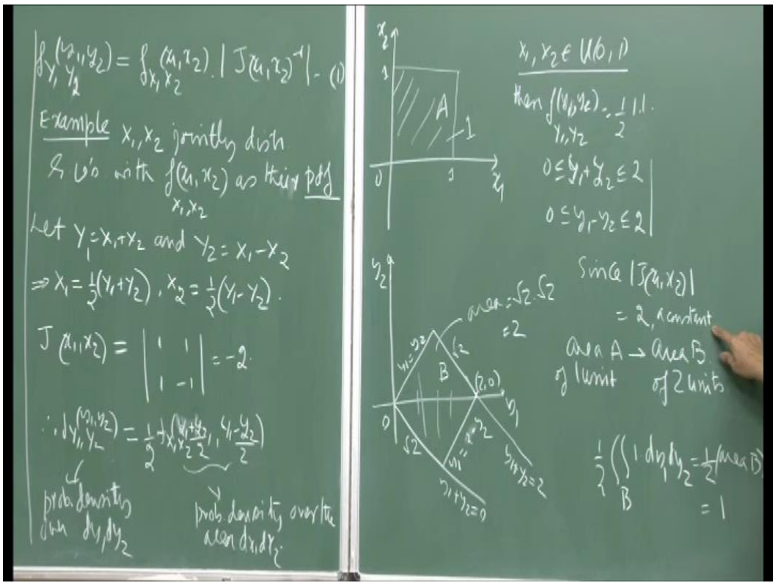

Let us consider this example. So, x 1 and x 2 are jointly distributed random variables with f x 1 x 2 as their p d f and let y 1 be equal to x 1 plus x 2 and y 2 is x 1 minus x 2. So, we defined two new random variables as function of x 1 and x 2. So, this implies that your x 1 is half y 1 plus y 2 and x 2 is half y 1 minus y 2. So, Jacobian if you want to compute, then it will be the derivative of this with respect to, so here when you differentiate respect to y 1, this will be 1 and differentiate respect to. That means, differentiating x 1 with respect to y 1 and y 2. So, this is 1 and 1, then you differentiate x

2 with respect to y 1 and y 2, it will be 1 and minus 1. So, the value of this determinant is minus 2 and if you take the absolute value, it will be 2 and the inverse value of the inverse of the Jacobian determinant of the Jacobian and with absolute value will 1 by 2.

So, according to this formula, our p d f for y 1 y 2 would be equal to half f of x 1 x 2. The p d f for x 1 x 2 when you substitute for x small x and small x 2 in terms of y 1 and y 2, so this is y 1 plus y 2 by 2 and this is y 1 minus y 2 by 2, right. So, what I am trying to say which I said few minutes ago is that you know you can treat if you have to take a small element of area which is d x 1, sorry d y 1 d y 2. Here in y 1 y 2 plane, this is d y. So, element of area we treat this density as the probability density over d y 1 and d y 2 and here, if you look at this part, this will be the probability density over the element of area d x 1 and d x 2. And in this case of course since we are taking very small element of area, I can treat I can say that this is half times.

The density is the relationship between the two densities. I am just trying to give you feeling about this anyway and then, you can see here if you take the particular case that x 1 and x 2 are uniform 0 1, both are distributed uniform and both are random variables, uniform random variables over 0 1. Then, you see your transformation and if I am taking y 1 as x 1 plus x 2 and x y 2 x 1 minus x 2, then you see here this p d f of y 1 and y 2 by that formula would be half into 1 because the uniform that I am treating as independent. So, I am taking 1 into 1, right. The pdf of` both the f x 1 and x 2 would be simply product of f x 1 into f x 2. So, both being uniform, this is 1. So, this is the formula you get and the range is y 1 plus y 2 varies from 0 to 2. So, you can get individual ranges which I have drawn here. So, the area when you consider x 1 x 2 variable, this is the area, right and this equals 1 the area, right.

Now, this gets transformed to this kind of region in y 1 y 2 planes, right. If you draw this y 1 plus y 2 equal to 0 and y 1 plus y 2 equal to 2, which are these 2 lines and y 1 minus y 2 equal to 0 is this and y 1 minus y 2 equal to 2 is this line. So, it is this area which you get. So, a gets transformed to b and you see the area here is 2 units, this is 1 unit and 2 units and this is what I want to explain that since Jacobian is a constant, therefore you see this probability density is same over the whole area and that is the feeling I want to give you. So, the relationship between the two areas here because now that the Jacobian is a constant. So, therefore, the area a, which is 1 unit goes over to area 2 in the y 1 y 2 plane. In other words, you can also say that because this density and into 1.

So, when you integrate over the whole of b, this whole thing we should add up to should integrate to 1, alright which is half area b which is 1. So, area b is 2 and when you equate the 2 p d f's for example, do this and integrate. So, see this area if you just do this, this is 1, right and this area this area also has to be 1 where I am in the integral. So, I am trying to say that this integral and this integral both, this has to be 1 and this must be 1, but this is related this half. So, therefore, this will be twice this. So, therefore, the density will turn out to be half. So, therefore, this density will be half of this because this much add up to this integration must integrate to 1, this integrate to 1. If you just took the x 1 and x 2 variables and these values half, so this is equal to twice this. So, therefore, the density for this one becomes half the density for x 1 and x 2. This is my own interpretation and I am trying to give you.

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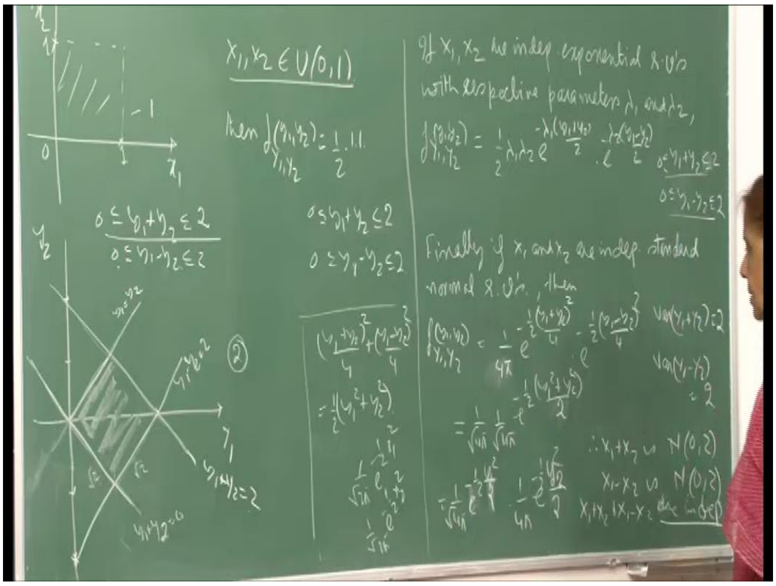

Now, consider when x 1 and x 2 are independent exponential random variables with respective parameter lambda 1 lambda 2, right and their wait I am treating, again I am treating them as independent. So, therefore, with a same transformation that y 1 is x 1 plus x 2 and y 2 is x 1 minus x 2, then the p d f of y 1 y 2 by that formula would be because Jacobian is again inject inverse of the Jacobian is half and the determinant and this will be lambda 1 lambda 2 product of the 2 pdf's for x 1 and x 2 with x 1 replaced by y 1 plus y 2 by 2 and x 2 replaced by y 1 minus y 2 by 2, and the limits here would be because x 1 goes from 0 to infinity, x 2 goes from 0 to infinity. So, this will be this since they both are no negatives. So, this is it.

So, now, you can compute the individual limits for y 1 and y 2. Now, finally, if x 1 and x 2 are independent standard normal random variables, then you see this will be your joint would be because you are independent. So, your joint anyway it will be 1 upon, what it is? Root 2 pi e raise to minus 1 by 2 and their standard normal. So, simply x 1 square is for this thing and 1 upon root 2 pi e raise to minus 1 by 2 x 2 square. Now, sigma square is 1 mean is zero. So, these are two individual p d f. So, you multiply them. So, that becomes 1 by a 4 pi 1 by 4 pi into 0 root 2 pi root 2 pi. So, the half the Jacobian is half and this product is 2 pi. So, 1 upon 2 pi into 2, this becomes 1 by 4 pi and then, this is e raise to minus half y 1 plus y 2 plus whole square by 4 y a because your x 1 is x 1. I am writing as 1 by 2. So, this should be [FL]. So, 1 by 2 and then, y 1 by y 2 4 whole square and then similarly, minus 1 by 2 x 2 square x 2 is y 1 minus y 2 by 2.

So, the square gives me y minus y 2 whole square by 4, right and the variance of y 1 plus y 2 is 2 variance of y 1 minus y 2 is 2, right because again y 1 and y 2 are independent. I have computed this for you, fine. We will come to this conclusion later on. Let me continue with this. So, now what I have done is I have written this expression y 1 plus y 2 whole square by 4 plus y 1 minus y 2 whole square by 4. See the coefficient here the half I have left out just these two terms. So, then they add up 2 because the product term here will be 2 y 1 y 2 plus 2 y 1 and y 2. Here, it will be minus 2 y 1 y 2 divided by 4. So, that cancels out and you get twice y 1 square plus twice y 2 square. So, therefore, 2 upon 4, that gives you half y 1 square plus y 2 square. So, this whole thing reduces to e raise to minus 2, then y 1 square plus y 2 square by 2, right and then, I can again write 1 by 4 pi as 1 upon on the root 4 pi into 1 upon on the root 4 pi.

So, now I am saying that this is 1 upon root 4 pi into e raise to minus 1 by 2 y 1 square by 2 into 1 upon 4 root 4 pi e raise to minus 1 by 2 y 2 square by 2. So, you see the p d f here separates out into two single variable pdf’s. So, I will conclude that y 1 and y 2 are independent and you see that. Therefore, the variance y 1 plus y 2 actually I can also compute the variance of y 1, right. So, variance of y 1 plus y 2 is 2. Why I am saying that variance of y 1 plus y 2, where am I concluding from here? No no no no, this is a wrong statement. That is why I am saying that this is not right. Yeah this is because x 1 and x 2 are independent. Therefore, I should have written variance x 1 plus x 2 is 2. Variance of x 1 is 1, variance of x 2 is 1 and these are independent. So, this is this. Similarly, variance of x 1 minus x 2 is also 2, right.

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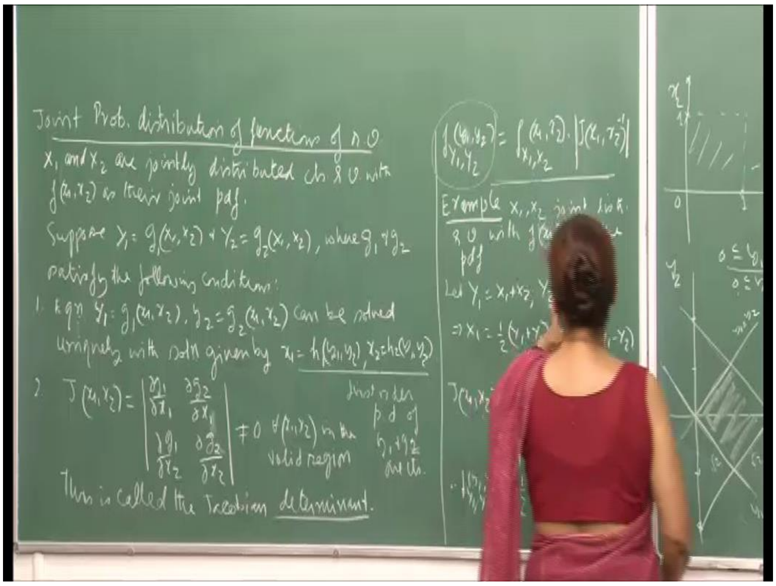

So, therefore, y 1 when you are looking at, so y 1 is what y 1 is. Y 1 is here. Where did I define it? Yeah, y 1 is x 1 plus x 2 and y 2 is x 1 minus x 2. So, therefore, y 1 and we have all also seen that the sum of random variables is again normal. So, here the mean is 0, variance is 2. So, y 1 is normal and y 2 is normal and therefore, this is accordance with because if y 1 is normal 0 2, then you are dividing by the variance square. So, y 1 square by sigma square is 2 sigma square.

So, this is y 1 square upon 2 sigma square 1 upon root 2 pi into root 2 because standard deviation will be root 2. So, that becomes 1 upon root 4 pi. So, this is all in accordance with this and therefore, you can say that a y 1 and y 2 are also independent normal variables and this happens only because you see that is why I took three examples for the same case. I first took x 1 and x 2 to be jointly uniform and then, we wrote down the p d f of y 1 and y 2. So, what that come out to be simply this, right which you cannot say it is constant in this area. That is all.

So, from here it probably will follow that this is a uniform. You think about it, but when you took the distributions for x 1 x 2 to be exponential, what you did not get is any separation here. So, therefore, here you cannot conclude that y 1 and y 2 are independent, but when you took the x 1 and x 2 to be standard normal independent random variables, then it turns out that x 1 plus x 2 and x 1 minus x 2 are also independent and normally distributed. So, this happens for a normal distribution for when x 1 and x 2 are normally distributed independent random variables. Then, these functions will also be normally distributed and they will be I mean for x 1 plus x 2 and x 1 minus x 2. I am not claiming that this will happen for any functions of normal random variables, but in case when the functions are x 1 plus x 2 and x 1 minus x 2, then they turn out to be independent. They will be normal. Yes, that of course we know from the property of normal distribution. Already we have seen it. So, this is the case.

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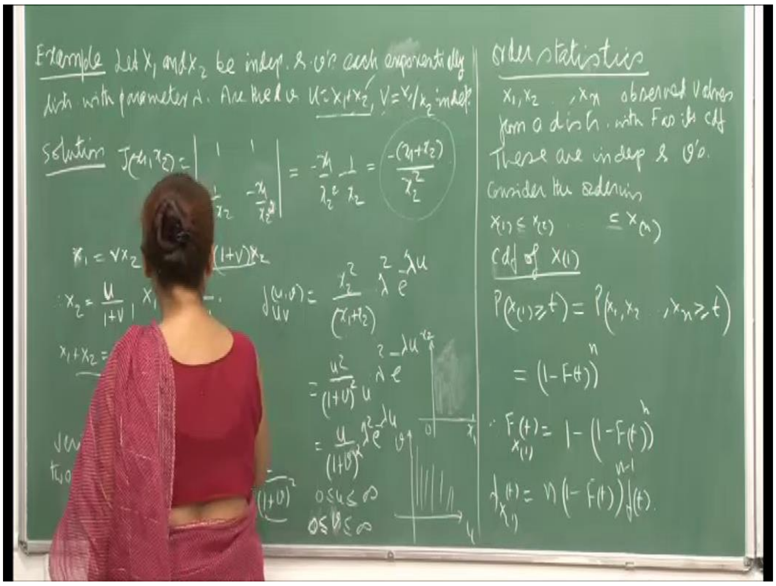

Let me continue with another example of a function of random variables. So, here x 1 and x 2 are independent random variables, each exponentially distributed with parameter lambda. So, now, with the question asked is, all the random variables u equal to x 1 plus x 2 and v equal to x 1 upon x 2 independent. So, therefore, we will find the joint density function of u, v and see if it can be separated out into a function of u and a function of v. So, like the Jacobian or the Jacobian is here, this is 1, the partial derivative or v is the partial derivative of 1 upon x 2 and minus x 1 upon x 2 square. So, therefore, the determinant is equal to this which can be written like this, right.

Now, compute to the inverse functions. So, here for the second equation, you see that x 1 is v x 2. Therefore, this now we substitute in this equation. So, for x 1, that will be if you write it down that is why I am doing here. So, if you write it for x 1 plus x 2 and x 1 is u into x 2, we just said that x 1 is v x 2. So, v x 2 if you write out here you do that and therefore, when you write x 1 as v x 2, so v x 2 plus x 2. So, x 2 outside 1 plus v, your x

2 becomes u upon 1 plus v. Then, from here you get x 1 is u v upon 1 plus v and if you write x 1 plus x 2, that comes out to be u. Well, that is already given to us. It is simply a verification, fine.

So, then you will sub that is the formula gives you the joint pdf of f u v which is Cobain inverse. So, x 2 square upon x 1 plus x 2 absolute value of this which is equal to this and then, because x 1 and x 2 are independents, so the joint p d f is a product. So, for each, the p d f is lambda into e raise to minus lambda u lambda x x lambda x 1 and then, lambda. So, here it will be lambda times x 1 plus x 2 which we have which are given to be as u. So, this is your p d f for and of course, you will substitute for x 1 and x 2 in terms of u and v. So, you can immediately see that x 2 square is u square upon 1 plus v square and then, x 1 plus x 2 is u. So, therefore, your final function is u upon 1 plus v whole square lambda square e raise to minus lambda u.

Now, I was trying to see we draw the picture, but again see the reason for a x 1 x 2 is a whole of first quadrant, and it looks like that for u and v also because u and v are also both no negative and both are extending to infinity. So, it appears that here since the regions are infinite, therefore, I cannot show you any swanking or anything and in any case, this is dependent on the point. So, this Jacobian is not a constant here. So, it will be dependent on coordinate values x 1 x 2 and so on.

So, therefore, I cannot do much here, but you see now this and of course, your limits for u are from 0 to infinity and for v from 0 to infinity and now, you can write this down as a product of two functions. So, lambda square e raise to lambda u into u is 1 and 1 upon 1 plus v square is the other function. So, since I have and remember I gave you this proposition in the earlier lecture that if there is a p d f which can be written out separately as a function of single variables, then each of them must be p d f themselves for the corresponding random variables.

So, now I want you to verify y that the two functions represent the p d f. That means, this is a p d f from 0 to infinity show that this integral is 1. Similarly, this integral from 0 to infinity is 1, right which you can do by heat iterative integration here and they both are no negative. So, therefore, we will conclude which I did not write here that u and v are independent. So, I will now talk about exercises 5 which is you know collection of problems from whatever we have been discussing in 3 to 4 lectures.

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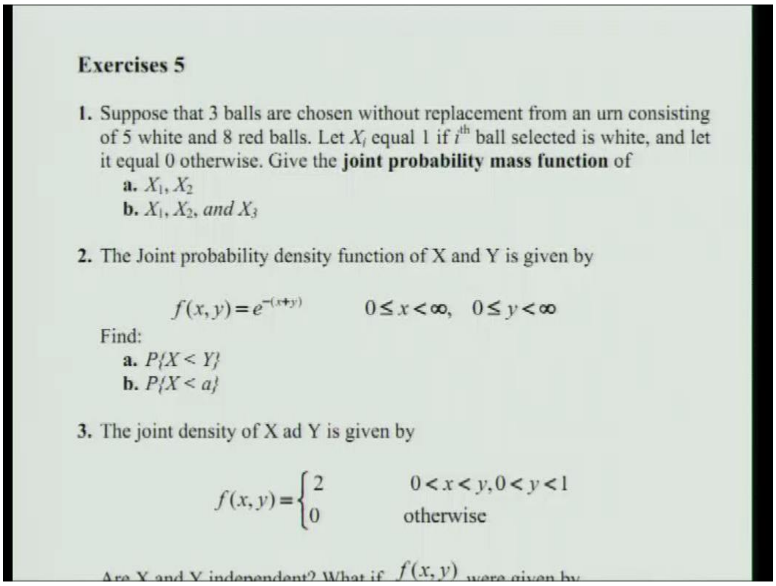

Let see again as usual I will try to give you some small hints and you should be able to work out the problems. In question 1, 3 balls are chosen without replacement from an urn consisting of 3 white and 8 red balls. So, X i equals 1 if ith balls selected is white. So, you know you have first, second and third balls which are chosen without replacement. So, if the ith ball is white, then you put X i equal to 1 and 0 otherwise. So, give the joint probability match function X 1 X 2. So, again you will make that chart we have shown you, right. You know rows will be before x 1 and columns will be before x 2 and then, you can write out for different values and then, I want you to write the joint probability match function of x 1, x 2 and x 3.

So, now in this case, you will have to simply write three values because it will be x 1, x 2 and x 3, all of them right. So, three-dimensional I have been discussing with you, two-dimensional so far. So, I thought let me include this and let see how you try this problem. Question 2: The joint probability density function of x and y is given by this function e raise to minus x plus y x and y between 0 and infinity, then find probability x less than y. So, now, this is a event. That means the region. So, you will draw the line.

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So, here it is simple. This is this. So, the whole of first quadrant is a valid region. Now, you want to find the probability. So, it will be under this region X is other way. This is X less than Y. sorry. So, it will be this region, right. So, therefore, fix your limits accordingly and you will be able to immediately write down the limits from here because X has to be less than Y. So, therefore, X cannot vary from beyond Y. So, it will be 0 to Y and then, Y of course varies from 0 to infinity and then, a probability X less than a. So, in the b you will have to find the marginal of X first and then, compute this probability.

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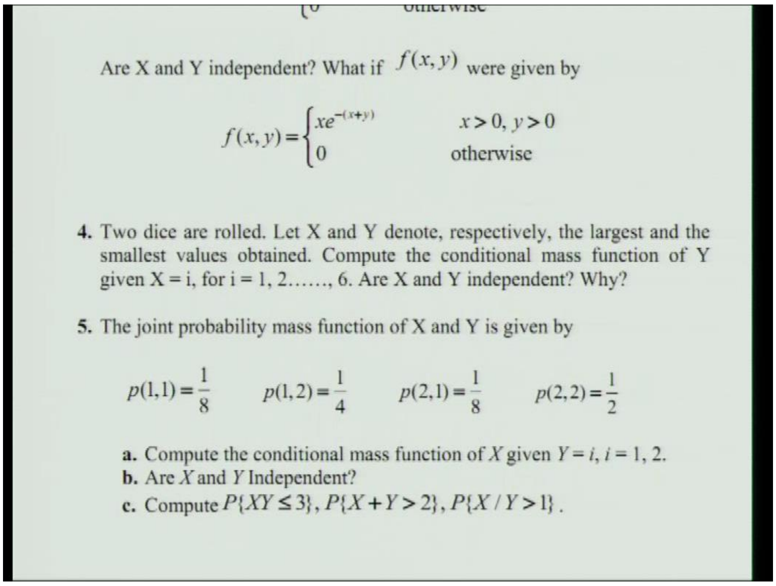

Yeah, this is problem 3. Now, problem 3 says you are given the joint density function of X and Y, which is f x y 2 if 0 is less than x less than y, y is between 0 and 1 0, otherwise x and y are independent. So, now, as I told you since the limits of x are dependent on y, my immediate reaction would be that now they are not independent, but you will have to find out the marginal, so that the joint is not the product of the marginal's. If x and y are given by this, a new function which is f x y equal to x into e raise to minus x plus y 0 then and if now the limits for x and y are independent of each other. So, in this case, again you can break up your joint p d f into x into e raise to minus x into e raise to minus y. So, therefore, they should turn out to be independent, yeah.

Next question 4 says that two dice are rolled. Let X and Y denote respectively the largest and the smallest values obtained. Compute the conditional mass function of Y, given X is i for varying from 1 to 2….6. So, let me use fixed value of i of X and then, say are X and Y independent. Why all these two answers? So, you are quite familiar with now rolling of two dice and how you write down the probabilities. So, therefore, you should be able to answer question 4.

Question 5: The joint probability mass function of x and y is given by. So, this is now discrete set of random variables and you are given the probabilities here. Compute the mass function of X given Y is i varying from 1 to 2. So, there will be two conditional mass functions. One for when y is equal to 1 and other is y equal to 2.

Are x and y independent? Apply the condition for independence and compute x y less than or equal to 3, probability x plus y greater than 2 and probability x upon y greater than 1. So, here you see the values of y are not 0, anywhere y takes the values 1 and 2 and x takes the values 1 and 2. So, all questions are valid and you should be able to answer them.

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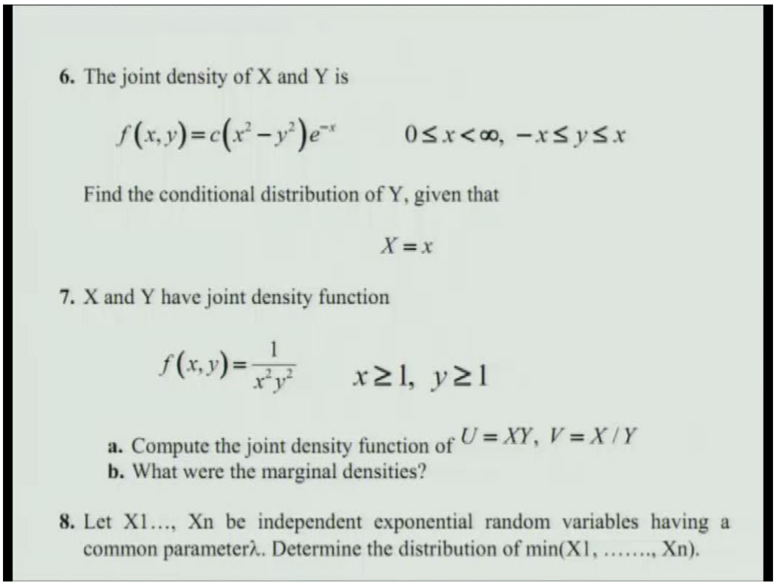

Six is again you are given a joint density function of X and Y and here you see it why varies between minus and x and minus x and x. Draw a region, find the addition distribution of y given that x is equal to x. So, I have just included all these things. They are different from each other, but then you get an idea when you solve these problems. X and Y have joint density function given by 1 upon x square y square. X and y are greater than or equal to 1. Compute the joint density function of U equal to XY and V equal to X by Y. It should be what the marginal densities are.

So, anyway joint these things, density function you will compute by using the Jacobian method, right and then try to draw the regions because for X greater than 1 and Y greater than 1, it is simply these things. When you see this is 1 and this is 1. So, in the original thing, this is the region, alright and here, of course I can give you a hint because when U is equal to XY, so you will have to write X in terms of U and V in terms of Y and then, you see that the V region, how the region is transformed. So, do it because I have given you an idea already and you have to then compute the marginal densities of U and V, ok.

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8. Let $X_1 \ldots, X_n$ be independent exponential random variables having a common parameter λ. Determine the distribution of $\min(X_1, \ldots, X_n)$.

9. If X and Y are independent binomial random variables with identical parameters n and p, show analytically that the conditional distribution of X, given that $X+Y=m$, is the hypergeometric distribution Also, give a second argument that yields the result without any computations

Hint: Suppose that $2n$ coins are flipped. Let X denote the number of heads in the first n flips and Y the number in the second n flips. Argue that given a total of m heads, the number of heads in the first n flips has the same distribution as the number of white balls selected when a sample of size m is chosen from n white and n black balls.

10. The random variables X and Y are said to have a bivariate normal distribution if their joint density is given by

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\times\exp\left\{-\frac{1}{2\left(1-\rho^2\right)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\frac{\left(x-\mu_x\right)\left(y-\mu_y\right)}{\sigma_x\sigma_y}\right]\right\}$$

So, eighth question. Eight is once should have been suffix, but does not matter. Let X 1 to X n be independent exponential random variables having a common parameter lambda through all of them comes in. That means, they are observed value from exponential distribution with parameter lambda determinant, the distribution of minimum X 1 X 2 and X n. So, this is what I have already discussed with you, right finding out the p d f of x bracket 1. That means the smallest to the n sample values.

If X and Y are independent binomial random variables with ith, so we just make the correction with identical particulars n and p. So, X and Y are the same binomial distribution. I mean the same binomial distribution show analytically that the conditional distribution of X, given that X plus Y is m is the hyper geometric distribution. There should be have been a full stop is the hyper geometric distribution also gives a second argument that gilds a result without any computations in the ninth problem. You are given to a binomial independent binomial random variables with identical parameters n and p. So, you have to find the conditional distribution of X, given that X plus Y is m and you to show that this is a hyper geometric distribution and this again added this problem because I have already shown you a similar one. I solved the similar problem in the lecture and also, gave a second argument that gilds a result without any computations.

So, the hint is given here and you can argue that is given a total of m heads is the number of heads in the first n flips has the same distribution as a number of white balls selected. So, you figure out the hint and then, see if it is useful, ok.

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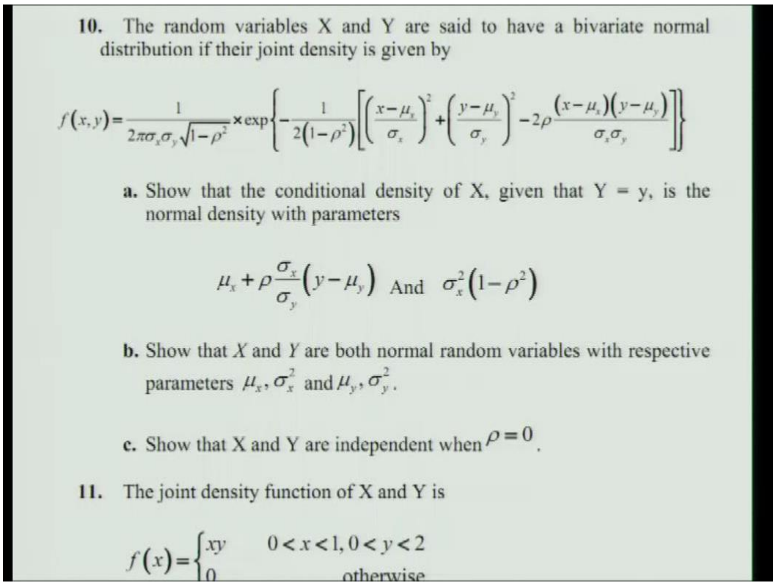

Now, the tenth problem is random variables X and Y are said to have bivariate normal distribution if the joint density is given by this. So, here I have not discussed in the lecture, but I thought you should be able to work on this. So, bivariate normal random variable distribution is of this form, where you have this squared term with respect to x with respect to y and then, you have the product term and of course, the rho part which will by the time you get to this problem, I think I would have discussed with you which is the correlation coefficient. So, this is the expression, but anywhere right now this is a.

So, now you have to show that the conditional density of X, given that Y equal to y is a normal density with parameters. So, you see the moment you fix your Y, then you can rear in the terms and write name in the form, so that this becomes a mean of the conditional variable. That means, conditional density of X given that Y is y and the variance will become sigma x square into 1 minus rho square.

So, it is just a question of you know manipulating the term and since, you already know what you have to show. Therefore, this is not going to be difficult, so that X and Y are both random variables with respective parameters mu x sigma x square and mu y sigma y square. So, here you see if X and Y are 2 normal bivariate, then the joint density function is given here above and then, you can show that when you do the integration for when you integrate f x y with respect to y from minus infinity to infinity, you will get mu distribution with mean mu x and variance sigma x square. And similarly, when you integrate with respect to X, you will get the a marginal of Y which will come out to be normal with mean mu y and variance sigma y square.

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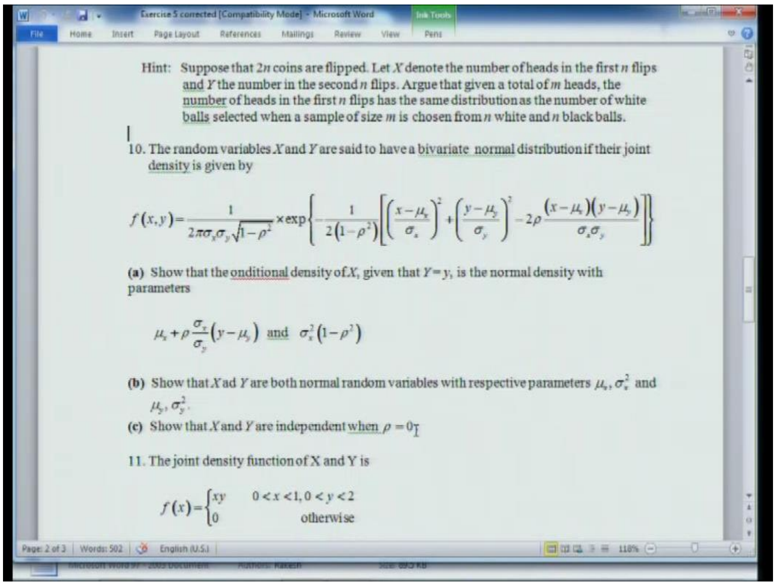
Hint: Suppose that $2n$ coins are flipped. Let X denote the number of heads in the first n flips and Y the number in the second n flips. Argue that given a total of m heads, the number of heads in the first n flips has the same distribution as the number of white balls selected when a sample of size m is chosen from n white and n black balls.

10. The random variables X and Y are said to have a bivariate normal distribution if their joint density is given by

$$f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2+\left(\frac{y-\mu_y}{\sigma_y}\right)^2-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}\right]\right\}$$

(a) Show that the onditional density of X, given that $Y=y$, is the normal density with parameters

$$\mu_x+\rho\frac{\sigma_x}{\sigma_y}(y-\mu_y) \text{ and } \sigma_x^2(1-\rho^2)$$

(b) Show that X ad Y are both normal random variables with respective parameters μ_x, σ_x^2 and μ_y, σ_y^2.

(c) Show that X and Y are independent when $\rho=0$

11. The joint density function of X and Y is

$$f(x)=\begin{cases} xy & 0<x<1, 0<y<2 \\ 0 & \text{otherwise} \end{cases}$$

So, part c says that show X and Y are independent when rho is 0. See now here if you look at the expression f x y, then by putting rho equal to 0 you see this coefficient on the root 1 minus rho square will become 1. Then, 1 upon 1 minus rho square will become 1 and the product term in the exponential will become 0. So, the joint density function will become product of marginal of X and Y, right. You can see immediately because I can write 2 pi as root 2 pi into root pi root 2 pi, then sigma x into e raise to minus 1 by 2 x minus mu whole square upon sigma x square into 1 upon rho root pi sigma y e raise to minus 1 by 2 y minus y mu y upon sigma y whole square. So, it will become product of two marginal. So, therefore, 2 x and y. By our theorem, X and Y are independent and the converse is also true that of course if that I have talked about the converse, the actual thing is that if X and Y are independent, then of course rho must be 0. That is what I am saying here, so that X and Y are independent when rho is 0.

So, here I am asking you to talk about the converse. The theorem is that if X and Y are independent, then rho must be 0. So, this is what you have to show and also, what I am saying is that the converse is true. That means, if rho is 0 for a bivariate, a normal random variable, the x, y being a bivariate normal distribution having a bivariate normal distribution. Then, if rho is 0, we can also show that X and Y are independent. So, we will discuss in lecture 17 also and then, later on I will show you that the covariance of a bivariate normal random variable, where x, y is given by rho. So, this we will discuss much later in lecture 23.

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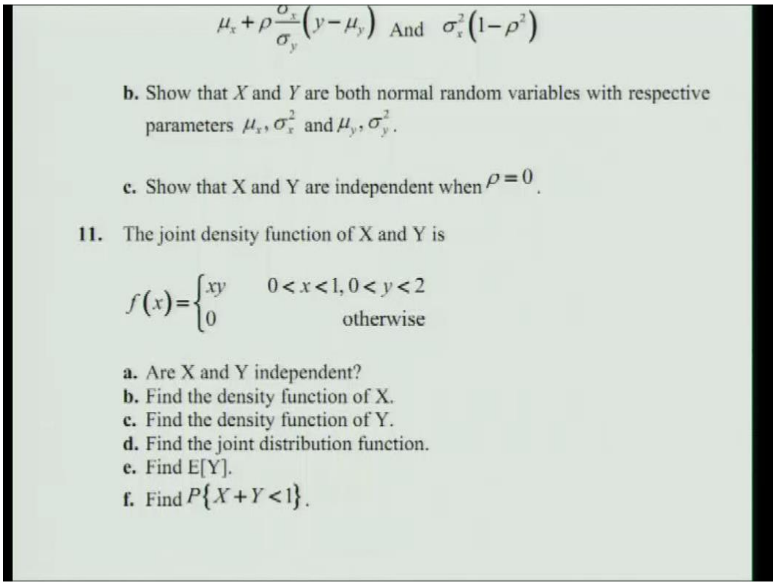

See in part c of question tenth, we have to show that X and Y are independent when rho is 0. So, what I have we have said so far is that if eleventh problem, the joint density function of X and Y is given like this and here, x varies between 0 and 1 and y varies between 0 and 2. First question is X and Y are independent? Yes, you can answer because the limits are separate and the joint p d f can be separated into X into Y, but I would like you to find out. So, anyway you are finding out the density function of X, the density function of Y and then, find the joint distribution function. So, you are asked to find the cumulative distribution function and then, find E Y and find probability X plus Y less than 1. So, again this is I have just included this as an exercise that you get more familiar with how you work out these different integrals.

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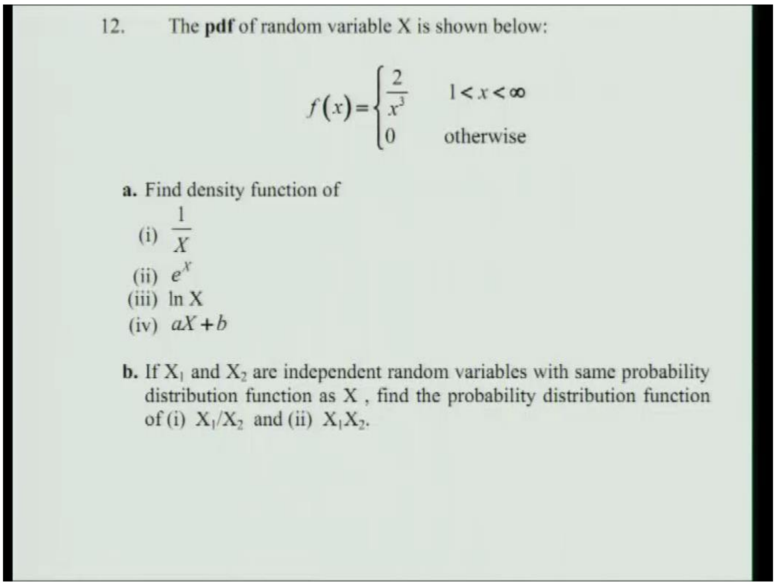

This is another problem number 12. The p d f random variable X is shown below. This is a single one. Find density function of 1 upon x. So, this I have included because I thought we have not discussed many functions of a single random variable. So, therefore, find the density function of 1 upon X, e raise to x, ln X. 1n x is again the log with base e and a X plus b. Again, it is simply an exercise to get familiar with you know how you find the limits and so on. The ranges for different functions and so if X 1 and X 2 are independent random variables with same probability distribution function as X, find the probability distribution function of X 1 upon X 2 and X 1 X 2. So, you may feel that somewhere other things are repeated. It does not matter as much as practice as you can to get a good feeling about how you handle this.