Convex Optimization
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Module No. # 01
Lecture No. # 03

Welcome back again to the course on Convex optimization. A course on Convex optimization is not feasible, unless you have some idea what Convex Sets and Convex functions. Now, we had already spoken over Convex Sets and Convex functions in the previous lecture, but let me tell you a bit more details study about Convex Sets, what are their important examples, and Convex functions, what are the important examples, what are the important properties, would lead us to understand Convex optimization better. Here, our aim is not necessarily to give a proof on each and everything that we do, except possibly the major once. But, here our aim is essentially Convex optimization and not really Convex analysis as per se, and what we are going to do here now is a part of Convex analysis. I would begin with by showing you a very very important book on this subject, is called the fundamentals of Convex analysis by Hiriarturrut Claudelemarechal. This is book on Convex analysis, has beautiful chapter on Convex Sets and Convex functions, possibly one of the best exposition of the subject, and I strongly recommend this books. So, we go back and re learn our notion of the Convex Set.

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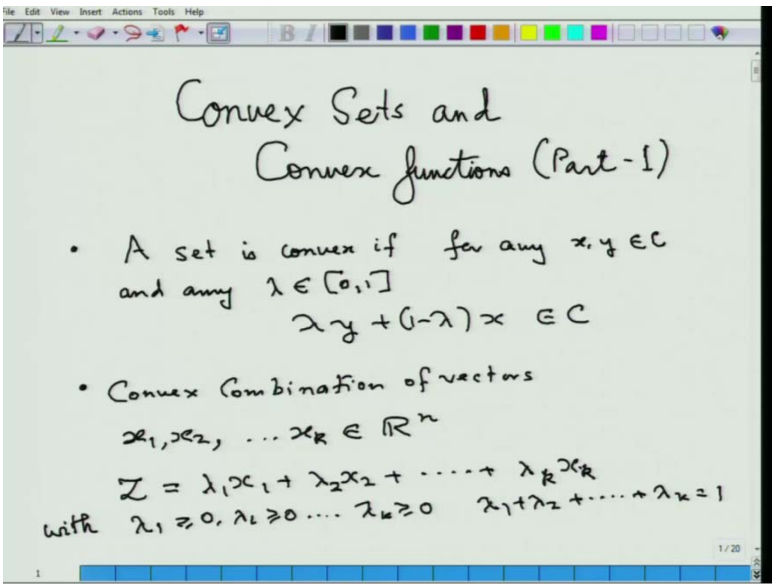

A set is Convex, if for any x y which is in the set c, and any lambda that is choose between 0 and 1, lambda times y plus 1 minus lambda times x must also belong to the set c. We have given several examples last time, even showing that the human body is not a Convex Set. And here we would look into, from more of the geometrical perspectives and right down some important examples of Convex Sets, before that I will just extend this idea of bit. Let us first define the notion of a Convex Combination, Convex Combination of vectors. Now, consider x 1, x 2, x k, some k vectors which are given in R in. So, what you do is, we are bothering about the question of defining a Convex combination. So, Convex Combination of this k element is an element z which is return as a lambda 1 x 1 plus lambda 2 x 2, I am assuming this is the known to everyone vector addition, lambda n x n, sorry this should be lambda k x k with lambda 1 greater than 0, lambda 2 greater than 0, all the lambda as a greater than equal to 0, and the sum of the lambdas are equal to 1. Because like lambda plus 1 minus, lambda some to on each of the lambda are greater than equal to 0, because the lying between 0 and 1. So, how do you define all possible Convex Combination from set c. You take any collection of finite number of elements from a set, do there Convex Combination, keep on changing the lambda. So, you will generate new combination; keep on changing the number of elements that you take from the set. A very fundamental and very basic result about a Convex Set is following.

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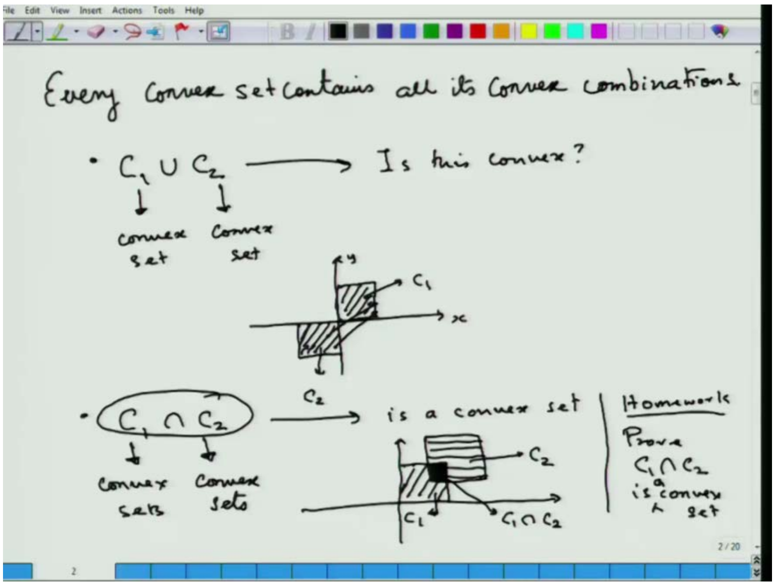

Every Convex Sets contains all its Convex combinations, before going to very specific example of Convex Sets and important one, we would like to sate certain specific properties this properties become important when optimization is a handled. So, let us look at certain basic properties, because when we have set, what are the properties you look for. See, when we have set, when we have class of set. So, Convex Set of all Convex Sets if form of class of sets in Rn, and what you look for is what happens if I combined to set that is if I take the union of two sets. So, if my c 1 is Convex Set, and c 2 is a Convex Set, the question is, is this Convex. You would be obviously sad to know that the very first property that we are considering here turns out to be negating property. So, c 1 union c 2 is not Convex.

Let us see, take a square here, take this square and take this square in naught two. Obviously, I am putting the x axis y axis for your convenience, but that is really not required. So, this is my c 1 and this is my c 2. So, this whole set that looks some sort of beta of zigzag. This whole set is like this, this is my c 1 union c 2, but then this whole set, so take any point here, and any point here and join it, it is outside. Any point here any point here, its line segment goes outside. So, c 1 union c 2 is not Convex Set. So, if that satisfies you, we will give you positive result. What about c 1 intersection c 2. So, if both of these are Convex Sets, then this intersection itself is a Convex Set. Very simple again take this one and take this square. Squares are obvious examples of Convex Sets, suppose this is y c 1 and this is y c 2.

So, this is an area which is an intersection, and of course this is c 1 intersection c 2. You can easily see that it is a Convex Set. I leave you as a home work, those who are listening to the course also have a bit of pen and paper with you, as you need to just figure something, everything will not be figured out, because I do not want, due to lose out on the fun. So, homework; prove c 1 intersection c 2 is Convex, or is a Convex Set. Let me tell you any mathematical subject, like this fascinating subject like Convex analysis, Convex optimization, cannot be understood unless you do prove things yourself. Prove is an integral part of mathematics, how mathematical discourse and of mathematical understanding. So, without understanding proves or whether learning to proof things, the fun over object is almost lost.

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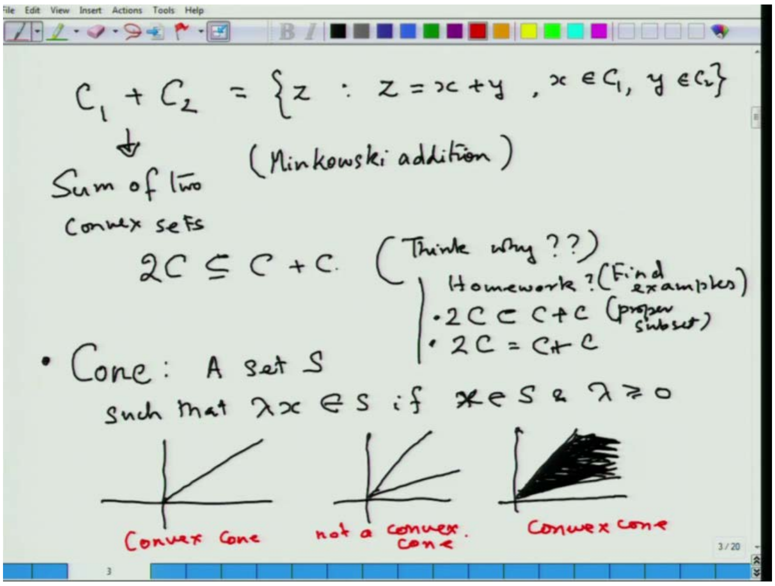

So, these are the good properties, now I will describes some other property which is essentially when you study optimization, is sum of two sets, assume for us that they are Convex Sets. So, assume c 1 and c 2 to be Convex, and I am asking the question what do you mean by term sum of two sets c 1 by c 2. Now, here you have to realize that c 1 is a subset of the vector space R n, c 2 is the also subset of the vector space R n. So, whenever I am talking about c 1 plus c 2 is the most natural definition is to consider all elements of the form z, such that z is equal to x plus y, where x belongs to c 1 and y belongs to c 2. So, this is sometimes calling the Minkowski addition of two sets, after the famous mathematician Minkowski. I do not want to define c 1 minus c 2, because you can define it likewise, by the difference of two vectors.

The critical thing to observe here is twice of c, if you have two sets c, you can add c plus c, but then twice of c is always a subset of c 1 plus c 2. Think why, and find then example where this is a proper subset of c 1 and c 2, and find an example where this is equal to c 1 and c 2, so another piece of homework. So, one thing that you can observe that, what is twice c you are taking the same element then adding them, but c plus c does not mean that you take the same element from c and take the same element from c. It means take just one element from and c can take another element c which could be different to the element that you have chosen before, and then add them. So, find an example where this is the proper subset, so this home work is find examples, and also find an example where this equals. So, these are the two things that you try to figure out, you will have lot of fun doing so, just bring on which sets in naught 2.

Now, once I do this, I would also try to define the notion of a cone. Cone is an important class of set use in optimization, and Convex cone is what will be of at most importance. So, what is a cone you can understand what is a cone, because if you taken a ice cream cone, you call it ice cream cone that what it is look like, but in mathematics this cone is not just, it ends up to the bream of the ice-cream, but you know just keeps on extending, when extend the whole thing in the imagination. So, cone is a set S such that lambda times x is element of S, if lambda is element, if x is element sorry if x is element of s and lambda is greater than equal to zero. So, for example, you take an example take any ray going out of the origin, this is the cone. A pair of rays is a cone, but as you can understand this not a Convex one, that is not a Convex in general. So, any cone which is Convex is called Convex cone, this is the cone. So, you can observe, you figured out of this definition is what I given will be satisfied. So, Convex cone, not a Convex cone. So, this is very basic thing, and now we go on to some important examples. The first and foremost example which is very important optimization is of the set R n plus.

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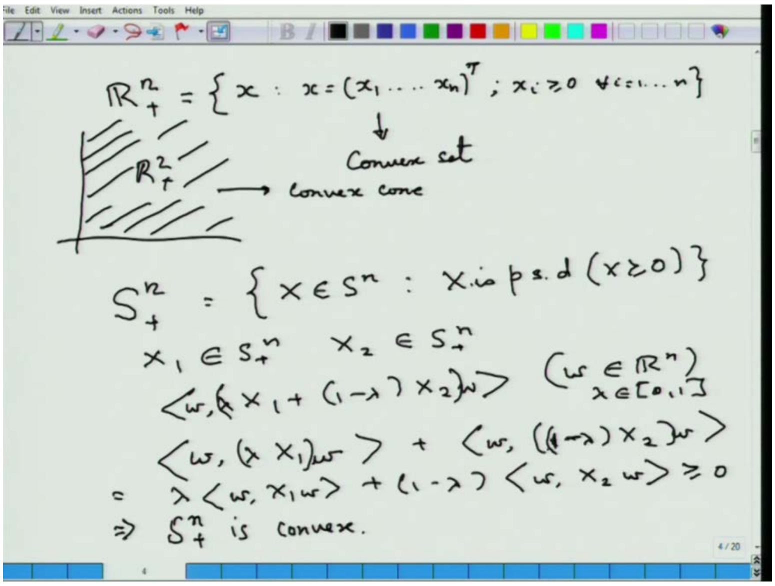

So, this is a Convex Set, is a set of all x, which is return as x equal to x 1, x 2, x n, but I should is id as the column. So, I give at as transpose T where x i is greater than 0 for all i from 1 to n. So, in our setting this is R 2 plus. So, the interesting fact is that, this is not only a Convex Set, important Convex Set, this is a Convex cone. Another important Convex Set is the space of all symmetric positive definite matrices, is a set of all x element of S n; such that x n, such that x is p s d that is positive semi definite or in other words, you can right it like this with an lower and ordering. So, how do you prove that this is a Convex Set. So, you take 1 x 1 from S n, and another x 2 from S n, infact S n plus. Now, what do you need to do is to show that lambda x 1 plus 1 minus lambda x 2 is also in a S n plus. So, you take the lambda of x 1 plus 1 minus lambda of x 2, in order to show it is in S n plus we need to show that this metrics is positive semi definite. So, what do you do is just do, take this by linear product.

So, you can write this as w, you must put a bracket on the standard, this metrics operating on this w, w times lambda x 1 into w, plus w times 1 minus lambda x 2. So, if you look at this, this is nothing, but lambda times. Now, both of these are in S n plus. So, both of these are the greater than equal to 0, because w is any element in R n. So, both of them are greater than equal to 0, lambda is greater than equal to 0 and 1 minus lambda is greater than equal to 0, because to show the Convex Combination lambda is obliviously chosen from 0 1, which is oblivious fact, which you need not even state repeatedly. In Convex analysis lambda is always between 0 and 1 when unless until mentioned. So, what you see here is this fact, that now all of this whole thing is greater than equal to zero. So, let me not write and just put this greater than equal to 0, implying the fact that S n plus is convex. We will have more opportunity talk about S n plus in the future.

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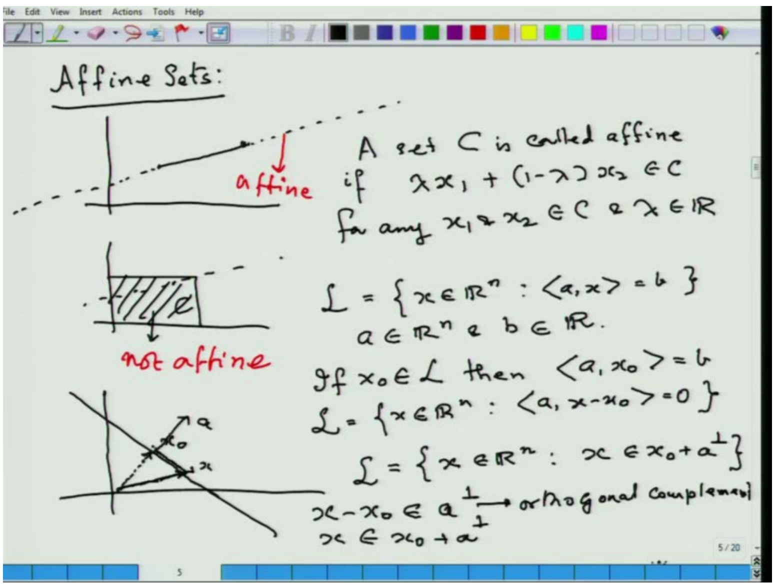

And let me talk about another class, important class of sets called affine sets, which are sub class of Convex Sets. So, what is an Affine Set. Affine set if you take two points say naught 2, the line segment joining game is a Convex Set which I do not have to tell you, because definition simply tells that. Definition line segment simply tells that this is a Convex Set. Now, think of the line which passes through these two points. Now, this line itself if you look at it, is a Convex Set. Now, it's very important to know that, how do I define such a line or if you have a plane or a say a figure like this, triangular triangles, what a 2 d triangle in 3 d, then I can a put it on a plane. So, how does that plane is a Convex Set. So, how do I define such as set. So, such sets like this, like the line passing through two points is the class of Convex Sets call affine sets, which as the following definition, and a set C is called affine if lambda x 1 plus 1 minus lambda x 2 is element of c for any x 1 and x 2 element of c, and lambda element of R.

Note that lambda no longer has any sign here. So, of course this is true for any lambda between 0 and 1, signifying that this is also Convex Set, but every Convex Set is not an Affine Set, because if you take say this set this. So, take two points here, take the lining passing through them. So, the line passing through them does not reaming inside the sets c, if this is my set c the line passing through any two points does not reaming the set c. So, this set is not affine while this Convex, this is affine and important class of Affine Set is the straight line, so you can write such straight lines as follows. So, take the set L, say set of all x in R n. So, the set of all x which satisfies his equation, where a is a member of R n and b is member of R. Now, what is important to me at this stage, is very simple to prove that this is an Affine Set, just by a properties of the inner product. Now, what is the geometry of this particular type of sets. So, if you look at the R 2 geometry for this particular case, if n is 2 then you are basically looking at some line like this. Now, if you look at this line what is the nature of a n, does a have anything to do with the line itself.

Now, take any line point x 0 on the line, then what would happen. Then of course, if x 0 is element of L, we can call this as linear Manu fold, so that is more technical term but forget it. So, if x is not a element is well than what happens is a of x naught is equal to b. Now, if that happens than I can replace in this equation b with a of x naught and then I can right L, as the set of wall x such that a of x minus x naught is equal to b. So, what happens is that if you take this x and take any x here. So, if this is my x naught vectors and this is my x vector. Now, this is your x vector and this is some x vector, if you take any x. So, this vector is your x minus x naught, and what does this equation say that this a is, this should be a mistake a x 1 x naught should be equal to 0, because you put b equal to a x naught then you transfer it. So, what it says on the x minus x naught vector. So, x minus x naught obviously is lying in this plane. You, can understand that this vector x minus x naught a is perpendicular the to the vector x minus x naught, but since x minus x naught lies along the plane from the geometry. So, which means that a is perpendicular to the plane itself. So, what does it mean.

So, what it shows that a is perpendicular to the plane itself x minus x naught, that is exactly what is the nature of the element a. So, a is an element, if you look at x minus x naught, this element is in the orthogonal complement of the vector a, it is in the orthogonal complement of the vector a naturally, that is what we had just say geometrically. So, which means a x is element of x naught. So, but if you take any element which is x naught plus a tilde, then of course, immediately will get this, x in R n which satisfies, that x is an element of x naught plus a tilde, a which is orthogonal complement orthogonal complement of a. This is terms only linear algebra which you are supposed to know orthogonal complement of a. So, any x which satisfies this, is satisfying this and satisfying this and hence L can also be described like this. So, this is also a Convex Set, but this a other in a Affine Set a very important class of affine set.

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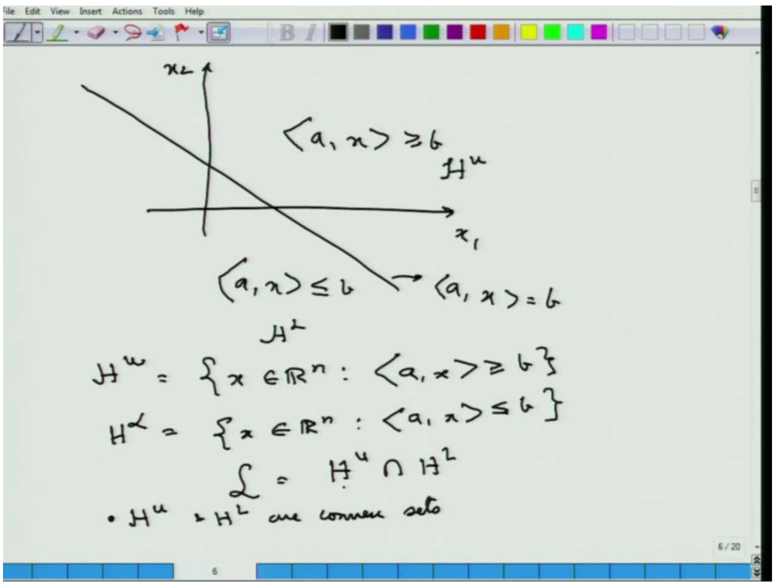

So, if you look at this L, what does a straight line do. So, if I call this as a x equal to b. Now, this straight line is dividing a plain into two parts. Any point that you take here would satisfy a x greater than equal to b, because this is equal to b any move up, it leaves the value of b and become bigger, any move down it leaves the value b and becomes smaller, and this part is called a x less than equal to b. So, basically I have divided this line a x equal to b, has divided the plain into two parts which we call H U and this is called H L. This is called upper half space, is a set of all x in R n such that a x is greater than or equal to b, the lower half space a H L is given by the set of all x in R n and L is the of course, H U intersection H l. So, we have now got a fairly decent amount of explanations about, or fairly decent amount of introduction to Convex Sets. So, after this will talk about certain important properties of Convex Sets than start describing Convex functions. Now, we have shown here that L is H U intersection H l, but of course, as a Home work I would ask you to show that H U and H L are Convex Sets and by knowing that the intersection of to Convex Sets is Convex you can immediately deduce that L is a Convex Set.

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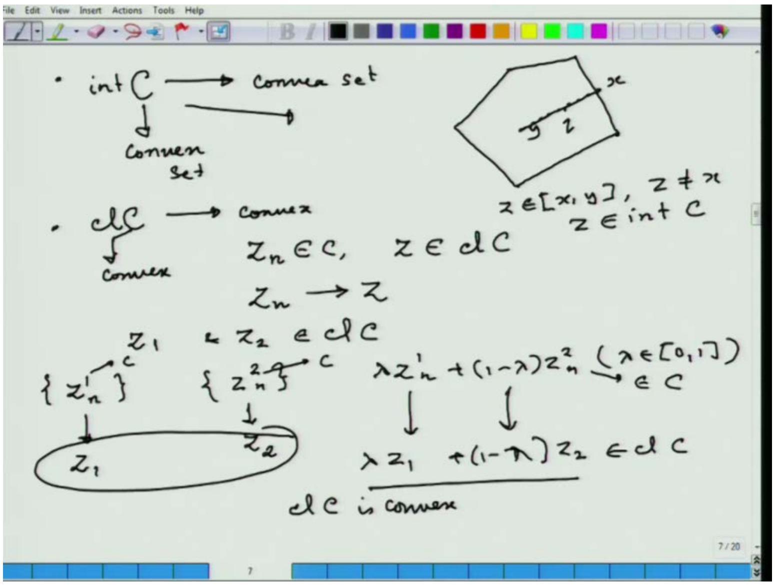

Now in important property of Convex Set is the following; if you look at the interior of a Convex Set, if you take a Convex Sets c and look at to interior. I am sure all of you know the definition of interior. In case you do not look of any standard book of analysis. So, if c is Convex then this set, the interior of the c is also Convex Set. Similarly, if you look at the closer over set c, if this is Convex this is the sets c, sets c itself, if this is

Convex then so is the closer. The prove of this is a very simple, that is take sequence z n, take sequence z n element of c going to element z in the closer. So, you see whenever you have z, you will have a always have a sequence z n going to z that is the definition, using this definition actually prove that the closer is also Convex Set. So, you take two elements z 1 and z 2 in the closer. Now, what would happen is, that you would have z 1 n, and z 2 n, this two sequences where z on an convergence to z 1 and z 2 n convergence to z 2. So, now, you make Convex combinations. Now, these are elements in c.

So, make Convex Combination lambda z 1 n plus 1 minus lambda z 2 n where lambda is between 0 and 1. Now, what happens, because these are elements in c this is also element in c, it is belongs to c. Now, when you take the limit, of course you can fix of lambda and take such sequences. Now, if you fix of the lambda that is you want lambda z 1 plus lambda z 2. So, you fix up of particular lambda and then take this sequence, if you take the limit, because it is a linear function. So, it will immediately give me at this whole thing convergence to lambda set 1 plus 1 minus lambda z 2. So, whenever z 1 and z 2, but since this is in c where the very definition of closer, this is also belonging in to the closer c. So, whenever z 1 and z 2 belongs to the closer, this also belongs to the closer proving that the closer c this Convex. Now, what is important is how do prove this to be a Convex Set.

The proofs of this realize on very important properties is call the interior property of Convex Set. I am assuming that this Convex Sets c has an interior, every Convex Sets c need not have a interior, we will just speak about that. So, take this set in naught 2, whole thing inside and with the boundaries. So, it is close Convex Set . So, you have this part, the white part as the interior of the set. Now, if you take any point in the boundary, see you take x in the boundary, and you take y in the interior, and you join them by the straight line. So, any point in x y, take any z from x y, such that z is not equal to x, then z must belong to interior of c. This is a very fundamental result, if you take any z here, which is not equal to x it must be in the interior. This is the very fundamental property it will be used to prove this, and many other things. So, as I told you that every Convex set need not have an interior.

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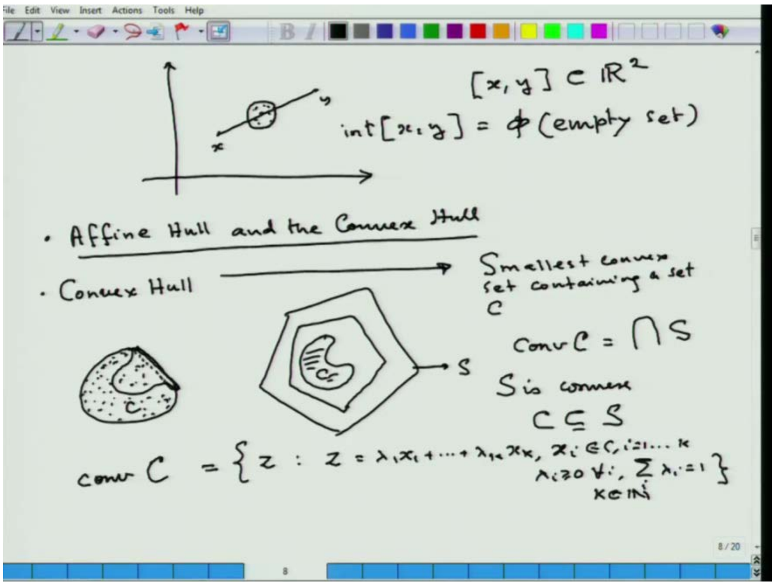

How do a support my statement. Let us take an example, in mathematics it is imperative if you want to say something does not told just given example, but if you want to something holds, then you need to have proof. So, you take the straight line, line segment joining these two points x y. So, this line segment is obliviously a Convex set. So, my question is whether this has an interior. Now, what do you mean by an interior in R 2, remember x y is viewed as set in a R 2. Now, if I take any point here, then by interiority means I have to take a ball, we have already described what is a ball. A ball around this point; such that whole ball should be in x y, but these does not happen, you take any point accepting the point x y. Take any point it does not happen; so x y, if you look at the interior, if I say int; the interior, this is empty, but then what is not. Is there something we can speak about its interiority, because if I look at x y for 1 dimension view point as an interior.

Then how do I speak about some sort of interior of this set x y, this brings us to two important notions, the affine hull and the Convex hull. So, keeping in with the tradition of Convex analysis and Convex optimizer. I will first define what is Convex hull, but affine hull is what would be a help here. You take any set a non-convex one; Convex hull is the smallest Convex Set which contains this set c, now this is my set c. Now, this set obliviously is not Convex which is very clear, but you see if I take this last two end points and join them up; and this new set, this boundary, this new set becomes convex. So, Convex R is a smallest Convex Set c containing, Convex Set containing the set c. So, you take any set c like this. You, take any Convex Set c, like this you take any Convex Set containing the set c, set c is here with this whole thing.

This is one Convex Set, say s which contains the set c. Similarly, you can take another Convex Set which contain the set c, and similarly go on the intersection all such set would be finally what we require the Convex hull. So, Convex hull; if I have to define it, is a smallest Convex Set containing c, containing a set c. So, which means if you have set c, the Convex hull of c which we define as Conv s, Convex hull of the set c which we define as the intersection of all the sets c s, where s is Convex and c is a subset of s, but these are only very quilted if things, means is there any way to represent. In mathematic representation of sets become very important issues, especially when you do Convex things like optimization. These representations help when you are a doing theory and also in computations. So, how do you represent the Convex hull over set c.

So, I hope you remember the beginning of the lecture as spoke about Convex combinations over set c. The, Convex hull over set c is the collection of all the Convex Combination of the set c. So, Convex hull over set c consists of all element z, of the form z equal to lambda 1, x 1 plus lambda k, x k where x i is element of c for i equal to 1 to k, and lambda i is greater than equal to 0 for all the i and summation over i lambda i is equal to 1. Now, this k is the element of n, there is this k can vary, so you can just take any finite number of elements from the set c, and make their Convex Combination put them aside in one set, the set that will be found is Convex hull, but you see here k can be changed, and this is the bad factor in the representation that there are infinite such k, it is very difficult to really visualize such a set. This problem was solved by this beautiful result of Caratheodory.

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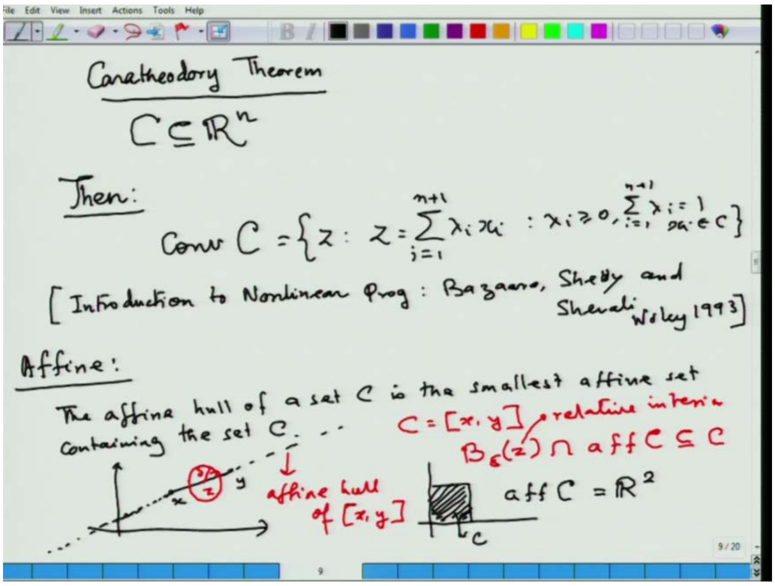

Caratheodory Theorem says, that if c is the subset of R n, then the Convex hull of c is a set of hull z. So, the z is equal to summation lambda i x i, i is from 1 to n plus 1, now here fixed it up, you just take n plus 1 number of elements, in lambda i is greater than equal to 0 summation lambda i is equal to 1, i is obliviously from 1 to n plus 1, and all the x size element of c. Here is a beautiful representation I do not have to this number, k which will arbitrary change with every z, take a z your k will change in general. Here, you take a z your k will change in general. Here, you take a z your k will not change and this is very fundamental result in Convex analysis. Proof is essentially for a mathematical audience and not really for the audience set we are targeting here. For proof you can see any book on good book on algebra or any book on Convex analysis, or books which engineer preferred is this one; Introduction to Non-linear Optimization or Non-linear Programming by Bazaara Shetty and Sherali. It is a Wiley publication, second edition is 1993 and I think there is also third edition.

So, that chapter two has a proof of, there is simple proof of this fact. So, now, what is an affine hull, you can obliviously make up the definition very simply. So, the affine hull of a set c is the smallest affine set containing the set c. Now, let us see what is the affine hull of that straight line, or the line segment that we drawn, not really straight line, a line segment. So, the affine hull is the smallest Affine set. This whole space R 2 is an affine set, so obvious it contains this one, and the next Affine Set, by the very definition of Affine Set you can understand in a Affine Set in the whole lines have to be in the set not just line segment. So, this line, that is passing through this two points x y that is the affine hull, that is the affine hull of. Now, look at this fact that if I take this x points some point z, and I take or pole of radius say delta.

Now, even though is not contained in the set x y, this set c equal to x y, this Convex Set x y, but if you observe it carefully, the intersection of this ball, center at z of radius delta is intersection with the affine hull of the set x y, or affine hull of c. This is contained in c. So, if such a point z where this thing happens, is called a point of, is a relative interior point. So, z is called relative interior point. Now, here instead of this straight line if I take this square, so it is a two dimensional set, no longer a one dimension set in or two. Here it is two dimensional set, if this is my c what is the affine hull. So, affine hull or a f f is denoted as a f f c, like con c Convex hull a f f c is affine hull. The affine hull of the set c in this particular case is nothing, but whole space R 2. So, what happens here is the following, is that once you dimension goes down, when you are a low dimension set, but in a high dimension space, your affine hull is a also lower dimension, and those are the sets where you do not have the interior, but you have to something like a relative interior. So, very important result in Convex analysis following.

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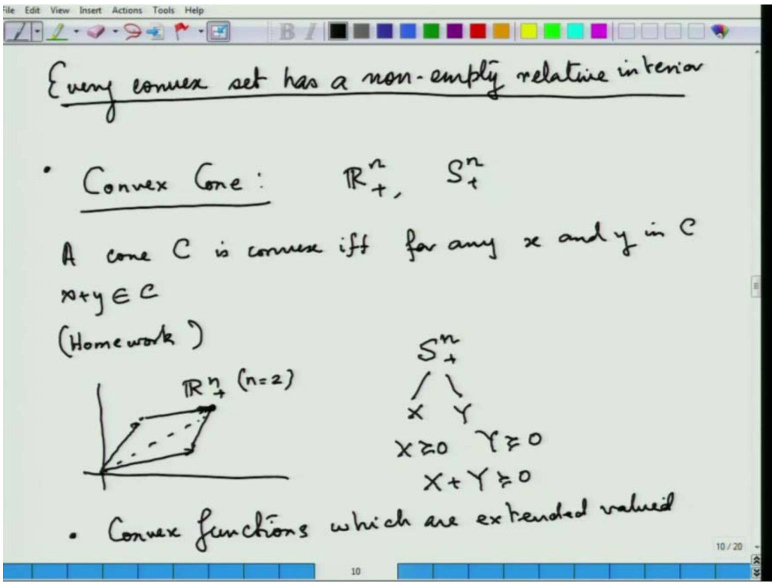

Every Convex Set has a relative interior; set has a non-empty relative interior. Relative interior is nothing, but the collection of all relative interior points. So, this is a very important statement, because if you take any full dimension Convex Set, like three dimensional Convex Set in R 3, then it always has an interior and this fact is extreme importance to Convex optimization. This is what is call as status consent qualification which is of extremely fundamental importance to the study of optimality condition in Convex optimization, and because those optimality condition are used in a algorithm. So, they this whole factor in a very meaningful role. So, now we have fare idea about what is the Convex Set, the Convex hull, the affine hull, the very basic facts about Convex Sets. Let me just tell you a fact about Convex cone. How do you characterize a Convex cone. Let us go back to this whole issue of cone. Now, the Convex cone that you know, two important Convex cones that we have spoken about is R n plus and S n plus.

Now, how do you characterize a Convex cone. A cone c is Convex iff and only iff, this is the short form iff at only iff, if and only if, for any x and y in c, x plus y is also in c, check it out with this. So, this is your home work to many home works today, what you need to figure out these simple things, to have a better idea. You take any points in R n plus. You see this point is also in R n plus, here of course, or demonstration as I always told you would be with n equal to 2, and of course you will take S n plus. So, if you take two element x and y, with x positive semi definite and y positive semi definite is a very simple fact or a very simple prove that this also p s d. So, with this basic fact we stop speaking about Convex Sets, and in tomorrow’s lecture we would view part two of this, we will speak about Convex functions. What I warned you that, we would really talk about Convex functions which are extended valued. Of course you know what is definition a Convex function and few important property, that every local minimum is a global minimize a Convex function over a Convex Set.

But we have not yet considered with all important, the notion of Convex functions which are extended value, and we really have to do it and we have no other choice, because functions, Convex functions which naturally arise in optimization or of this nature, and we cannot say no to it. So, with this we stop today’s lecture, which was essentially on Convex Sets with their basic properties, next lecture would be on Convex functions and we would speak about them in part two of this lecture, and part three which would again will come back to Convex Sets, but it will essentially be a combination of Convex Sets and Convex function, which is of extremely fundamental thing of all separation of Convex Set, because as many mathematicians think that optimization theory is a long corolla you have separation theorem, possibly it is, but not exactly so. So, we would like to go into on the third lecture, this is the first part, second part and the third part is on separation of Convex Sets in which we will do some proves.