Foundation of Optimization
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Lecture - 20

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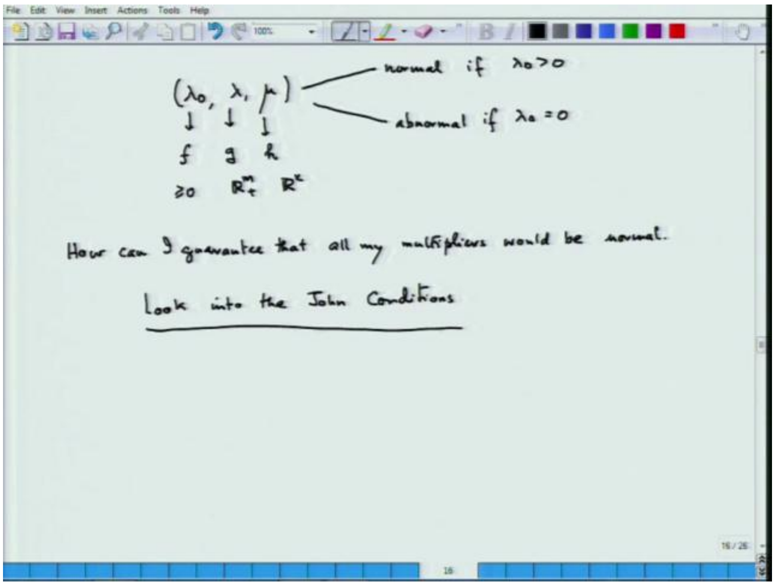

So in the last classes we were talking about finding rather how to compute things, how to talking about normal and abnormal multipliers so, just to recall that the Fritz John multiplier set can be written like this is associated with f and this is associated with g, and this is associated with h, this is greater than equal to 0 this is in r n plus and this is in r k plus r k whatever be the number of equality constants.

So, and we were calling this to be normal if lambda not is strictly bigger than 0 we were calling this to be abnormal now how can I guarantee that all my multipliers would be normal that is a very very fundamental question. So, let me ask this question how I can guarantee that all my multipliers would be normal all my multiplier would be normal that is the worst thing we are all surrounded by waves which are not good for us anyway.

So how can I guarantee that all my multipliers would be normal now in order to guarantee that I would look into the equation look look into the Fritz John condition to the John conditions so, it would be good if I shift to the black board to explain you this fact so, what happens is that this is my John condition the first line of the John condition

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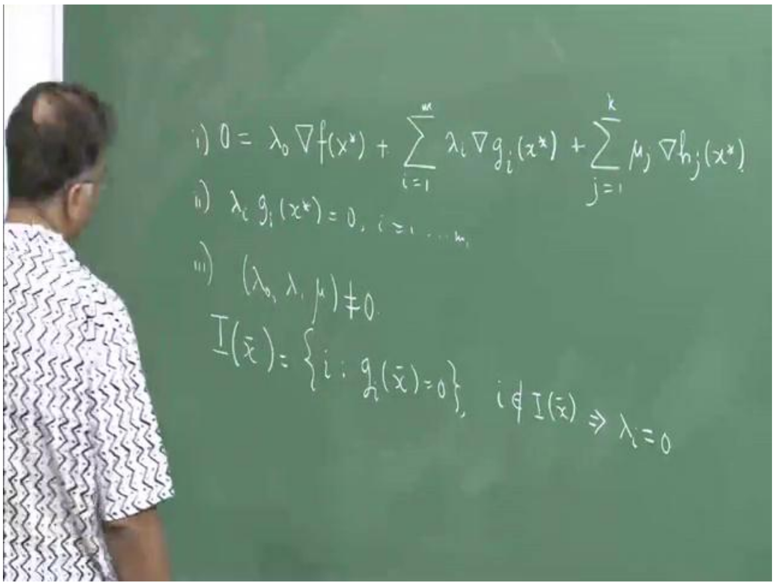

So if x star is my solution local minimum so, my John condition says that this expression is equal to 0 grad g i x star plus summation I is equal to or j is equal to one to k mu j grad h j x star and of course, you know the other fundamental conditions that lambda i g i x star would be equal to 0 this is one and this is two and of course, the most important of all the conditions is as we have written in this this is what it says.

Now I know that if i is not in the active index set that is if I define this particular set called the active index set which we have already mentioned before so, I is some number some index between one to m and take all those indexes for which g i x bar is equal to 0 so, if i s not element of i x naught then this condition number two would imply that lambda i is equal to 0 because the product has to be 0 if one of them is not one of them is strictly less than 0 the other has to be 0 so, that is the meaning of complimentary slackness that both of them cannot hold with strict inequalities at the same time so, I can then rewrite this upper part in a slightly simple way.

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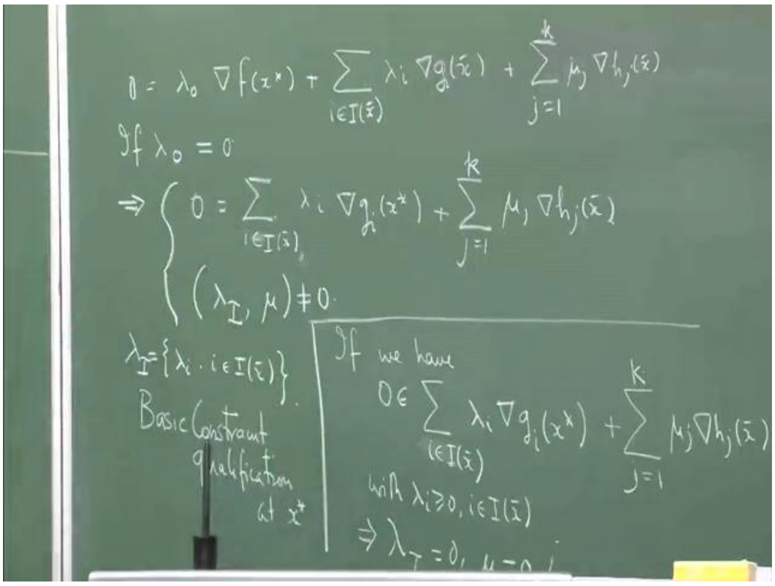

That here I place this whole summation with the fact that i belongs to i x bar sorry j is equal to one to k mu j. Now let us examine the situation what would happen if lambda not is equal to 0 so, if lambda not is equal to 0, you would imply that 0 must be equal to and lambda i which means the set of all the vectors all the components this is a vector consisting of all the components of lambda corresponding to i element of i x bar so, lambda I is equal to lambda i these vectors where i belongs to I x bar so, this vector now because lambda.

Naught is 0 but, the whole vector cannot be 0 whole set of multipliers so, these two parts have to be non 0 lambda i and mu this part has to be not equal to 0 right so, it means that if lambda naught is equal to 0 then this would be 0 with this not equal to 0 this vector not equal to 0 so, lambda naught equal to 0 implies these two conditions so, if this condition fails then lambda naught is not equal to 0 can never be equal to 0 which means what is the meaning of failure of this condition which means if we have 0 element of summation i element of i x bar lambda i so, if we have this with lambda i greater than equal to 0 then it implies that this lambda i vector is 0 and then mu vector is 0. So, whenever I can have a condition like this then this lambda i and mu j all of these all the lambda i s with I belonging to i x bar and mu j with j in j from one to k that should be 0 so, this condition is sometimes referred to as a basic constraint qualification.

so, it is a qualifying condition on the constraint which is always guarantying me that lambda naught cannot be equal to 0 so, you can make up your own theorem which would say so, we say that this means that the basic constraint qualification at the point reference point x star at the local minimum .

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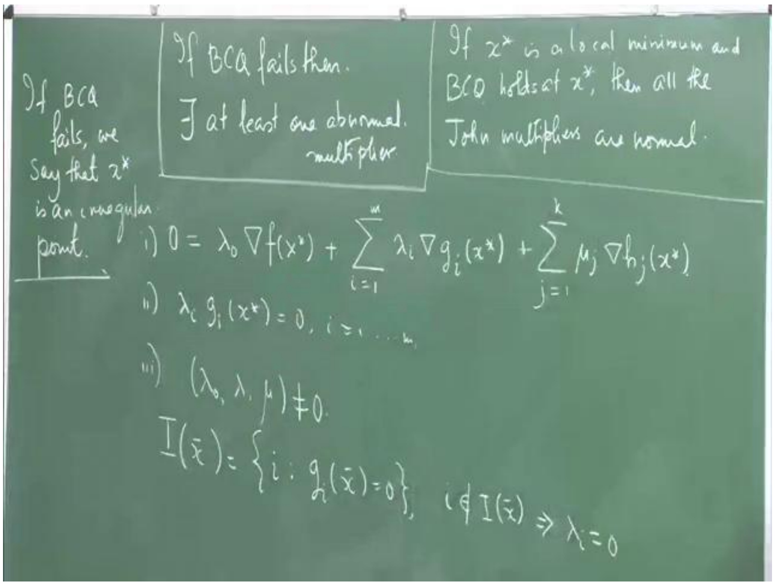

If x star is a local minimum and b c q basic constant qualification or b c q holds at x star if this happens then if with this holds at x star then all the John multipliers is normal. So if b c q fails means if this happens that is lambda and mu is not equal to 0 but, still we get this equal to 0 this summation sum of these vectors then it means that there would exist at least one normal abnormal multiplier if b c q fails then there exists at least one normal multi least one abnormal multiplier then there exists which is a sign used by mathematicians at least one abnormal multiplier. Now the interesting question is if I have a situation where my b c q has failed and I know that there exists an abnormal multiplier can I still say or show by an example that even if I have a abnormal multiplier, but I can find another multiplier vector set of multiplier vector another set of lambda naught lambda mu for which lambda naught would be strictly greater than 0 and corresponding to the same point x star so, if b c q fails we say x star is a irregular point.

Now it is a part of ongoing research as to how to address the situation when my b c q has failed when my b c q has failed how do I address the situation can I guarantee that or under what conditions can I guarantee that in spite of the failure of the basic constraint qualification there would exist at least one multiplier which would be a normal multiplier and that is a piece of interesting research because all all the so, called constant qualifications which are weaker than b c q there are like abedy gignard which we will not discuss in this course so, all of them which are weaker than b c q can only guarantee that there would exist one normal multiplier but, they cannot guarantee that all the multipliers are normal the strong or rather the weakest possible constraint qualification which will guarantee that all the multipliers are normal is the basic constraint qualification the basic constraint qualification also has an alternative form obtained in 1967 by Mangasarian and Fromovitz that is why this is also known as Mangasarian Fromovitz constraint qualification alternative equivalent form of b c q.

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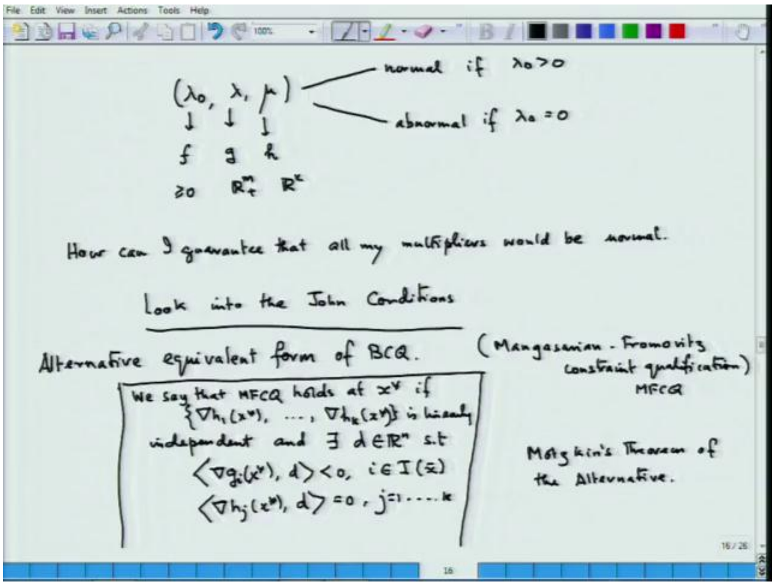

So, what is the meaning of this how to get this alternative equivalent form let me write down the alternative equivalent form first and then we will see how to obtain that we say that m f c q, c q means constraint qualification I will write what is m f that is called Mangasarian Fromovitz so, we say that m f c q this is in short m f c q m f c q holds at x star of course, x star is a local minimum or feasible point. I am not repeating that fact if set of vectors is linearly independent which k x star.

And there exists d element of r n such that actually these are nothing but, applications of the separation theorem but, we have not done separation theorem in that great detail that we have done in the convex optimization course so, we would just show the theorem of the alternative that is used to come from b c q to m f c q so, i x bar is the same as the i x bar given on the board and sorry not x bar its x star how have I come to this that is a very very important thing for this we actually apply what is called the Mofzkin's theorem of alternative ok.

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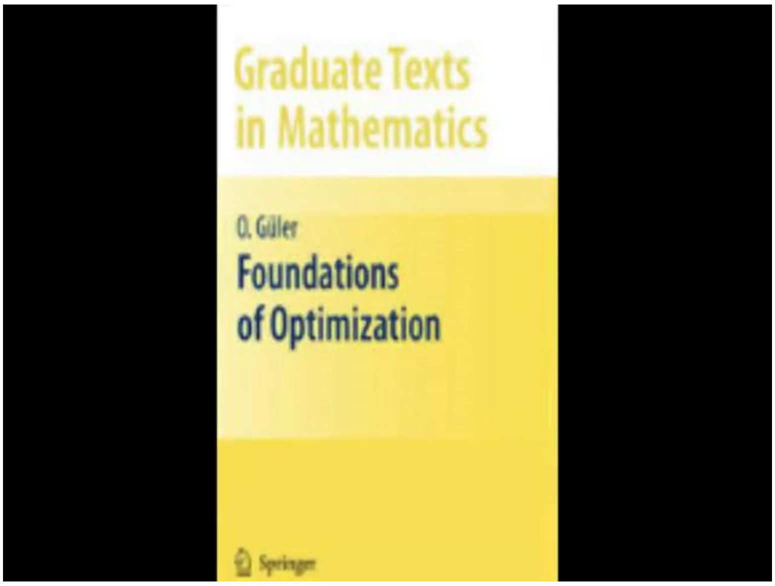

Now let us first mention we will mention the Mofzkins theorem of alternative form this book foundations of optimization by Osman Guler and I have already mentioned this book at the beginning please note take care that this book is very very useful for anybody who likes to do some advance study in optimization excellent excellent book now we will mention that and we will apply the Mofzkins alternative theorem to come to a conclusion but, please note one very important fact the fact is the following that this fact stands independent of what we are going to do because this has to be linearly independent suppose these gradients are not linearly independent they are linearly dependent then there must be some mu j some mu j non 0 for which this is equal to 0 so, then I can take all the lambda i’s to be 0 and add it with this to get total equal to 0 so, which means that whenever this is linearly dependent that is whenever k is bigger than or equal bigger than n strictly bigger than n then the m f c the b c q will always fail so, this number of constraints are also very very important at this juncture that if k is strictly bigger than n it would imply the failure of b c q.

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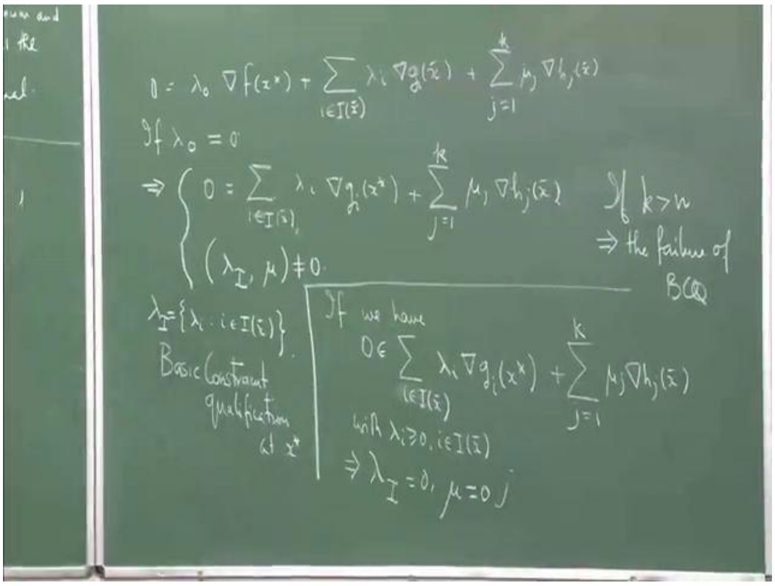

So, if your number of equality constraints is very large and they are not linear then it is a very very clear fact that the b c q would fail. So, in many many application problems the b c q is actually failing so, we cannot always say that all our multipliers would be normal. So, this is a very mathematical or theoretical practice to have normal multipliers so, let us write down the m f c q once again and then show that it is equivalent to what is being said as the b c q so, b c q and m f c q are equivalent so, I will go from the b c q to m f c q and m f c q to b c q.

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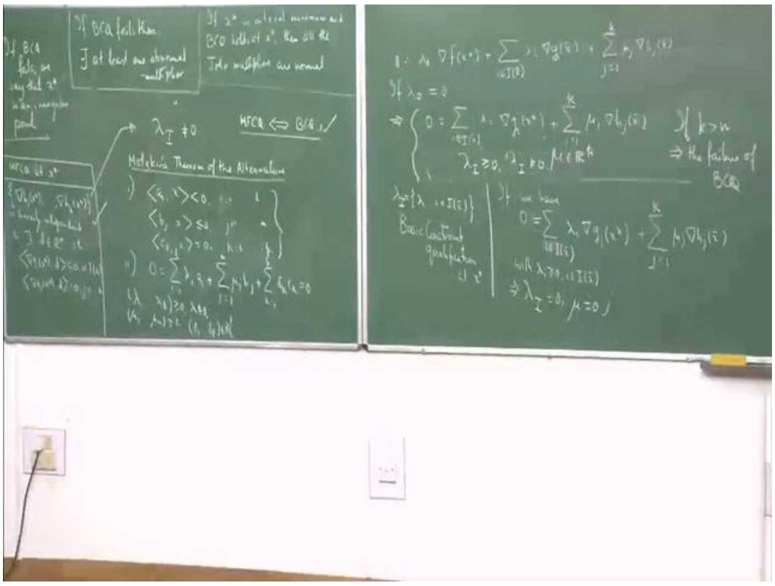

So, m f c q at x star so, which means grad h one x star grad h k x star is linearly independent and there exists d element of r n such that grad of g i x star d so, a this is what we will have now let us assume that the m f c q is true m f c q is true means.

If if this is linearly independent then which means that all the lambda i's here cannot be 0. So, if this is linearly independent if this is true so, I am assuming that m f c q holds at x star then this would imply that lambda of I this is not equal to 0 this is this is automatically implied at ok.

Now what what would these two conditions imply that these two conditions from Mofzkin's theorem of the alternative I am just writing down the Mofzkin's theorem Mofzkin's theorem of the alternative now in Mofzkin's theorem of the alternative there are two systems and both of them cannot have solutions at the same time that is the meaning of the theorem of the alternative.

So in the Mofzkin's theorem of the alternative suppose I have this system sorry this is I equal to one I am just writing whatever is written in Guler so, that its absolutely clearly written now this is one system so, there are combination of equality strict inequalities less than equal to type in equality and the second system is this here you would actually take lambda I that is lambda one to lambda m one this is all greater than equal to 0, but lambda is not equal to 0 and you will have the vector mu which is mu one mu two mu m this is greater than equal to 0 while that delta one, delta two, delta p, this is in r r p. So, these are the two systems either there is an x which solves this if there is an x which solves this then this system will not have a solution. So, here in the Mangasarian Fromovitz constraint qualification there are these two systems these two systems does not have a solution which means that this plus I am just have to bother about I do not have to bother about this I have to bother about these two which means at this is holding with lambda i greater than equal to 0 of course, which if this if this system has a solution that is there is a d which satisfies so, this system cannot have a solution which means that there cannot be any lambda where all the lambda i's are greater than equal to 0 but, lambda vector is not equal to 0 and mu j is in r k such that this system has a solution that is sorry 0 size for those lambda and mu there this whole all the vectors would sum up to 0. So, there cannot be any lambda of this sort that there cannot be any lambda right which means that if this system has a solution, then this system cannot have a solution, which means there would be at least one non 0 element here and by the application of the Mofzkin's alternative theorem we know that this lambda is has to be always greater than equal to this lambda all the lambda's has to be greater than equal to 0 and the whole vector lambda i cannot be equal to 0 so, that that we immediately know which means that this system this b c q does not hold now on the reverse suppose you have a system where you know that the b c q is holding which means so, what does it means that if this system has a.

Solution then this system means this system sorry I am making a mistake I will just come back once again what I am showing is that if this system has a solution just by looking at this which means the first system of Mofzkin's alternative theorem has a solution. So, the second system will not have a solution so, this system has a solution so, this system cannot have a solution and in this system we will have lambda I not equal to 0 by given by the Mofzkin's alternative theorem so, this once I know that this is not equal to 0 and this is equal to 0 so, I know that this system there is so, there is lambda I vector not equal to 0 so, there cannot be any lambda I vector equal to 0 not equal to 0 and any mu in r k for which this will be equal to true which means if this system has a solution this system cannot have a solution. Which means this is this is what will happen which means this the b c q will hold. Now suppose the b c q is holding this system is holding that is this system this system has failed now because this is we have assumed separately that this linearly independent now if this is linearly independent I am guaranteed that lambda has to be greater than equal to 0 because if not I said that this is holding that is there cannot be any non 0 lambda i's and mu j's which will be make this vector equal to 0.

So if I now say that no the lambda i‘s can be all 0 if all the lambda I s are 0 then some of the mu j's has to be non 0 if this system does not have a solution right which means grad h j x bar would be linearly dependent once again this is a very very crucial fact so, I know that there is no no solution for this system now because of the fact that grad h j x bar are all linearly independent if this system was if this system because of this system is not having a solution and all the grad h j x bar are linearly independent I must have tau to be greater than equal to 0 because if tau is all equal to 0 and this is linearly dependent then I can put mu j. I can put mu j equal to some mu j would be non 0 and then I put all the lambda i's 0 and make this 0 which will be contradictory to the fact that this system cannot have a solution non 0 solution there cannot be lambda i's and mu j‘s all non.

0 and which add up to 0 which means that the linear independence of the gradients is imposing the fact that this system does not have a solution with the additional fact that

lambda i this vector cannot be equal to 0 then now by going I know now that this type of a system does not have a solution though this system has a solution which means now going back there would be a d for which this system would have a solution so, what we have.

Shown is that if this is assumed this is a part of m f c q that m f c q in fact so, if m f c q holds all this holds this will be true this has to be true this cannot have a solution right and if b c q holds it is immediate that this grad h s i would be linearly independent because if they are not linearly independent I told you what can happen then I can show that the b c q has failed so, b c q has hold this is linearly independent and if this is linearly independent then if this is linearly independent then we have applied Mofzkin's alternative theorem on this part to conclude this thing and of course, lambda I cannot be all equal to 0.

So here suppose this has a solution means if lambda i is equal to 0 so, what we have done is the following is that if m f c q is holding that is if the there is a d such that then this system this system cannot have a solution with of course, the vector lambda I would be not equal to 0 right and m f c q holding is we have already assumed that this is linearly independent because if this is linearly dependent then this system anyway has a solution so, linear independence first is has to be given because if linear dependence is there on this then b c q anyway will never will always fail so, this plus this shows that b c q will hold if Mangasarian holds mangasarian m f c q holds now suppose b c q holds b c q holds and that is there is no such lambda i's if I there is no such lambda i's for where this this non 0 lambda I s for which this will be equal to 0 so, whenever this happens these are all equal to 0 so, there exists no lambda i there is no lambda vector with lambda greater than equal to 0 lambda greater than equal to 0 and lambda not equal to 0 there is no such vector for which this is holding. So, which is basically this line no such vector lambda i which is not equal to 0 and greater than 0 and mu mu not equal to 0 for for any mu whatever with so, there is no such vector lambda i greater than equal to 0 and lambda i not equal to 0 or mu not equal to 0 because you see if there exists a vector mu not equal to 0 for which this is holding and if all the lambda i's are equal to 0 then the b c q will fail so, there cannot be any lambda I here not equal to 0 and lambda I greater than equal to 0 that is lambda I all of them are greater than equal to 0 and one of them is strictly 0 for which this is holding for which this equation would hold.

So, b c q actually failure of b c q is actually some sort of a statement like this because if I take because what would happen is that once this statement is true that this system cannot have a solution with this and this not equal to 0 and this not equal to 0 because if this system if because if this this system has a solution and if this is equal to 0 then it will mean that this is this is linearly dependent and linear dependence means the failure of b c b c q. So, which means that if this if you say that b c q holds then you must have linear independence of this right and when you have linear independence of this it will always imply that this will be not equal to 0. So, these are very crucial point to understand so, this is what is called the failure of the b c q now if the b c q fails but, now if you see that b c q is holding then this has to be always nil always linearly independent now once you say that it is linearly independent which means this system cannot have a solution because if this system has a solution with lambda i greater than equal to 0 and lambda i not equal to 0 which means that this system has failed so, this system cannot have a solution failure of this system sorry the satisfaction of this system that whenever I have this this would be equal to 0 is a failure of this system is a failure of this system I will just write down so, this the satisfaction of this system that is b c q holds then this system fails then this system fails.

Component wise greater than 0 this system actually fails so, if this holds then this system is obviously failing which means that again if this system holds and whenever I have this this is equal to 0 then this system is failing means that system is failing which would imply something like this this system has a solution and this system has a solution and obviously this system if this system has this is the system which is true then this is any way linearly independent which is that so, finally, we have m f c q is equal to b c q and that would end will end the talk here today and the next class we will prove that for every linear programming problem which is for for us a very important class of problems.

The multipliers would always be normal it can never be abnormal and then we will go on to the study of quasi Newton method using the knowledge of constraint optimization that we have got today but, it is very important to note the relation between b c q and m f c q through the Mofzkins alternative theorem that is why I repeated it constantly to know the difference and this is a very important difference to sorry not to know the difference I would say that to know how to do the equality so, how to apply the alternative theorem so, very important to note this fact of satisfaction of this is a failure of this and hence a failure of this system and something of this type would be true so, it is not so, trivial as you think think over it at your home and we can again discuss if that is necessary.

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