Introduction to Probability theory and its Applications

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Lecture - 1

Basic Principles of Counting

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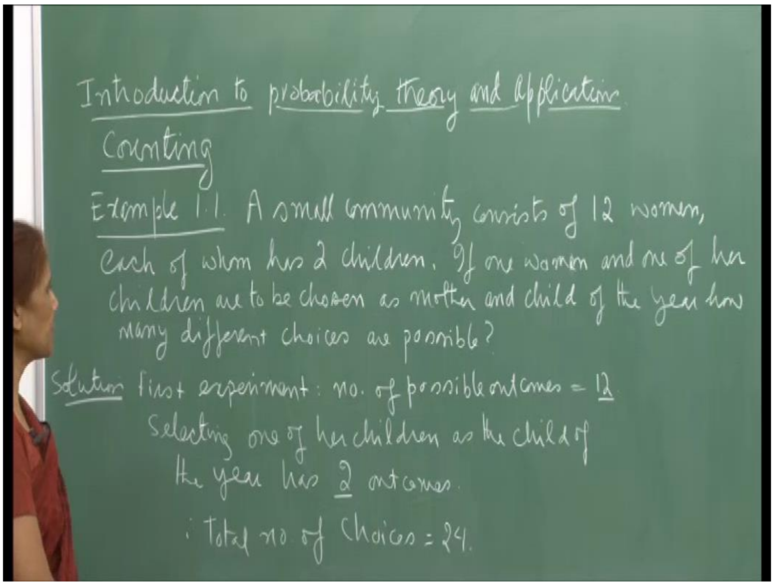

So, we will be talking about the course on probability theory and its applications. Somehow, when you utter the word probability to a layman, it does not sound very familiar or you know, one does not feel very comfortable with it. But my attempt in this course would be, that at the end of the course you feel, that you can understand what all goes behind computing the probability of an event.

So, basically probability theory is estimating the possibility of outcome of an event. So, this word possibility, that what are the, how possible, how probable the event is, is actually done by counting. And so therefore, before I start giving you axioms of probability theory, I would like to begin with the basic concepts of counting because that helps you in estimating the possibility of the occurrence of an event and I will try to explain what we mean by an event and so on. So, as we go on, hopefully, that you will understand all these terms.

So, let me just begin by, so therefore, the first topic I am going to talk about is counting and I will start with an example. So, suppose there is a small community, which consists of 12 women and each of whom has 2 children. Now, if one women and one of her children, so it has to be the pair, a women and her child. So, if the pair has to be chosen as mother and child of the year, how many different choices are possible? So, let us just, very simple way of counting.

So, what I am saying here is, that first experiment would be number of possibilities of the, first experiment would be choosing the mother. Out of the 12 women, we choose one of the woman as the mother of the year. And so the number of possible outcomes is 12 because there are 12 women and one of them will be appointed or chosen as the woman of the year.

And then we will, so second experiment would be selecting one of her children as the year as the child of the year. So, once you have chosen the mother of the year, then she has 2 children. So, only one of them can qualify to be the year of the child, I mean, child of the year. So, therefore, the possible outcomes are 2. So, then we say, that the total number of choices is 24.

So, I sort of broke up this event, though I will call it event of choosing the mother and child of the year by first saying that I will chose the mother, and then one of her children would be chosen as the year, child of the year, right. So, therefore, the number of choices are 24, right.

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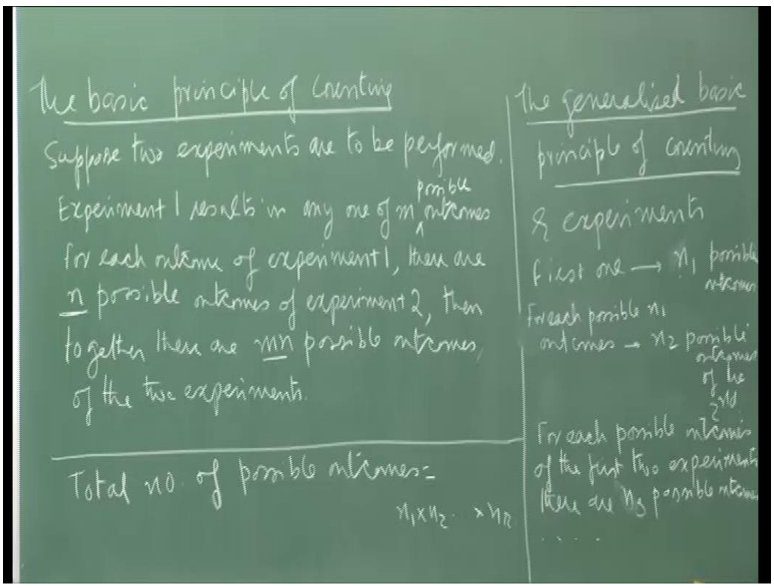

So, if, if now I want to sort of formulate this into the basic principle of counting, then the basic principle says that if there are two experiments to be performed and experiment 1 results in any one of impossible outcomes. So, I have way of finding out, that whatever my first experiment and the possible outcomes of that event are m.

So, for example, here my first experiment was choosing the year, mother of the year and since there are 12 woman, out of which I have to choose the mother of the year. So, there were 12 possible outcomes of this experiment. Any of the 12 women could have been chosen as the mother of the year. So, the first experiment results in m possible outcomes. Now, for each outcome of experiment 1 that means, for each possible outcome, each of the m possible outcomes here then I want to know what are the possible outcomes of the second experiment?

So, in that example mother had 2 children. So, it could only be one of the 2 children who would be chosen as the mother of the year. So, now here we will say, that if for each outcome of experiment 1, there are n possible outcomes of experiment 2, then together there are mn possible outcomes of the 2 experiments, right.

So, I hope this, once you understand this basic principle, of computing, of counting, then you know, things become simple because you have to first, first find out what are the possible outcomes. And then we will talk about, when we define the concept of an event, then we will try to find out in how many ways that particular event can occur. So, here the total number of this thing are this.

Now, if you want to generalize this concept of counting, so this will be generalized basic principle of counting. And here, we will now say, that suppose I have r experiments, which could be 1, 2, 3 and whatever the number and then again we will start saying first one, the first experiment results in n 1 possible outcomes. So, the first experiment results in n 1 possible outcomes. Now, for each possible n 1 outcomes of the first experiment, right, for each possible n 1 outcome of the first experiment there are n 2 possible outcomes of the second experiment, right. And then the third stage we will then count for each possible outcomes of the first two experiments, right, because two experiments have taken place.

So, for each possible outcome of the first and the second, that means, now you have n 1 possible outcomes of the first experiment, n 2 for each of the experiment outcome here, you have n 2 possible outcomes. So, total number of outcomes become n 1 n 2. Now, for each possible outcome of n 1 and n 2, the third experiment, if possible, outcomes will be n 3 and so on. So, you will go on counting this and therefore, by a simple arithmetic you can see, that the total number of possible outcomes would be n 1 into n 2 into n r. That means, at each, for each experiment whatever the possible outcomes, you will multiply them all. So, this gives the generalized principle of computing.

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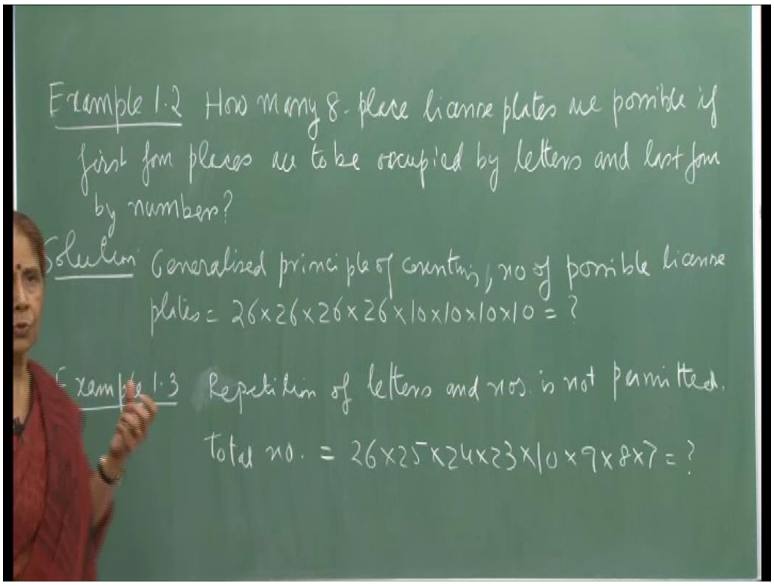

So, let us look up at this example, example 1.2. How many 8 place license plates are possible if first four places are to be occupied by letters and last four by numbers? That means, the first four can be any of the alphabet A, B, C, D. So, there were 26 choices for the first, for first four places and then for last four have to be the numbers. So, which means, there can be any of the digits 0, 1, 2, up to 9, right.

So, now again, as I quoted the generalized principle of counting, so number of the first experiment would be choosing the first place of the license plate, right. So, then I have 26 choices because 26 alphabets are there. Then, again, for each of the alphabet I choose here, for each of the 26 alphabets I choose here, I can again choose the 26 alphabets again, right. If I have a letter A here, for example, then I can choose any of the A, B, C, D, 26 alphabets here. So, again the next outcome, that means, the outcome for each outcome here, there are again 26 choices of the second experiment and so on. So, again for the third.

That means, now once have chosen the first 2 alphabets here, then again for any each of these outcomes, 26 into 26, I can again choose in the third place any of the 26 alphabets. So, then the third choice is again the number of possible outcomes are the number of choice, choices are 26. And same, the same argument goes for the fourth place.

And similarly, for the numbers I have chosen 1 of the 10 numbers here. Then, again after having chosen all these, I can again choose any of the ten numbers here and then in the third place also any of the ten numbers and so on.

So, this will be and you can just see, I have left a question mark for you to compute the number it will be a big number. So, this many license plates can be there if you have this kind of arrangement that the first four places have to be occupied. So, it is eight letter license plate number and so the first four places have to be occupied by alphabets and the last four by digits, right.

Now, if I change the experiment and I say, that repetition of letters and numbers is not permitted. So, the moment I say that once I have chosen a letter here, then the same letter will not be repeated here, here or here. So, then by a way of counting will be this. Now, for the first place I have any of the 26 choices, any of the 26 letters can be chosen here. But once I have chosen a letter here, then that same letter is not permitted to be chosen here. So, now, my choice at for the second experiment for every possible outcome here, goes down to 25 because whatever alphabet I have chosen here, I cannot chose it again. So, therefore, my choices here are 25.

And once I have chosen alphabet for the first place and the second place 2 alphabets have been selected, then both these are not allowed to be chosen again. So, therefore, my choice for the third place would come down to 24. So, out of 26, two letters have been already chosen and they are not allowed to be chosen again. So, this would, choice will be 24. And then similarly at the fourth place I will be having a choice of 23. So, if you just keep using this generalized principle of counting, which I enumerated sometime ago, then you see, this is how you will, right, done the possible outcomes.

And similarly, for the numbers, for the first place I will have choice of 10, but once I choose a number here, then my choice is limited to 9, and after this again I can only choose out of the 8, which are not repeated, which have not been already be chosen. And similarly, for the fourth place my choice will be left limited to 7 numbers, which have not been repeated. So, this is the kind of…

So, the generalized principle of counting helps you to come out with the possible number of outcomes of an experiment, right, or event here. What we want to say, because this was, as I am counting, I am saying this is eight experiments choice of each letter for each place of the license plate and the choice of number I counted as experiment. So, had eight experiments here and used the generalized principle of counting to find out the possible number of outcomes right.

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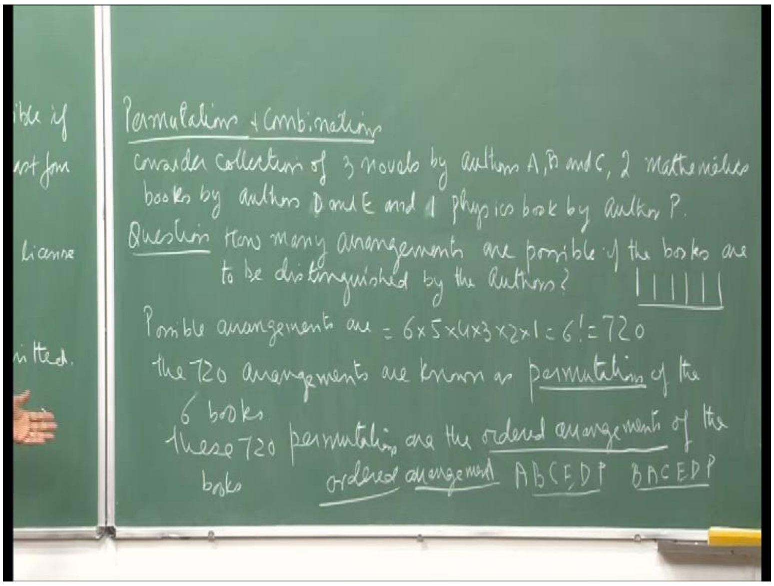

Now, another possible another way of counting is done by through permutations and combinations, and let me now here try to explain the difference between permutations and combinations.

So, here let us say, we have collection of 3 novels. And so I will just say, that authors are A, B and C, then 2 mathematic books and again I will distinguish them or differentiate them by the authors and call the authors as D and E and one physics book. Now, I will call the author as P. So, there are 6 books. And since I am distinguishing books by the authors, so it is not just novels, I am saying, that the, I mean the authors are A, B and C. So, they are different novels I am distinguishing. Similarly, I am differentiating between the 2 mathematics novels and of course, there is one physics book. So, the question is how many arrangements are possible if the books are to be distinguished by the authors.

So, I am, I am wanting to make arrangements of these books and arrangements would be, would be differentiated because the, I will be, I will be referring to the books by the authors. So, it is not just novels or a mathematics book. So, therefore, possible arrangements, now here again let us just say that we are, yeah, ok, right. So, here that means, in the first place if we now count the thing, that means, I can, I can say, that there are 6 places, right.

So, now, the choice for the first place, that means, I am just arranging the books like this in a line, in a row, suppose. So, the choice for the first place would be any one of the 6 books, right, because I am differentiating the books by the author. So, therefore, for the first place the choice is 6 books. Once a book is placed here, then there will be 5 more left, right. I am saying again. So, therefore, out of the 6 experiments I am just saying what are the possible outcomes. So, once I have chosen 2 books here, then I am, I am left with picking out the book for the 3rd place from among the remaining 4 books and then 3, 2 and 1. So, the total number is 6 factorial, which is 720, right.

Now, these 720 arrangements are known as permutations of the 6 books. And the permutation word is essentially saying, that the permutations are the ordered arrangements of the books. Now, ordered, I mean, that if I am writing, say for example, ordered arrangement, I want to say ordered arrangements. So, if I am writing here, say it could be, that I am, right, I have chosen all the 3 novels here, right and then of course, let us say, mathematics books you have D and E.

So, it may be, you have this and you, right now if I have the order B arrangement, B A C E D P, then you see this arrangement is different from this. Because now here is the 2nd author, if we call the book by B as 2nd novel, then the 2nd novel is occupying the 1st place and this is 1st, the novel by the 1st author is occupying the 2nd place. So, here the arrangement and therefore, this, this order is also considered here in this arrangements and therefore, I will say, that this is different from this order, this arrangement is different from this order, that arrangement B A C E D P.

So, this is the idea, that 720 permutations are the ordered arrangements of the 6 books. And so how you place them depends on, that means, I am distinguishing between which author is occupying the 1st place, which author is occupying the 2nd place and so on. So, therefore, this way it will be 720. But if you do not distinguish between the, if you do not distinguish between the authors, that means, I just treat the 3 novels as novels and in that case, this and this arrangements would be the same, and right. And in fact, you can see how many possible arrangements.

See, I can have here C A B E D P. Now, you can tell me 3 more because these 3 letters, A, B, C themselves, I can arrange in 3 factorial ways, that will be 6, 6 ways I can arrange these 3. And therefore, all those arrangements, once if I do not consider the, I distinguish, do not distinguish between the 3 novels by the authors, as long as they are just novels, then that arrangements, all those 6 arrangements will be counted as 1 arrangement.

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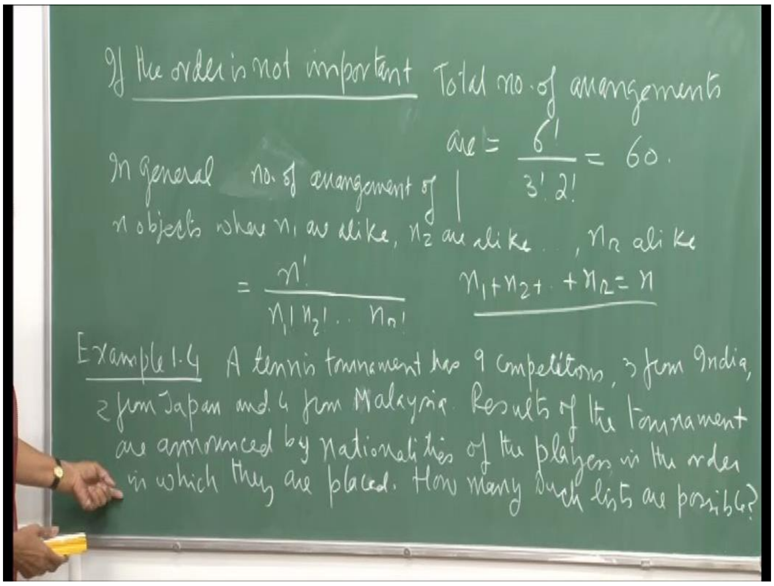

So, when we are looking at the arrangement of the books, the novels, 3 novels, 2 maths books and 1 physics book. So, once we do not order, we do not worry about differentiating the books with respect to the authors. In that case, you see, as I tried to show you, that the 3 novels, since they will not be distinguished by the authors, so then for every arrangement, for every arrangement in which only the arrangement of the, arrangement of the author has been changed, the other books remain the same. In that case, 6 of those arrangements will amount to 1 arrangement because the 3 novels can be arranged in 6 different ways.

And if I leave the math books and the physics book intact, then all 6 arrangements in which only the arrangement of the novels have been changed, then those 6 arrangements will amount to 1 arrangement. And similarly, corresponding to each of those, if I leave everything intact and only disturb the arrangement of the math books, then since I am not differentiating between the author, those arrangements will not be different because that 2 math books, whether the author D or E, whichever comes first does not matter to me. So, in that case, I will then divide.

So, that means, when I am counting the arrangement where the order is not important and that comes out to be 6 factorial is, was the arrangement, total number of arrangements where the order was important. But if I do not care about the order, that I mean, I do not distinguish between the authors, in that case the arrangements will become 60. So, this is what now I am trying to come to. So, the permutations, the order was important. And then I had total number of permutations. That means, that the order of the objects was also being considered right.

So, now, the general principle, that I am trying to ((Refer Time: 18:08)) that in general, number of arrangements of n objects, where n 1 are alike, n 2 are alike, and n are alike, n r are alike, that means, there are only r different kinds of objects. And since, so therefore, surely here n 1 plus n 2 plus n r adds up to n, right. So, have n objects. But n 1 of them are the same, then n 2 are alike and similarly, n r of them are alike. And in that case, when I am wanting to make, arrange the n objects, then the total number of objects would be, because n factorial would be the arrangements of n objects when I am distinguishing between each of them, right, whatever the way of distinguishing the object, whatever it is. So, that would the total arrangement.

But since I had tried to show you through this example of arranging the books, that if n 1 are alike, then those n 1 can be arranged in n 1 factorial ways and they will amount to the same arrangement because I am not differentiating between the order. So, therefore, I will divide this total number of arrangements here by n 1 factorial, n 2 factorial and n r factorial. So, that will give me the total number of, so that becomes the number of combinations. Well, that means, arrangements where the order is not important would be the number of, total number of combinations.

Now, let us look at this example. So, a tennis tournament has nine competitors, competitors. So, and then 3 from India, 2 from Japan and 4 from Malaysia. Now, if the results of the tournament are announced by nationalities of the players in the order in which they are placed. So, it is not, you know, you are not distinguishing the place by the names, only their nationality is being considered. In that case, 3 from India would be just 3 Indians and it does not matter which one gets the first position, third position or whatever it is. So, here it will be just all the 3 players would be considered the players from India.

So, another way of looking at it is, that you can think of these people as one entity because they, they are just representing the same country. So, similarly 2 from Japan would be just distinguished by their nationality and not by their names and 4 from Malaysia.

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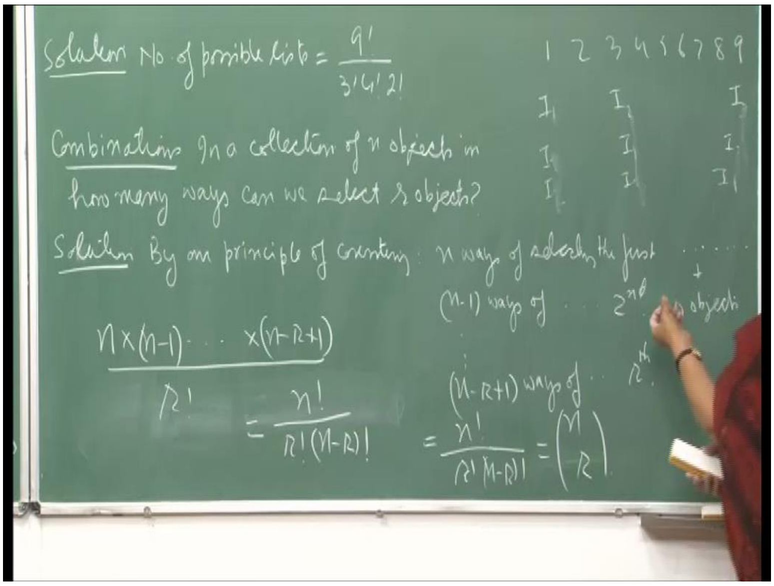

So, therefore, if you want to now find out how many such lists are possible, that means, what, what is the arrangement of the position, that these 3, the players from the 3 nationalities occupying in your list, first, second, third, up to there will be 9 positions. So, then the total number of lists, that you can have would be 9 factorial. So, 9 is the total number of players. And if I was going to distinguish them by the names, in that case, they would have been 9 factorial arrangements of the way in which the order, in which players would occupy position in the tournament. But since we are not considering the names, we are only considering the nationalities.

So, for example, we then divide this number by 3 factorial, 4 factorial and 2 factorial to get the total number of lists. So, the 3 Indians can occupy any of the positions and they will be the same. So, that means your 6, 6 lists in which if 3 Indians were occupying different positions in the sense, that I just arrange, rearrange the position, that are that means, 3 Indians, if they just appear. See, suppose you consider the 1st position, 2nd, 3rd, 4th, 5th, 6th, 7th, 8th, 9th, same thing. So, if this was Indian, this is an Indian and let us say, this is an Indian, right.

Now, what are the 3 names? If I see, if you had a name, here is the Indian 1, Indian 2 and Indian 3. Then, I can have Indian 2 here, Indian 1 here, Indian 3 here and this way you can go on, right. Indian 2, Indian 3, Indian 1 and just see, that you can write, you can rearrange these 3 in 6 possible ways. So, other arrangements will be the same.

But if these 3, I can arrange in 6 different ways, but for me they are the same because in my list it will appear as I, I, I, I, I, I. So, this will not be there, right. So, all these arrangements will be the same and so all these 6 arrangements will account to same. Therefore, I am dividing this number by 3 factorial and similarly, for the 2 from, I mean, sorry, here, in this case here, that was the example for the books. So, here I am dividing this number by 3 factorial, 4 factorial and 2 factorial to get the possible number of lists. So, just trying to make the concept of counting clear, so that you can, then you know, when you have to compute the possibility of the occurrence of an event, all these things will come in handy.

So, now again, I am trying to enunciate this principle, that if you have a collection of n objects, in how many ways can we select r objects. That is again the same thing because you can say, that you can put the n objects in a row and then you are picking up r out of them. So, that, that is the different arrangement. This is another way of saying the same thing, right.

So, here, by your principle of counting, when you have, you can say, that you have arranged all these n objects in a row and then you are picking of one of them. So, then for the first place, that means, when I want to pickup r objects from this list of n objects, the first object, there are n possible ways of selecting the first. So, we are counting the same thing again by a different way. So, here you have n ways of selecting the first object. Then, since you have already selected the first object, you are left with now n minus 1 ways of selecting the second object, right. And similarly, it goes on depleting. And finally, for the rth object that you want to choose, you are left with n minus r plus 1 because you have already selected r minus 1 objects. So, n minus r minus 1, this is the number. So, this many ways you can select the rth object, right.

Now, since it does not matter again because I am not distinguishing between the r objects, that means, the order is not important once. You see, out of these n objects I have collected r objects. So, it does not matter which one came out first, which one came out second, as long as the set of objects is the same. So, that means, my way of selecting may have been, you know, because I picked, when I picked up the 1st object, that counted in the 1st object, 2nd and 3rd, so on. But then finally, what I have with me is the set of r objects and it, though in, in that set the 3rd or the 4th object could have been selected 1st, does not matter, right. So, therefore, what we are saying is, now you have n minus r plus 1. So, that means, the total number I should have written the number here, let me write somewhere here.

So, the order is, when the order is immaterial, I am saying, that this is, let me, let me just write it here. So, that means, what I am saying is, that the first object I could pick in any of the n ways. Then, for the second object to pickup I had only n minus 1 choices available with me and so on, up to n minus r plus 1, right. So, this number. And since what we are saying is, that the, it was not important the order in which the r objects were selected, it is only the final subset of r objects, that I have with me. So, therefore, I will divide this by r factorial and so you can, right.

This number, you see, if you have up to this, this number, you can write as n factorial divided by r factorial n minus r factorial. So, this is the final. So, that means, this is the number of combinations I can have, because by selecting r objects I can call an arrangement, that was here. That means, the n objects that I have, I pick up r, I put them first and then the remaining n minus r. So, essentially the r objects, that you pick, it does not matter in what order you pick.

So, therefore, the total number of ways in which I can arrange my n objects gets divided by r factorial and obviously, the last n minus r objects also. The order is not important because finally, you have divided these n objects into r n n minus r, right. So, it is simply, which r objects get selected, it does not matter in what order they got selected. So, therefore, the total of number of ways in which you can pickup r objects from n objects would again be given by this number and I would, normally we say n choose r.

This is the notation for this number n factorial divided by r factorial n minus r factorial. So, this is n choose r.

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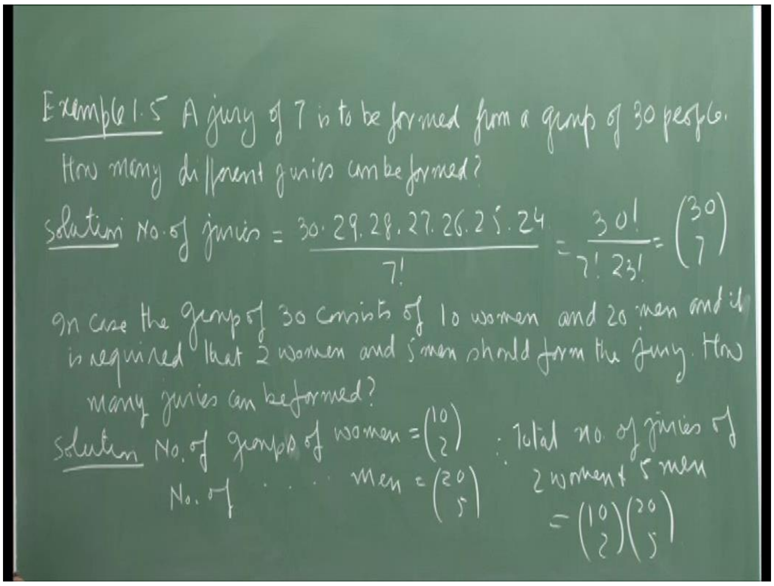

So, again let us take up an example. So, a jury of 7 is to be formed from a group of 30 people, how many different juries can be formed? So, just carry over what we discussed just know. So, there are 30 people and you have to pick up 7 people from this set of 30 people, which will form a jury, right. And so we want to know how many, the question is how many different juries can be formed. So, this is like we discussed, n objects, you want to pickup r objects from there and how many ways are there, how many possible combinations are there, that is what we are looking at. So, obviously, you can see, that the number of juries would be.

So, first, first person in the set can be chosen out of the 30 people. Then, once you have chosen that person, you are left with 29 choices and again, after these two you are left with 28 and so this number is 24. And now, what we are saying, that it was, it was not at all important as to when we have selected the 7 people. Since it is the subset of 7 people, it does not matter which one of them got selected first. So, the order is not important. It is only, that finally I have this subset of 7 people. So, therefore, 7 we will have to divide by 7 factorial to get the possible number of juries because this 7 set of people that you are forming the jury, you are not looking at when the first, when the, in what order they got selected, that is not important, right. That is what I am emphasizing again and again, that here the order of selection is not important, it is just that you want a subset of 7 people and therefore, this will be the total number of arrangements.

And so here with this number we can write, because this is up to 24. Remember, this is n minus r plus 1. So, here your n is 30 and your r is 7. So, when you want to divide, from 36 you are left with 24 and so this is the total number if it was important as to in what order people got selected for the jury. So, it is not important, it is not being considered, it is immaterial. So, therefore, I divide the 7 factorial, then this number can be written as 30 factorial divided by 7 factorial into 23 factorial, which by our notation is 30 chose of 7.

Now, in case the group of 30 consists of 10 women and 20 men and if it is required, and if it is required that 2 women and 5 men should form the jury. So, now, we are saying, that out of 10 women 2 must get selected and out of the 20 men 5 men should get selected to form the jury. So, in that case the number of groups of women that you can form out of, you know, by choosing 2 out of 10, this will be 10 chose 2. Here, again the order is not important. I just want to form, select subset of 2 people, 2 women from out of the 10. So, that will be the total number of ways in which I can select the 2 women from the 10 women.

And similarly, I can, 20 choose 5 is the number of ways in which I can select 5 men from the set of 20 men. And so the total number would be, because for each subset, that I chose here of women, there will be this many. So, I am now using here by generalize principle of counting. And so the total number of ways in which I can, the number of juries of 2 women and 5 men can be formed is this. So, just through examples I am trying to make the concept clear.

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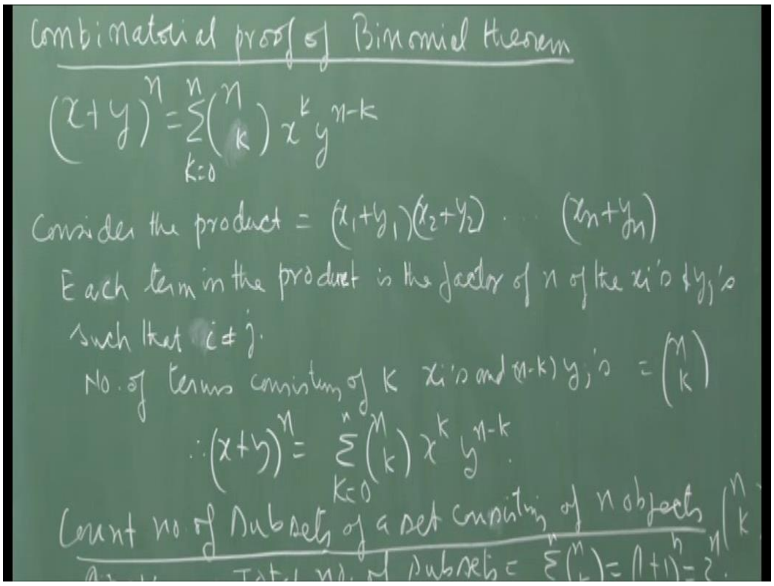

And now, let us see, that you can, all of you have used binomial theorem and you know, that if you want to express, if you want to expand this number x plus y raise to n, then the formula is n choose k x raise to k y raise to n minus k, and you have done it by induction and so on, the proof. Now, let me give you this combinatorial proof of binomial theorem using this concept of counting, that we have learnt here. We can use this and show, that the expansion of this term can be written in this way.

And so let me consider the product x 1 plus y 1 into x 2 plus... So, there are n terms here, right. Now, when I am multiplying each term here, final in the product would be consisting of n factors, factor of n of the x i's and y j's, that means, each term of this, project, product will contain, when you count the number of x i's and y j's, that total number would be n because you have n factors, right. But certainly, in any factor when you look at the x i's and y j's, if the, if an x i appears, then in the same index will not be for y because you see, x 1 and y 1 are in the same term here. So, therefore, when I multiply, either I will be multiplying x 1 by the other terms and so then y 1 will not appear in that, right.

So, therefore, in any product, the x i's and y j's that you have, if an x i appears, the corresponding y j will not appear in that product, right. So, in other words what I am saying is that the any term of this product, which will contain, let us say, k of the x i's and then the remaining n i n minus k of the y j's would be such that. So, you are choosing, essentially, each term here would contain, let us say, when I am looking at the term containing only k of the x i’s and so remaining n minus k or y j’s. So, then how many ways can be there? How many such products can be there in which...

So, you know this is containing the each term in this product has n factors containing x i’s and y j’s. So, then if I am looking at all the terms, which contain k of the x i’s, right, then that means, I am out of the x 1, x 2, x n, these n x’s I am choosing k of the x i’s, right. And then of course, the moment I choose my k x i’s, then the y j’s got to the n minus k y. y k’s, y j’s got, get select automatically because in the ones, which are not appearing in my x 1, x 2, the product x i’s that I have now taken, now remaining indices will be, will go to the y j’s right. So, here, that means, and that was the way of choosing k x i's out of the n x i’s x 1 x 2 x n in n choose k, right.

So, therefore, the number of terms, which contain k of the x i’s is this many terms. And so this whole product, either what will happen, say for example, if you look at it, product y 1 y 2 y n, no x i appears. So, either, so the k can vary from 0 to n. The terms will contain either 0 x i’s, then 1 x i, 2 x i’s and so on. So, you want to add up all such things. You are counting the total number of terms in this product and that total number will be given by summing up n choose k from 0 to n. And when you choose like this, then that means, your k of the x i’s are here and n minus k are you are adding up, right.

So, these things, then I put, put of x i equal to x for all i and y j equal to y for all j. So, in that case, your k of the x i’s, that you took from here, they all become equal to x, x raise to k y raise to n minus k. So, then therefore, this product and when once you put the x i’s together equal y i’s are together, then this whole product coincides with this. And so you have a nice way of, you know, proving the binomial theorem.

Now, I can, immediate application of this is, that if you want to count, you know, have n objects and you want to count how many subsets can be there of these n objects, you, when of course, the empty also be considered as a subset. That means, nothing gets chosen from the n objects. So, you have subset consisting the empty set. Then, you can have subsets consisting of only one element from one object, from these n objects, subset containing 2 objects, 3 objects and so on.

So, if you want to count the total number of subsets, that you can form of this n objects, then since I told you, that n c k gives you the number of subsets of size k, right, and so you want to count this number n c k from k varying from 0 to n. And which, if you look at this expansion here, essentially of putting x is equal to 1 and y is equal to 1, so that this all becomes 1. And so you are summing up on this side when you put x is equal to 1 and y is equal to 1, this reduces to 2 raise to n. So, the total number of subsets. This is another nice way of counting the number of subsets that you can form, given n objects.

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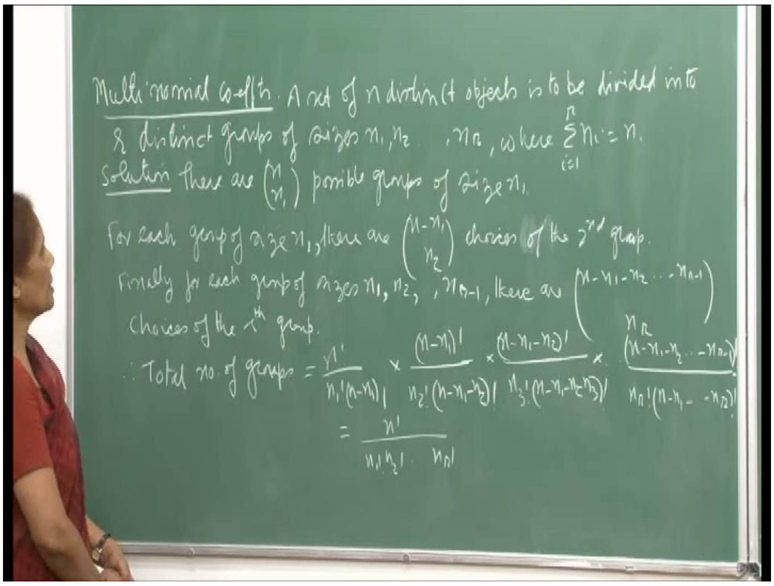

So, extending this concept that I just enumerated for you for the binomial theorem, now we can take over, go up to multinomial coefficients. So, this is a set of n distinct objects is to be divided into r distinct groups of sizes n 1, n 2, n r for the binomial theorem, r was 2. So, now it is n 1 plus n 1 n 2 n r where of course, all the n i’s adapt to n. So, I am dividing this n objects into r sub groups. And the size of each group, the first sub group is n 1, size of second group is n 2 and so on, right.

So, then using the same principle I want to choose form n, n 1 possible groups. So, the number of possible groups of size n 1 is n n 1, n n choose n 1, right. Are you also, this is, actually this is also written as n c n 1. So, combinations, so n 1 combination, that means, you want to choose n 1 objects out of n. So, what are the possible number of ways? So, any of these notations is acceptable, fine.

So, the first group will be, the number of groups of sizes n 1 would be n choose n 1. Then, since I have already chosen n 1 objects out of n, so then I want to choose the second set of objects, n 2 consisting of the set of objects must be n 2 in size and then. So, out of n minus n 1, I want to choose n 2. So, therefore, each group of size n 1, the number of choices of the second group is n minus n 1 choose n 2, right. And so you extend this concept and so finally, the last choice would be, because now you will be left with n minus n 1 minus n 2 minus n r minus 1, r minus one subsets.

r sub groups have already been chosen. So, then these many, you are left out of this, you want to choose n r objects. So, again the number of possible ways is n minus n 1 n 2 minus n r minus 1 choose n r right and. So, we will say, that the total number of groups is the product of all these, right. And so via notation this would, well the first one would be n factorial divided by n 1 factorial n minus n 1 factorial. Then, the second one would be n minus n 1 factorial divided by n 2 factorial n minus n 1 minus n 2 factorial and so on.

So, if you see, that this will be my total expression and then the terms cancel out because you have n minus n 1 factorial here and this is n minus n 1 factorial. So, it cancels out. So, these terms will cancel out, you know, in pairs. And so in the denominator you will be left with n 1 factorial, n 2 factorial and n 3 factorial up to n r factorial.

And this is what, again what we are saying is, that we are deciding whatever objects we have, putting in one group we are saying they are all alike, the same principle you are using and so we, the arrangements will not matter in whatever way you choose, in whatever order you choose, it is immaterial. So, therefore, again I am, I can arrange my possible objects, n objects in n factorial ways. But since I am, I am grouping them into r different sub groups and each group, the number of objects is, we are not differentiating between the objects in one group. So, then the total number of ways would be n factorial divided by n 1 factorial n 2 factorial and n r factorial, right.

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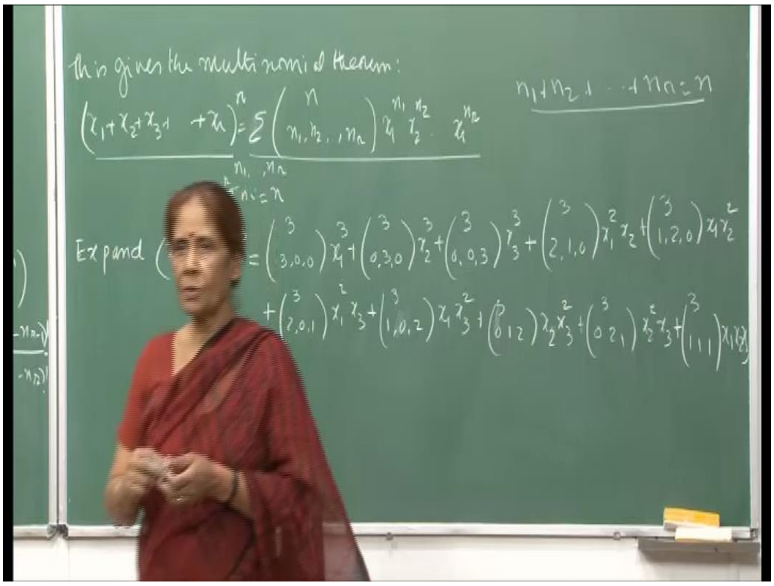

And now, this helps us to expand the examples. This gives the multinomial theorem, that if you want to expand x 1 plus x 2 plus x 3 plus x r raise to n, then it will be, because remember, in this product, just as I showed you for the binomial theorem, you want to choose. So, because each product here will consist of some powers of x 1, some powers of x 2, some powers of x 3 and x r. And so here I am now saying, that this n 1, n 2, n r will vary from, because every product here will contain 0, x 1, 1 x 1, 2 x 2 to 2 x 1 and so on and different. That means, whatever the powers, each term here you can see in the, in the product this term will appear and so your n 1 plus n 2 plus n r must be… Because each term, this is raise to n.

So, each term in this expansion will contain n x i's together. That means, you add up the indices of x 1, x 2, x r, they should add up to n and that is what is given by this, right. So, therefore, using this concept this is how you can write the expansion, right. And if you look at this example, where x 1 plus x 2 plus x 3 raise to 3, then you see, I can choose my n 1 to be 3. So, that means, this subsets contains just the power of x 1, powers of x 2 and x 3 are missing here, right. So, therefore, the expansion, this is 3, 3, 0, 0, x 1 raise to 3, then here again you are choosing a subset from 3, which consists only of powers of x 2. So, 0, 3, 0 and so it will be x 2 3.

So, I showed you the, you know the idea behind this. And then now I am using it repeatedly to say that how I can write the expansion of this, right. And so therefore, you can see, that the way I can choose my numbers n 1, n 2, n r. So, if they add up to n, so in the case when you have your r is 3 and your n is also 3, so then these numbers must add up to 3, right. You are dividing your number 3 in three possible ways, so that they add up to 3. So, here these are the possible, right. And the final thing is 3, 1, 1 when each x 1, x 2, x 3 has only power 1. So, this is the expansion.

And now, in, in, in the assigned exercise sheet, that we will just show you I am asking you how many terms are there in the, in the multinomial expansion and remember, because which is actually counting number of subsets, right, here essentially accounting the number of subsets, which you can have, you know, like you have r, this thing here x 1, x 2, x 3, x r. So, essentially my question is how many terms there in this expansion. And so I have already discussed this case with you for the binomial, how I use for answering the number of subsets of, total number of subsets of n objects. So, here you have to use the same concepts and tell me how, how many terms are there in this multinomial expansion. So, let me just show you the, discuss the exercise sheet.

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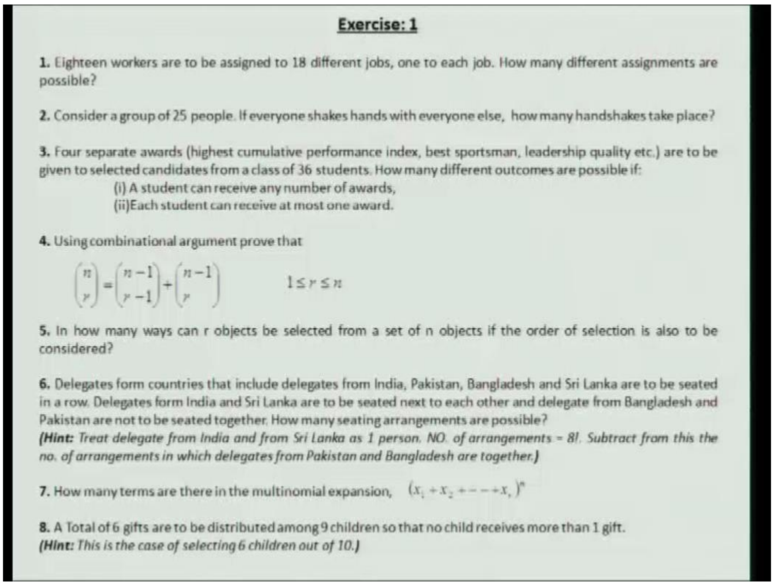

Exercise: 1

1. Eighteen workers are to be assigned to 18 different jobs, one to each job. How many different assignments are possible?

2. Consider a group of 25 people. If everyone shakes hands with everyone else, how many handshakes take place?

3. Four separate awards (highest cumulative performance index, best sportsman, leadership quality etc.) are to be given to selected candidates from a class of 36 students. How many different outcomes are possible if:

(i) A student can receive any number of awards,

(ii)Each student can receive at most one award.

4. Using combinational argument prove that

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \qquad 1 \le r \le n$$

5. In how many ways can r objects be selected from a set of n objects if the order of selection is also to be considered?

6. Delegates form countries that include delegates from India, Pakistan, Bangladesh and Sri Lanka are to be seated in a row. Delegates form India and Sri Lanka are to be seated next to each other and delegate from Bangladesh and Pakistan are not to be seated together. How many seating arrangements are possible?
***(Hint:** Treat delegate from India and from Sri Lanka as 1 person. NO. of arrangements = 8!. Subtract from this the no. of arrangements in which delegates from Pakistan and Bangladesh are together.)*

7. How many terms are there in the multinomial expansion, $(x_1 + x_2 + \cdots + x_r)^n$

8. A Total of 6 gifts are to be distributed among 9 children so that no child receives more than 1 gift.
***(Hint:** This is the case of selecting 6 children out of 10.)*

Question 1 is straight forward, 18 workers are to be assigned to 18 different jobs. So, the important part here is, that each job is different from the others and therefore, this will be ordered arrangement of the 18 workers. So, you can write down how many possible assignments are possible. That means, an assignment would mean, that you assign one particular worker to a particular job. So, then in how many ways you can do this is the idea here, right.

Now, consider a group of 25 people. If everyone shakes hands with everyone else, how many handshakes take place? So, that means, that persons shakes heads with another person, then both have shaken hands with each other. So, just keep that in mind. And you can immediately write down what the answer will be for number 2.

Now, here in question 3, four separate awards it can, it can be, you know, somebody getting the highest marks, highest cumulative performance index, best sports man, leadership quality, etc. So, you can have 4 different awards given to these, to the students. The idea is to select from a class of 36 students, students who can be given this award.

Now, in case student can receive any number of awards, then it will be different number of arrangements, that you can have or different ways in which 4 awards can be given to student or more than 1 student. In number 1 the condition is, that the students can receive any number of awards. So, you have to do a counting that way. In number 2, we say, each student can receive at most one award. So, that means, either a student receives an award or student does not receive an award. So, this, this is the way you have to count.

Now, question 4 is interesting. Using combinational argument prove, that n chose r. That means, selecting r items from n items is equal to selecting r minus 1 items from n minus 1 plus selecting r items from n minus 1. So, you can, if you write down the expression for n minus 1 chose r minus 1 and plus n minus 1 chose r, you can show, that these two add up to n chose r.

But I want you to give a combinatorial argument and you can see here, in the first term it says, n minus 1 chose r minus 1. That means, I am keeping away one particular object or item from the n that are there. Then, I am selecting r minus 1. So, that means, the first set of numbers, n minus 1 chose r minus 1 gives you the number of ways, which you can pick up, r minus 1 objects from n minus 1 when a particular object is being selected. So, when you add that to this selected group, then it will become r object from n objects.

Now, the second says, that n minus 1 chose r. That means, that particular, see the thing is, either a particular object or item, that you have picked is either there in a collection of r objects or it is not there. So, the first case says, that yes, it is going to be collection of r objects. So, once you add it to the set of r minus 1 objects that you have selected, then it becomes r. In the second one, what you are saying is, this is a set of, this is a set of selections in which that particular object does not appear. So, you have separated that objects and then from the remaining n minus 1 you are choosing r. So this is a kind of combinatorial argument you are giving to prove the identity.

And so you see, I have already shown you, that you can, by combinatorial you can give, show, you prove the binomial theorem. Now, similarly you can try to prove number of such identifiers through combinatorial arguments. Then, question 5 we have already discussed. In how many ways can r objects be selected from a set of n objects if the order of selection is to be considered?

So, question 6, one delegate each from 10 countries that include delegates from India, Pakistan, Bangladesh and Sri Lanka are to be seated in a row. So, the arrangements have to be row delegates from India and Sri Lanka are to be seated next to each other and delegates from Bangladesh and Pakistan are not to be seated together, how many seating arrangements are possible.

So, the idea here is, that you know, you, you have 10 different positions, people are sitting in a row and you want people delegates from India and Sri Lanka to be seated together, right, which means, that I can treat those two as one person. In that case, it will be 9 people, then who have to be arranged because delegate from India and delegate from Sri Lanka have to be together. So, that means, the total arrangement is 9 factorial. But since the people are sitting in a row, that means, the arrangement, that first the Indian delegate is sitting and then the Sri Lankan, or the Sri Lankan delegate is sitting first and then the Indian, so this will count as the two different arguments and therefore, the total number of arrangements would be 2 into 9 factorial in which the delegates from India and Sri Lanka are together.

Now, you do not want people, delegates, Bangladesh and Pakistan to be sitting together. So, now again consider the situation when they are sitting together. So, we will subtract the number of… So, in that case, now look at the arrangements in which delegates from India and Sri Lanka are together and delegates from Bangladesh and Pakistan are together. Then, you know, you will have 8 different positions to arrange because these 2 delegates will be together. That means, they can be treated as 1 person, right, and therefore, it will be, total number of arrangements will be 8 factorial.

But then again you have to, it can be 2 possible, you know, like as I told you, the arrangement can be India-Sri Lanka or Sri Lanka-India. Similarly, it can be Bangladesh-Pakistan, Pakistan – Bangladesh. So, therefore, essentially you will be subtracting from 2 into 9 factorial, the number of arrangements in which all the four, I mean, delegates from Bangladesh and Pakistan are together and delegates from India and Sri Lanka are together. So, that will be 2 square into 8 factorial. And so you subtract this number from 2 into 9 factorial, that should give you the required number of people, number of ways in which you can have the required arrangement.

How many terms are there in multinomial expansion of x 1 plus x 2 plus x r raise to n? So, here again I am wanting you to do this exercise. See, the whole idea, either way I explain to you this expansion was, that you are essentially dividing your number n into r smaller numbers, n 1, n 2, n r ((Refer Time: 51:18)) sum up to n. So, in how many ways can you partition this set of n number into r subsets, so that they all add up to n. This is the whole idea.

And so you will see, that from this identity x 1 plus x 2 plus x r raise to n. You have to put all these x 1 x 2 x r equal to 1 and so you get r raise to n is equal to the number of terms that appear in the expansion, multinomial expansion of this term, right.

Then, finally, a total of 6 gifts are to be distributed among 9 children, so that no child receives more than one gift. This is exactly the same as, you know, in part 2 of 3, right, because here also we said that no student should get more than one award, should get at most one reward, award. And so here also we are saying, that the total of 6 gifts are to be distributed among 9 children, so that no child receives more than 1 gift, ok.