Foundation of Optimization

Prof. Dr. Vishnu Narayanan
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture - 38
Special Lecture

Hello everyone; so last time we talked about various model formulations, and we saw how to formulate some interesting models as integer programming problems. We also lived that you know there are problems, where discrete variables are not part of the input, but in order to model this problems effectively, we do need to use these discrete or mainly 0, 1 type of variables and we saw a couple of examples of that. So, today we are going to switch kiers, and we are going to study the structure of mixed integer programming problem. So, suppose we want to solve a particular mixed integer programming problem, then we need to know, how the solutions look like; so hopefully this lecture will help in deciding or determining the structure.

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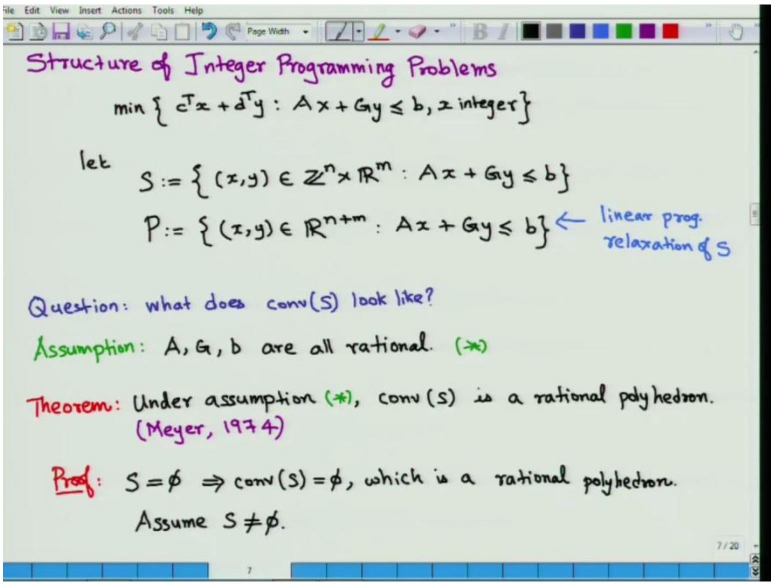

So, let us consider mixed integer programming problem of the following kind. So, I have I want to let say minimize some objective function. So, the objective function, I am going to call it c transpose x plus d transpose y subject to A x plus G y less than or equal to b x integer. So, this is a generic mixed integer programming problem. Now, what we will be concerned today is we would not be concerned about the objective value at all, what we are going to look at is we are going to look at the feasible region, and we are going to study properties of the feasible region.

So, let us, let us introduce some notation just so, that everything is going to be fairly straight forward. So, I am going to denote by capital S is the set of all x comma y where x is integer and y is continuous such that, A x plus G y is less than or equal to b and I am also going to introduce another object, which I am calling P which is nothing but the set of all x comma y in R n plus m such that, A x plus G y less than or equal to b.

So, here I am not specifying what are the dimensions of A, G and b and we can sort of assume that there of appropriate dimension. So, that is understood and P is P has a name P is called the linear programming relaxation of S that is what I have done, what is the only difference between S and P is that in P, I do not require x variables to be integer x can be continuous. Where as in S, I require x to be integers. So, therefore the feasibility region of P is larger feasibility region, and clearly P is a polyhedron. So, it satisfies I mean optimizing over P is something, which is, which is very easy.

So, now our question. So, again we have not stated our formulae what the question is. So, what is the question? So, question we want to answer, question that we want to answer is the following what is or what does the set convex all of S look like right. So, this is what we want to answer. So, the main question is what does the convex all of S. Now, in order to answer this question we will making few assumptions I mean one major assumption. So, the major assumption that we are going to make that we are going to make is that A, G and b are all rational right. So, we are going to assume that the entries of all the matrices A and G, and as well as the vector b are all rational quantities.

Now, first of all it may seem like restrict able assumption, but in reality it is not because if you want to solve an integer programming problem using a computer, you need to be able to represent a number some sort of input, and you can only represent rational numbers in general. Second is that we will see that if some of these turn out to be irrational then they can be a problem. So, we will see at the end using examples, that they complete problems if some of the data is not rational. So, this is what is the setting.

So, now under this setting when A, G and b are all rational, we can answer the question. So, so we can answer the question satisfactorily and the answer is as follows. So, we can state it as a following result. So, if I call it star then I can say that. So, under this assumption the convex all of S is a rational polyhedron. So, this says that well if all data is rational you can always find a rational polyhedron such that, the convex all of S is equal to this that the rational polyhedron.

So, essentially in theory it is one can solve integer programming problems, if you know how to solve linear programming problems because in theory, all you have to do is that find the convex all of S, and once you have that it is a polyhedron and optimizing a linear function over it is just linear programming. So, this is very useful result and this was proved by this result is due to Meyer in 1974. So, this is the result and most of today's classes you spend in trying to prove this theorem. So, let us see how the proof works. So, first you need to rule out a trivial case.

So, the trivial case being S being the empty set; so if S is empty then clearly the convex all of S is also empty and this is a polyhedron, empty set is a polyhedron. So, which is a polyhedron, which is a rational polyhedron. So, now, we are going to assume that S is non empty. So, under this assumption let us see how we can prove that convex all of S is indeed a rational polyhedron.

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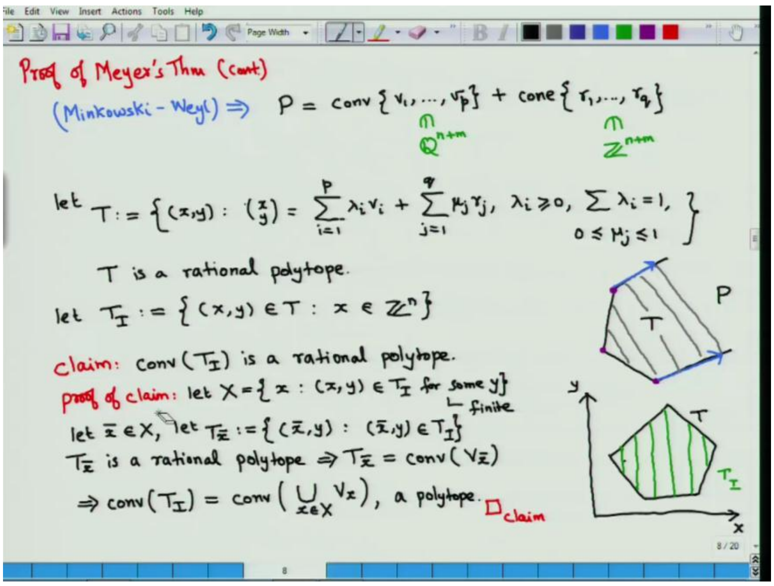

So, here we continued, we know that S is a rational poly no we know S is not empty and we know that P is a rational polyhedron. Now, this is a famous result due to Minkowski and Weyl. So, this is a result due to Minkowski and Weyl which implies that I can since, P is a polyhedron I can write P as the convex all of some finitly mini points v 1 dot, dot,

dot up to v small P plus the cone generated by r 1 up to r k. So, this is guaranteed by the Minkowski Weyl theorem, what we will do is we will see why this we will first assume the Minkowski Weyl theorem, and prove Meyer's theorem and then we will go back if we have time to see why the Minkowski Weyl theorem is true.

Now, I can write P as a convex all of a finite number of points plus the cone generated by and the finite number of points, and these are all.

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Yes. So, all these are rational numbers now, what we are going to do is. So, clearly what we are going to assume is that all these g i's are rational numbers Q n plus m and now since, it is a cone scaling does not affect anything. So, I am going to assume that the r's are all integer because I can scale because it is a cone fine. So, this is, so we are assume that the extreme points are rational and the extreme rays are integer. So, that is that is fairly straight forward.

Now, what we will do is somehow we will do we need to write convex all of S, as the finite number of points plus cone generated by finite number of rational vectors. So, that is the whole idea behind this proof. So, in order to do that, we need to specify how we are going to construct, this points as well as directions. So, for that let us do the following let see that let capital T be the following set. So, it is it is the set of all points x comma y such that I can write my x, y as the following way.

The sum i going from 1 to P lambda i v i plus sum j going from 1 to Q mu j r j, where lambda i's are greater than or equal to 0, summation lambda i is equal to 1. So, essentially lambda i lambda i's form a convex combination and 0 less than or equal to mu j less than or equal to 1. So, instead of allowing all possible mu j's greater than or equal to 0, I am restricting mu j's to be less than or equal to 1. So, that is that is why I am going to call this set T. Now, first of all you can convince yourself that since this is the convex all of the v's plus the bounded combination of the r, r, r j's, T is a compact set.

So, for example, suppose if we have the following polyhedron. So, let see the following polyhedron this is P, what a set T is going to be I have these three extreme points and these are my r's. So, my T is going to be something like this is my T. So T is a compact set and you will agree that T is a rational polyhedron, it is also easy to see that T is a polyhedron because all this is polyhedron and I am projecting it to the variables x comma y. So, this is essentially polyhedron. So, T is a rational polytope.

Now, what I am going to do is I am going to pick. So, I am going to show that, I can find all the extreme points of the convex all of S, lie inside T that is what I am going to show. So, for that what I am going to do is I am going to define let T sub I be the following set. So, T sub I is a set of all x comma y in T where x is an integer right. So, this is T sub I and now, I am going to claim that I am going to claim that the convex all of T sub I is a rational polytope.

Now, again from a picture this should be fair always, but we need a formal proof right. So, here we have some points, which are some mixed integer points and what we need to show is that you know, the convex all of that is a rational point I think the figure will be useful for this. So, let us look at the following figure. So, this is x and y. So, this is this is a simplest possible case, when there is one integer variable and one continuous variable. And suppose, the T is something like this so, this is our T, what are we looking at how do the mixed integer points look like.

Now, x has to be integers and y can be anything. So, the mixed integer points inside the set look something like this. So, green points here make the T I right and it is fairly easy to see here at least from this picture that the convex all of T I is a another polytope. Now, let us see y is too formerly, what I am going to do is, what I am going to do is look at each of these x values here this 1, 2, 3, 4, 5, 6. 6 different x values I am going to look at each of these 6 x values. So, let us see let x, capital X be the set of all small x such that small x comma y belongs to T I for some y. So, it is the lift of x whatever that you would like to call it into T I.

Now, this is a finite set because t is a bounded polyhedron t is bounded. So, this is finite. So, capital X is a finite set now, let us take a, let us take a particular x bar in capital X. Let us take a particular x bar in capital X and. So, suppose x bar is this gale here. So, what I am going to look at is this corresponding green vertical line segment, vertical line segment corresponding to x bar. So, T x bar is the corresponding vertical line segment, x bar comma y such that x bar comma y belongs to T I. Now, it is clear that T x bar is a rational polytope and because I am fixing x bar, I am taking some I am taking this my rational polytope T and I am fixing my x. So, it is intersection of two rational polytope with the polyhedron. So, it is a rational polytope.

So, T x bar is a rational polytope is a rational polytope. Which means, that I can write T x bar as the convex all of some finite set of points. So, let me call those finite set of points V x bar right because it is a rational point. Now, what I am going to do is I am going to take x bar is a left most integer point, look at the v the corresponding v. So, I am going to take all the v's. Now, there are finitely mini x's each mini's is finite. So, the union of that is a finite set and I can say that the convex all of T I is nothing but the convex all of the union, of all this v x. Now, this entire thing inside the union is a finite set. So, which means that this is a polytope.

So, here, we are done with the claim, the claim. So, watching setup to show that the convex all of T I is a rational polytope that, that is start. Now, what are we going to do we are going to use this T I, and show that all the extreme points of the convex all of S the convex all of all integer, mixed integer points in P lies in this set T I. So, so that will help us or that will be the result essential result that we are looking for. Now, let us look at any point. So, now we are back to the proof of the theorem.

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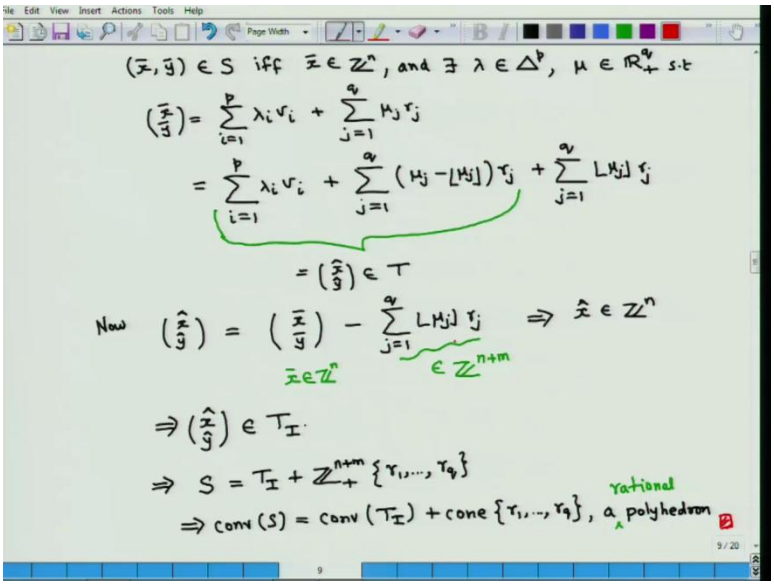

So, for any x bar y bar in S or rather, let me rephrase this x bar y bar belongs to S if and only if what are the conditions x bar must be integer, and x bar comma y bar must lie in P or in other words, from our notation I should be able to write x bar, y bar as summation i going from 1 to P lambda i v i plus summation j going from 1 to q mu j r j right. So, if and only if x bar lies in z n and there exist some vector in the in dimensional simplex, and mu's are positive right.

So, this is this is just from definition now what I am going to do is I am going to do a really simple thing, I am going to rewrite this as summation i going from 1 to P, lambda i v i plus summation j going from 1 to q. Now, I am going to rewrite mu j as mu j minus mu j round on r j plus summation j going from 1 to q mu j round on r j right, I am just separating the integer and fraction part of this mess.

Now, you will quickly see why that is a case. Now, if you look at the first two terms, I am writing this as the convex combination of the v i's plus a non negative combination of the r j's such that the multiplier is between 0 and 1. So, by definition this so, this thing I can call, let me call this as equal to some x cap, y cap which belongs to my set T, capital T right because I define capital T. I define capital t as the convex combination of the v i's plus non negative linear combination of r j's, where the multipliers are less than or equal to 1. So, now this is in my capital T.

Now, let us do a rearrangement I am going to write my x cap, y cap as x bar, y bar minus summation j going from 1 to q mu j r j right, there's nothing I have just rearranged it. Now, here let us look at what is happening here x bar is an integer, this whole thing is integer assume by scaling that r j's are integer and mu j round on is integer. So, this whole thing is integer. So, I can see that this. So, this x bar is integer and this whole thing belongs to the integer, the set of integers in R n plus m so that means, that my point x hat must also be integer right. So, clearly now x hat y hat belongs to T and x hat is integer which means that x hat, y hat belongs to T I.

So, from what we have gathered now what we did was we have took arbitrary x, y in S and I am written it as some point in T I plus some non negative integer multiply multiples of r j. So, essentially what I have done is I have is that my conclusion is that I can write S as T I plus I am going to write it as Z n plus m plus. So, just to denote that it is a set of all integer non negative combinations. Now, all you need to do is to take convex all on both sides and once, you take once you do convexity on convex it is a simple exercise to show that, the convex all of S is nothing but the convex all of T I plus the cone generated by r j and because of this is a polytope a polyhedron sorry. And it is also rational.

So, that is Meyers theorem which says that if I start with the rational data, the convex all of all mixed integer points in my polyhedron is also a rational polyhedron. So, this is, this is Meyers theorem. Now, let us look at what happens can we relax this assumptions a little bit. So, can we relax the rationality assumptions or is rationality crucial. Now, one thing is from the proof rationality looks crucial right because we need to assume that, this mu j round on multiplied by r j are integer. So, so this rationality looks crucial we are we are in fact, using rationality here otherwise we cannot assume that this can be scaled to an integer point

Now, this both the mu j itself is an integer.

From mu j it is integer, but we need r j to be rational. Otherwise we need not be able to scale it to an integer vector. So, the extreme directions need to be rational. So, there are very interesting examples when you know.

So, mu j is a rational point.

Yes, yes what I mean is all rational coefficients here. So, so let us look into couple of examples where we do not have this luxury and see what happens here. So, first example where rationality does not hold is.

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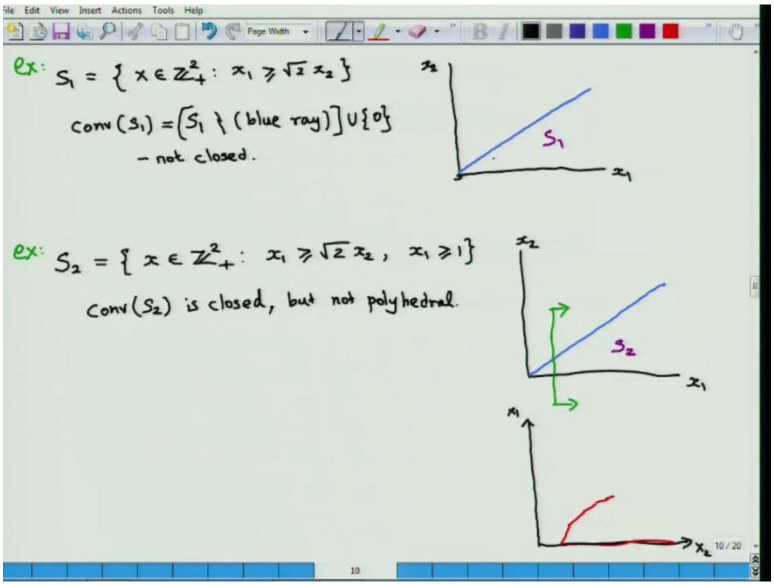

So, what I can do is I can look at my let say S 1 is the set of all. So, now there are only integer points is the simple example such that, or rather non negative integers such that x 1 is greater than or equal to square root 2, x 2. So, this is my first example. So, here is x 1, x 2 and this is my, it is straight. What is the convex all of S 1. So, what is the convex all of integer points inside this region that part except the blue line, blue, blue.

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So, this is S 1 minus the blue line.

Blue.

Well the origin is included.

Now, blue line where points where in x point x plus to be external.

They cannot be integer cannot be integer.

So, now, we are everything other than the blue line including 0.

So, I will just to make it just to make it clear I will also say. So, the convex all is not a closed set in this case. So, it cannot be a rational polyhedron clearly, it cannot be a polyhedron even. So, it is not a closed set. So, that is the first example, second example is a slight modification of this set. So, S 2 is the set again x in z 2 plus such that x 1 is greater than or equal to root 2 x 2 and x 1 is greater than or equal to 1. So, I have x 1, x 2 same this constrained and I also have this constrained. So, this is my feasible region. So, this is S 2, S 1. Now, here what I have done is I have purposely cut off the origin.

Now, what you can see is that a well interestingly S 2 is a closed set, you can actually compute that S 2 the convex all of S 2 is a closed set. So, that in S 1 the problem with closure comes because I mean in each of these case.

Where are the blue ray is not there.

Blue ray is not there that is fine, but what happens is that there are, there are integer points arbitrary close to the blue ray. So, in the first case since the origin is there, the blue ray is. In fact, in the closure of the convex all of S 1, but the blue ray is not there in the closure of the convex all of S 1.

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Yes.

And take a neigh neighborhood around it.

Yes.

There will be at least one point of the convex all of S 2.

No, no, not for arbitrary small neighbor you can always find a neighborhood where.

Arbitrary is small neighbor arbitrary is a small neighbor it is a bounded point it looks like at least not bounded.

No, see here is a problem what happens is in the in the first case is the origin is the part of the convex all I have.

Wait, wait I have here we have look into the part that day or no x is a element of z 2 plus.

Yes.

That is a important issue.

Yes. I have integer points, which are, which are that is arbitrary closed. What is that?

Those who are not integer points and I have.

I have integer points arbitrarily closed to this blue, blue ray, but I do not know where they are they could be arbitrarily far away, they could be arbitrarily far away.

From where integer point.

From 0, so they could be anywhere on this ray that ray there would not be integer points, but there will be because of there is slight approximation there will be integer points, which will be arbitrarily closed. So, I can take the convex combination of the origin and those points. And then show that this blue ray is going to be in the closure of that for the first example, but for the sorry, but for the second one you cannot, you cannot use that argument. In fact, I should say that it is not obviously, show that it is closure. You mean, you need some additional result. I mean I do not know what is an obvious way to prove it I can give you reference of how to show that in fact, a closed asset.

I mean. So, you do not take any points on the blue line.

But you are taking whatever I m must send if there could be there could be integers of S k as close as I want but.

But what we, but what we cannot guarantee.

Not guarantee as integers cannot be as closed, but why not integer closed.

Because of the result approximation here approximating irrationals by rational which is P p over q. So, that point P comma q is going to be very closed to the line x 1 root 2 x 1 equal to x 2.

P comma q you can have arbitrarily closed this should have lambda or should it would be root of z 2 plus

It is a grad

It is a grad yes.

Grad

Now, if there is a line going through

Now, I have to really look at the grads

Yes!

One with the grads

Yes, yes.

The grad distance is fixed.

That is true, but this line is irrational slope. So, now, you can. So, some now we know that you can approximate square root 2 to arbitrary prostrations by rationals. So, let us have.

We have the rational points very near the blue line blue line.

Very, very near.

So, there will be rational points of the form P over q, this arbitrary closed to root 2. Which means P is arbitrarily closed to root 2 q. So, there is an integer point P comma q. Which is arbitrarily closed to the line x 1 equal to root 2 x 2.

At least for so.

So, that is that is why.

But I cannot geometrically see if I look at the grad and you have that line

Yes!

Which does not have continual reaction on points

Yes!

And it will be not called an integer points.

Or rational omen it.

Right!

It is just going through

Right!

Now!

And you have truncated the 0 part

So, I cannot see how well I am really looking at the grad points only

That is right.

So, then I am trying to take the convex all basically I am joining everything and we are doing.

So, what happens here is if I if I would write let to draw the convex all it is going to. So, I mean as I said I mean it is you require a small you require a result to show that, but the convex all of this S 2 looks like the following. So, it is obviously, this is part of the convex all and then it is going to look something like this. So, these are all really small line segments, these are all connecting integers. So, it is going to be a closed set, but not a polyhedron. So, what we can what people have shown is that this is closed, but not polyhedron. So, this is called a locally polyhedral set. So, it is emphasized with.

You mean to now say that because I am com combining integer points, but well that will be a countable set two plus.

Yes!

Countable set.

And whatever you have under S 2 is countable.

Yes!

Now, you basically you are connecting those things this you are making some convex combination of this things.

Yes!

You are telling that the boundary of this set.

Yes!

Would can be as close as I want to line is to.

Yes!

There are infinitely mini extreme points, but still it has some properties of polyhedron, it is locally polyhedral in the sense there, if I intersect with any polytope then intersection is another polytope, but for the first one the convex all of S is not closed, but the closure of the convex all of S 1 is a polyhedron. So, what I am saying is that even with this really small things make a rationalities, even in the polyhedral there are things which we well I mean. Now, it is it is almost settled we really know how irrational the convex all of irrational polyhedral integers of a polyhedral look like right now we know. So, that is what I wanted to stress and talk about the rational and what is a problem, when data is not rational. How much time do I have?

It is probably 43 minutes you have some 6, 7 other minutes.

So, in the remaining time I would.

Here is the start time.

So, in the remaining time I would like to I would like to use the remaining time to show one part of the proof of the Minkowski Weyl theorem, so which we have used in this earlier.

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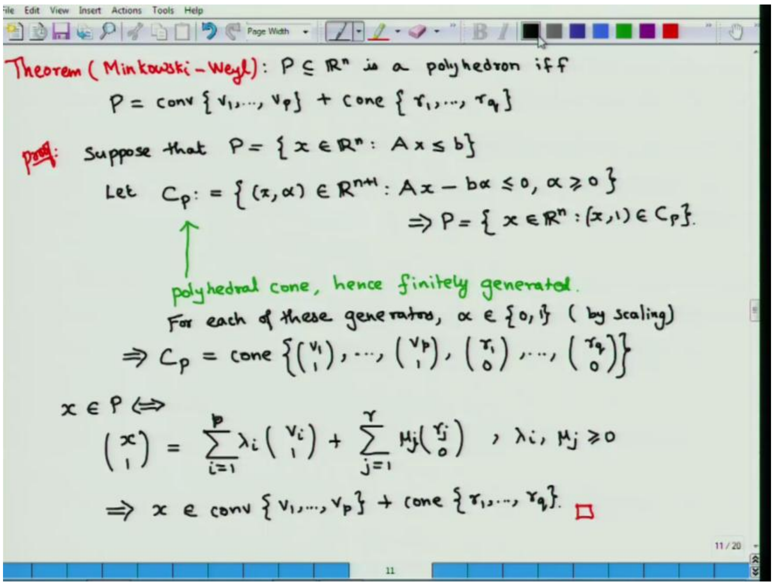

So, what this results says is that the set P in R n is a polyhedron if and only if I can write P if and only if I can write P as the convex all of up to some finite number of points plus the cone generated by another finite number set of points. And of course, it can be spendable to say that P is a rational polyhedron if and only if I can write it, like this for rational points v i's and r j's. So, this is what Minkowski Weyl theorem.

So, what I am going to do is, I am going to show one part, I am going to show that this P is a polyhedron then P can be written in this way, that is the main thing that we have used and the proof of the converse is also fairly similar to this. So, I will, we will stick with one direction. So, suppose that I can say, I can write P as a some polyhedron in the form of linear inequalities A x less than or equal to b. What I am going to do is I am going to do left of P left of P to a higher dimension and look at the cone that, that corresponds to P. So, what do I mean by that? I mean that let C p be the following set.

We try to have some homogenization of the set.

It is the homogenization set that is right, yes. So, x comma alpha in R n plus 1 such that A x minus b alpha is less than or equal to 0 alpha is not meant right. So, this is the homogenization and if C p is this, then p is nothing but the set of all points x. Such that x comma 1 belongs to C p this is fairly strict. Now, C p is a polyhedral cone and since, it is a polyhedral cone it is also finitely generated. So, this the here is a polyhedral cone hence, finitely generated.

Now, what we can do is a following if you look at each of this finite generators, what is the value of alpha, alpha can be either be 0 or positive by scaling I can assume that for the positive ones alpha is 1 right. So, for each of these generators, generators alpha is either 0 or 1 by scaling. Now, so.

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I can write my C p as the cone generated by the points of the form and I am just collecting.

So, what are you doing you are putting alpha is either 1 or 0.

So, the ones.

So, why are you do it if they if I have 1.

So, alpha can be either 0 or positive for the ones which fair for which alpha is positive, I am scaling.

So, scaling it to one right you can just need that.

So, I can easily do that. So, I know what is C p now and I know, what is p if I know what is C p. So, I can so, if x belongs to p if and only if I can write x comma 1 as. So, I can write x comma 1 as summation I going from 1 to P lambda i multiplied by plus summation j going from 1 to r mu j r j 0. Now, here if you look the second equation and obviously, lambda i mu j's must also be greater than or equal to 0. So, if you look at this second last line of equation. So, I must I see that lambda i's must in fact, add to 1. So, essentially lambda i is not just greater than equal to 0, but lambda i's form a convex combination. So, this implies that x belongs to the convex all plus r k. So, that is the.

If the lower part is automatically 1, then the upper part is the summation lambda i is equal to 1 anyway. Yes, that is right. So, that is that is why this result is true. So, with that let us stop it.