Convex optimization
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Lecture No. # 02

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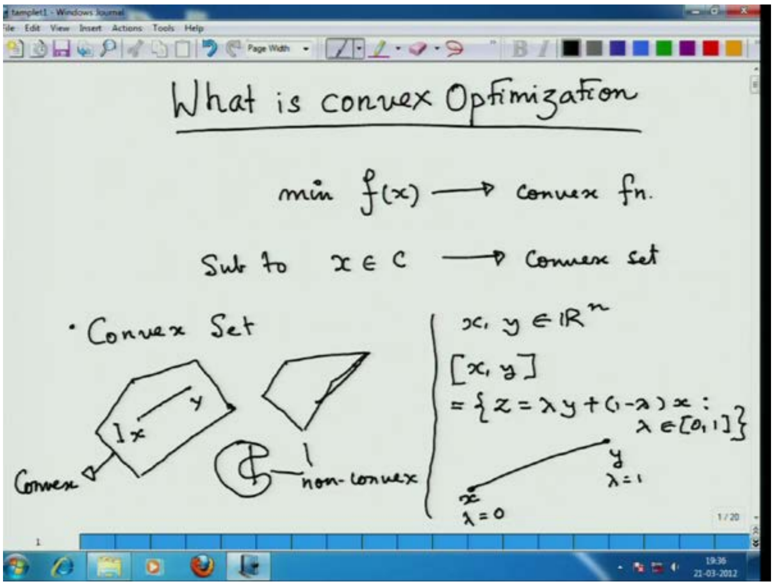

As we had said in the last lecture, today we are going to talk about certain basic facts about convex optimization itself that is we are going to discuss today. The minimization of a convex function f, subject to x belonging to a convex set c. So, this is what I call a convex function, and this is what I call a convex set.

Now, of course, the question is what is the convex function and what is the convex set. Now, before I start the full fledged discussion, let me tell you that, I would like to show you a book, which I think everybody should have a look at it. This book is called optimization Insights and applications written by Jan Brinkhuis and Vladimir Tikhomirov, and I have spoken about a book called stories about maximum minimum written by Vladimir Tikhomirov. This book is a fabulous book whether you want to do convex optimization or even a non convex one, has a huge amount of insight. It is published by Princeton University press in the Princeton freeze in applied mathematics, a quite a recent publication and it is just a mind blowing read in. In fact, it is something everybody who wants to do anything with optimization should have a copy in their desk.

So, going back we will first define what is the convex set. Simply, put convex set means that if I take a set in generally in R n, but as always we will look into the pictures in R 2 and take two points x and y in this set, any two points and join them by a line segment, and this line segment should also remain inside the set. Now, if you look at this set whatever points you take, whenever you join them they remain inside this set. You take a set like this. I take a point here, I take a point here, I tried to join it, that goes outside or set like this, it is more clear if I take these two points and I join them, a part of the line segment is outside the set. So, this set is convex, these two are non convex. If you look at your own body, your human body; the human body is non convex, because if you take a point here I take a point here you join them, it is completely outside the body. So, human body is non convex thing, is a non convex set.

Now, how do I formulize this definition that if I join any pair of points by a line segment, the line segment has to be in the same and that is that sort of set is called the convex set. So, to begin with I will talk about the given any two points x and y in R n, the definition of a line segment. The line segment is usually denoted as with this symbol, is the set of all z, which is expressed as lambda times y into 1 minus lambda times x, where lambda is a number between 0 and 1. So, it means if I take this two points x and y, and if I put lambda equal to 0 here, I am getting x and if I put lambda equal to 1 I am getting y. So, as I vary lambda from 0 to 1, we move along this line from x to y. So, it is clear that this is nothing, but the simple geometric line segment that we know. So, even when you are talking about two points in three dimensional spaces, we are talking about this line segment.

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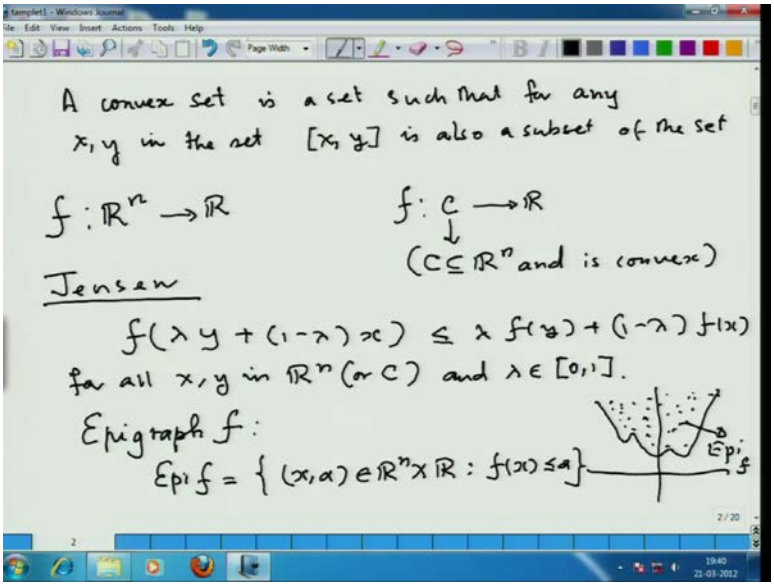

Now, formally a convex set is a set, such that for any x y in the set x y and segment is also a subset of the set. So, this is just the thing in English language which you now understand very well. Now, what is the convex function. a convex function can be defined from R n to R, or it can be defined from a set, convex of set C to R where c is a subset of R n and is convex. Now, the most original or the most or I would say the earliest definition of a convex function was due to Jensen, W B Jensen, and Jensen gave this definition which says that, if we consider from R n to R or even from C to R, so f of lambda y plus 1 minus lambda x. See this is very well defined if you take a convex set c and y in x 2 elements of c, then lambda y plus 2 elements of C then lambda y plus 1 minus lambda x, whenever lambda is written 0 and 1 is elemental c. So, this functional operation is well defined this has to be less than lambda times f y plus 1 minus lambda times f x, for all x y in R n or C and lambda in 0 1.

This is the definition of Jensen, but much more modern definition can be given in terms of epigraph. Epigraph over a function f is everything that lies above the graph. So, if this is the function f and this is the graph then epigraph is the dotted portion along with the outer curve, anything above the graph is called the epigraph. So, this is the epi f, epigraph of f. So, the epigraph of f which we denote as epi f, is the collection of all elements x alpha, where x is in R n and alpha is in R; such that f of x must be less than or equal to alpha, which is very clear from the diagram.

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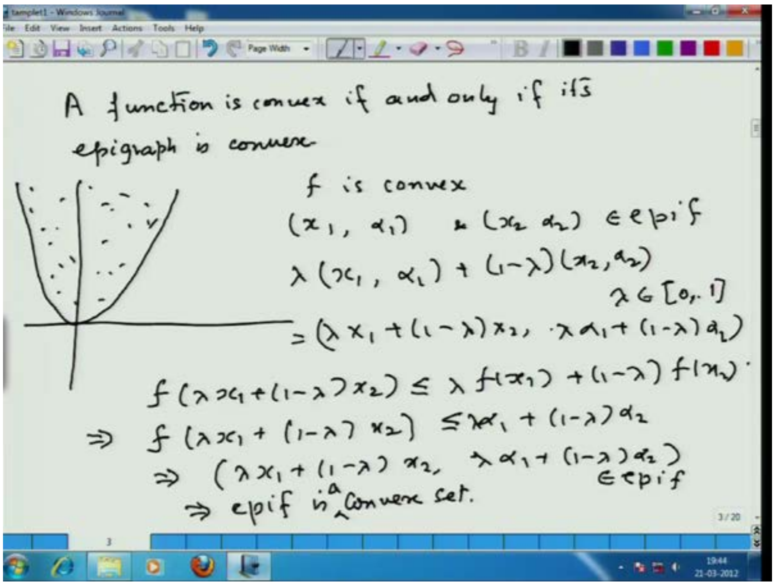

Now, what you can prove is that a function is convex, if and only if its epigraph is convex. Now, proving this is a simple fact, because you look at this function whose epigraph is convex, and you can prove that this definition satisfies the definition that we had given on this page. So, going ahead that we tried to prove that if f is convex then epigraph of f is convex. So, let us assume that f is convex, now you considered two elements x 1 alpha 1 and x 2 alpha 2 from the epigraph of f both of them. Now, if you make a combination lambda x 1 alpha 1 plus 1 minus lambda x 2 alpha 2, where lambda is some number between 0 and 1, then this just means lambda x 1 plus 1 minus lambda of x 2 comma lambda time's alpha 1 plus 1 minus lambda time's alpha 2.

So, I have said that epigraph of f is convex, now since f is convex I have to prove that epigraph of x is convex. Now convexity of f tells me that lambda x 1 plus 1 minus lambda x 2 is less than lambda f of x 1 plus 1 minus lambda f of x 2, you must remember 1 thing if a lambda is attached here your f x 1 lambda gets attached to f x 1, if 1 minus lambda with x 2, 1 minus lambda on this side gets attached to f x 2 , but since x 1 alpha 1 is in epigraph, f of x 1 is less than alpha 1, and since x 2 and alpha 2 is in the epigraph, f x 2 is less than alpha 2. So, which means that f of this would imply that f of lambda x 1 plus 1 minus lambda x 2 is less than equal to lambda alpha 1 plus 1 minus lambda alpha 2, by the very definition of epigraph, go back once again, check the definition look at this definition f x is less than equal to alpha, all such x alpha for which this occurs. So, this is an x, this is an alpha, this is in R n and this is R, and then this would imply that lambda x

1 plus 1 minus lambda x 2, lambda alpha 1 plus 1 minus lambda alpha 2, this is in the epigraph. So, I have taken two arbitrary elements from the epigraph and showed that their convex combination lies in the epigraph, which proves that epi f is convex, or is a convex set to be more precise.

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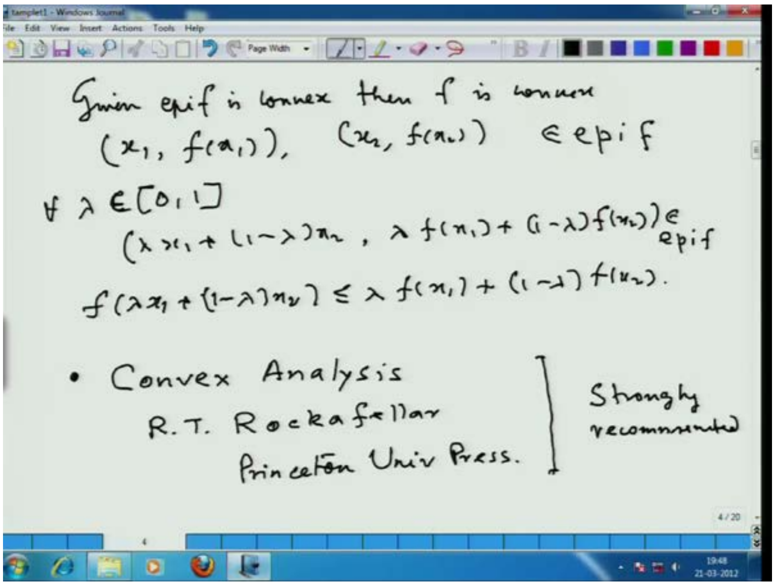

Now, suppose I want to go back and prove that given epi f is convex, then f is convex. Now, by the definition of epigraph if you take two points x 1, then x 1 and x 2, x 1 f x 1 and x 2 f x 2, both these points are in the epigraph, they belong to the epigraph of f, both of these two points are in the epigraph, and you know that epigraph f is given to be convex. So, which means that for all lambda, you take between 0 and 1; for all lambda between 0 and 1, lambda x 1 plus 1 minus lambda x 2, lambda f x 1 plus 1 minus lambda f x 2 is in the epi f. Again, by the definition of epi f it simply means that epi f of lambda x 1 plus 1 minus lambda x 2 is less than lambda times f x 1 plus 1 minus lambda times f x 2. So, in many cases the modern definition of convex functions are given in this way that, a function is a convex function if the epigraph is convex set. So, here we go back to the user definition of a convex function, I just rub this to make it

Now, let me tell you about the text from which you would be able to know convex c t or convex analysis at its best. So, I would refer to the most legendary and classical text in this area called convex analysis. It was published by Princeton University in 1970 and it was republished in 1994 as Princeton landmarks in mathematics, and this is by one of the famous mathematicians in this area Ralph Tyrrell Rockefeller, whose name is almost synonymous with convex analysis. So, anybody who is an optimization student, who is a graduate student in optimization should have this book with him, and as a my own researcher in optimization that even if I am stuck with some difficult in convex analysis in my research, I just have to go to the book of Rockefeller hang around for few hours, and the answer you can actually figure out the answer.

So, that is the power of this book. Now, let me tell you one thing is that, this book is not a book which has to be had from cover to cover, no mathematics books are actually, they are not story books that you read from cover to cover. You can follow if you read the places where you want, but this book specifically is not a cover to cover book as written by the author himself in the preface, that you really to this book is something which is to help you when you are in trouble, but this is book is a must for all optimization researches and graduate students in optimization, I would write here strongly recommended. Now, if you look into this book, I will just digress of it. I understand that here I have a diverse audience as I realized, whether I had said in the last lecture, because of this diverse audience. I would like to reduce a bit rigger that is required to do the mathematics of the subject, but however to give the taste or what convex analysis is or convexity is, we need to go away and understand the idea of an extended value function. If you go and open this book convex analysis, in the very first chapters dealing with convex functions

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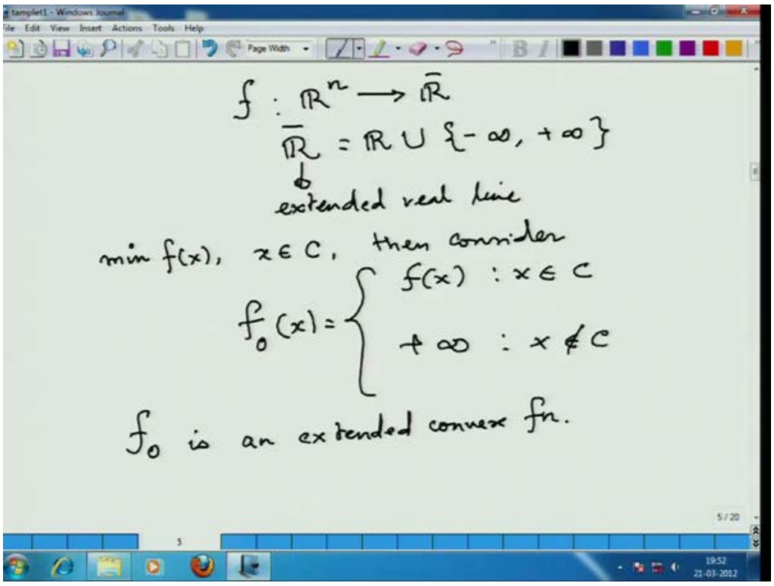

You will see that Rockefeller speaks about convex functions which are defined from R n to R bar, where R bar is R union, the two infinities; minus infinity and plus infinity, it looks strange, but this is called the extended real line. Now, why you need to speak about extended real line, the reason is essentially as follows; you will realize as we go on that the unconstraint problems where there are no restriction on the decision variables, such problems are much more easy to tackle, then the problems which have restricted on the decision variable, like we saw last in the last lecture. So, how do I theoretically convert a problem, which is a problem with restrictions on the variable to a problem which does not have restriction in the variable.

To do this if you look at the problem that we are dealing with minimize f x, x element of c, then this is a constant problem. This has a restriction that x is expressed to C then considered this function, which takes the value f x when x is in C, and takes the value plus infinity when x is not in C. Though, you might find it bit difficult in the beginning to appreciate this, but those who know some optimization has read, they have an exposure to under graduate non-linear optimization. They would realize possibly that this is what is called a penalization, that if you violate the constant I impose infinite penalty on you theoretically, and this is nothing, but theoretical model of the penalty function method which is quite a method in solving constant optimization problems.

So, this f naught if f is convex, then this f naught is an extended convex function, but when you define extended convex functions you have to have certain rules on infinity and minus infinity. So, we will not get in to all this at this moment, we will come into the rules as and when required. So, when we will study convex functions and convex sets in detail we will start from tomorrow, we will go into a bit of this issues. So, that even if you get in to this books like convex analysis or further books that I will say, you will be comfortable enough to handle this things. So, now let me get back straight into the modeling of this problem.

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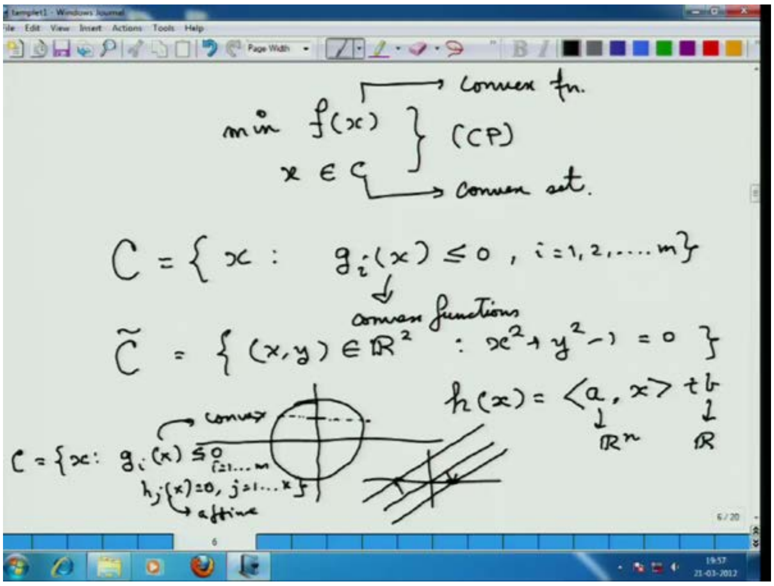

Now, this problem which we will start referring to as C P, where f is a convex function and C is the convex set, is called an abstract version of the convex optimization problem f, because though f has a representation C does not have a representation. This is a convex function and this is a convex set. Now, let me tell you very simple thing is that, the set C in most applications is represented as a set of all inequalities usually, that all these g i s are themselves a convex functions. This is the standard thing, you might ask me what about equalities, equality constants are quite well known to you possibly in calculus, when you learned Lagrange multiplier rule you talk about the quality constant. So, where is your equality constant. Let me talk about a set C.

So, I talk about this set c hat, and then I talk about in naught 2, so that you visualize it, x y naught 2 such that x square plus y square minus 1 is equal to 0. So, if you look at this, this is a convex function, and then if you try to sketch this set, this set is nothing, but this 1 only the circle nothing inside, only the circular ring, but then if you take a point here and take a point and join it, except these two point the whole line is outside the circular ring this is not a convex one. So, if you put convex equalities here you are not going to get in general convex set. So, what happens, what sort of equalities you need to put in. The equality constants that need to be put in to have a convex set at the end has to have a particular form, they are of this particular form, they are called affine function. So, they are usually written in this form for a given a.

So, this is in R n and this is in R. So, a linear function plus a translation, for it is a translation over linear function. So, basically linear function you see, this is the linear function y equal to x, any linear function has to pass through 0 in naught 2, and then you just translate it down. So, this is or this or that which you want to translate it that. So, this is an affine function, this affine function need not be linear, but every linear function; obviously, is affine. Every affine function is a convex function, because you can just figure it out when you do not have to really too much work in figure in this out. Now, if you additionally define a set C, set of all x such that, of course x is in R n, I do not have to tell you repeatedly, g i x is less than equal to 0 for all i from 1 to m, and h j x is equal to 0 for all j from 1 to k, where this is convex and this is affine, then this set C is a convex set, and that is what we essentially want at the end of the day. Now, let me go back to straight to the optimization issue.

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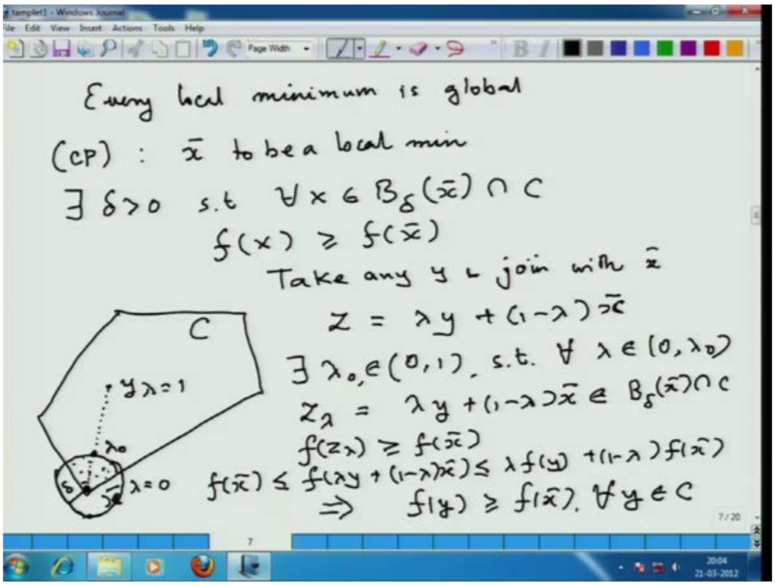

What I want to show you now, is that every local minimum that is the convex program C P, convex programming the term why I am calling it a program, as a historical basis, but I will tell you that history later on, but let me just go in and do the math, that every local minimum is global. There is as no global minimum, every local minimum is a global minimum. So, if I take the problem C P and consider a local minimum x bar to be a local minimum, if I considered x bar to be a local minimum what I am supposed to do, it means that there must exist a delta greater than 0; such that for every x which is in the ball center at x bar of radius delta, and whose points also line see that, this set the intersection of C and B delta x bar, for all such x f of x is bigger than f of x bar.

Now, this is my convex set, let this may this x bar my local minimum, what I have showed that, there is at there must be by definition, there must be a very bad drawing, delta and for any points in this intersection. This is a set C, for any points in this intersection f x is bigger than f x bar, this is the definition of a local minimum. Now, you take any arbitrary point y in this set C. Take any arbitrary y, now join this y with x bar. Now, if I am supposed to do. So, if I am to join spelling does not look very nice it is j o i n. Now let me construct the line segment, so any point on the line segment connecting y and x bar can be written as lambda y plus 1 minus lambda x bar, which means when lambda is 1 I have y, when lambda is 0 I have x, x bar. So, that means, there is lambda is equal to 1 and that is lambda is equal to 0. So, when I am moving dropping the value of lambda from 1 to 0, I am actually moving along these line segments.

So, as I move along this line segment I will come to this threshold point, whose lambda is say lambda naught which cause most to Z, corresponding Z naught corresponding to lambda naught y plus 1 minus lambda naught x bar. If, I reduce the value of lambda from lambda naught, all the points lye in this line. So, what I can say is that there exist a lambda naught, element of element of the interval 0 1; such that for all lambda between 0 and lambda naught or Z lambda Z defined by the lambda, lambda y plus 1 minus lambda x bar. So, now, I am considering only those lambdas which are lying in this interval, this must be in b delta x bar intersection C, this is clear from the geometry here from the picture here. Now, once I know this what would I have that f of z lambda is bigger than f of x bar. Now, which means f of x bar is less than f of z lambda means f of lambda y plus 1 minus lambda x bar, but now this lambda is still lying between 0 and 1. So, by definition it is lambda f of y plus 1 minus lambda f of x bar.

Now, if you do the manipulations f x bar cuts out from here, and because lambda is some quantity between 0 and 1 which excludes 0 and x excludes one. So, lambda is the positive quantity I can divide both sides by lambda, because this f x f x cancel off to get 0 on this side. I would simple have this fact, it would imply that f of y is bigger than equal to f x bar. Now, you observed that y was just an arbitrary element; y was just an arbitrarily taken element in c. It need not be inside this, it could be in either else. So, for any arbitrary y I have been able to prove that f of y is greater than equal to f of x bar. So, it is true for all y in c thus proving that x bar is a local, not just a local minimum, it is a global minimum of the function f over the set C and thus it shows that for a convex function every local minimum is global, and you must have observed that here we have not bothered about the different ability of the convex function, we are not really caring whether the function is differentiable or not, and this fact is a very important fact and you will see how important it is as we come on and as we proceed along and study more about convex optimization.

So, we have just learned about the very important fact that for convex problem every local minimum is global. So, this fact is very fundamental, and you will see how it helps us in rest of the talk, but it is very important as I have already mentioned that this class of problems convex optimization problems which have huge applications. Now, instead of going into specific applications, because that would be talking about domains which are not my domain rather than, basically if I am talking about, say problem in electrical engineering or problem in mechanical engineering, there are lot of problems which can be modeled as the problem C P. Now, instead of that let us look into some of the most important forms of convex optimization problems, and these forms, these are the classes which has played a major role in application, because these are the classes of problems which appear repeatedly in applications.

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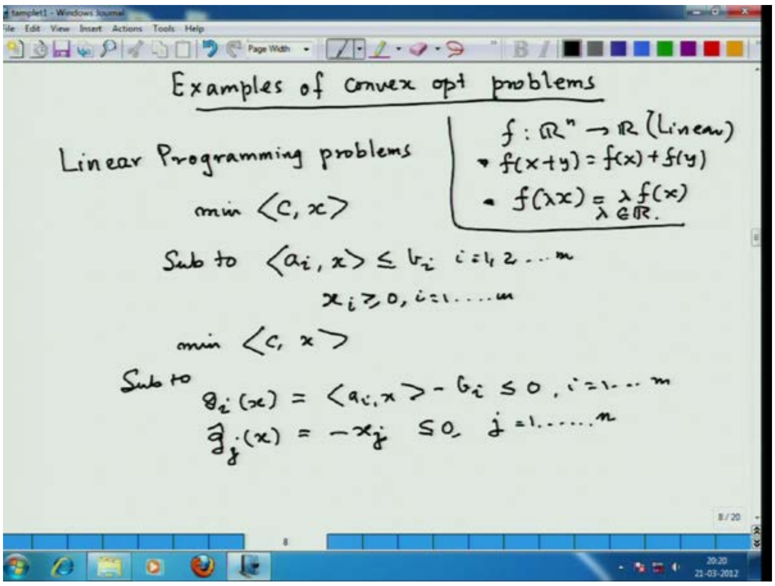

So, one of the most important class of problems is the one which is possibly known to many of the viewer's viewing this talk, is the class of linear programming problems. In this class of problems, you seek to minimize a linear function, say linear function you have only this part you do not have a translation. Every linear function can be expressed as inner product which is a very simple fact from linear algebra, and I am just assuming that everybody knows what is the definition of a linear functioning, in case you do not, you can take any function from R n to R, this function is said to be linear, if these two properties hold; first is the property of antiquity that f of x plus y is same as f x plus f y, and the second property is homogeneity that if I take any real number lambda and scale of this vector, scale up or scale down and this is same as lambda f x where lambda is in R. So, this is the definition of a linear function, also you minimize this subject to linear constraints, also in most practical problems there is a requirement that, all the x is are greater than equal to 0. Now, of course this is the convex optimization problem, because I can pose this as. So, here I have m inequality constants and also some other group of inequality constraints n inequality constraints, which are heard as minus x i or x whatever it is j you want, so the m plus n constraints. So, you can actually put this thing, put this a i x minus b i this whole thing inflammatory form

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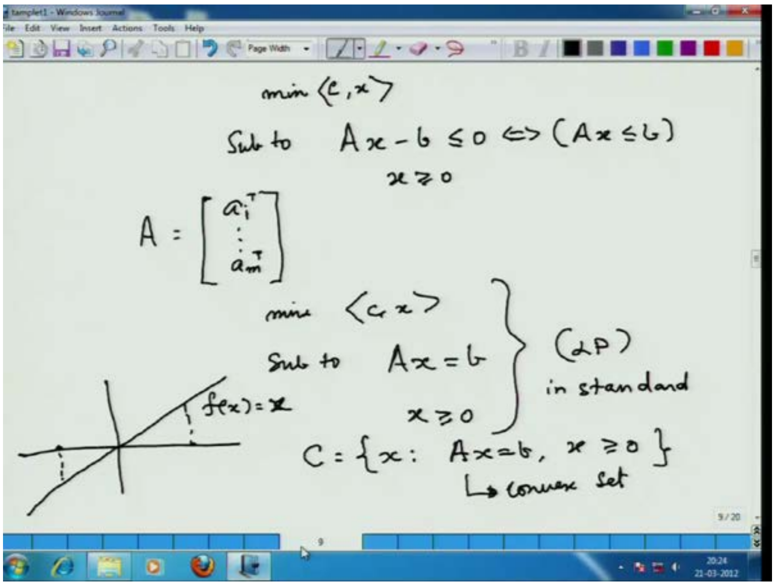

And you can write this as here when I am writing less than equal to 0, I am meaning that the component every component of this vector is less than equal to 0, and then I can also similarly write it like this or even equivalently write this as. So, what is this matrix a, this matrix a is a matrix whose rows are the vectors a 1 transpose dot a m transpose, this is actually the matrix a. Now, what you can do is you can actually add an additional slag variable, that is you can add an additional S and you can get a equality. So, the standard form of a linear programming problem is to minimize c x subject to, this is called the l p in standard form, and all linear programming book study this problem. We will have a scope to talk about this problem in detail, more from a convex optimization prospective. Of course, this is the sub class optimization.

The interesting part of this problem you see all these are nothing, but this is clubbing together of a class of affine functions and this is of course, minus each x i is a convex function, because minus x i itself is an affine function, linear function rather. So, this is you see, there are convex inequalities and affine equality. So, this visible set is a convex set. The set c which is in this case set of all x in R n such that, A x is equal to b and x is greater than equal to 0 component wise, then this set is a convex set. Now, this problem is interesting in the fact that this linear function has to be always minimized over a constraint set, you cannot minimize it over, just the whole R n, for example if you take the whole real line R, and you look at the constant function f x equal to x. These are linear function, but it does not have maximum nor has a minimum.

So, it is unbounded on the whole real line, unless you would say a constraint. Suppose, you restrict variables about this interval, then you know this is the minimum point, this is the maximum point, this point where minimum is achieved, this is the point, but maximum is achieved. So, the linear programming problem is essentially a convex problem which is a constraint problem, you have to have some restrictions like the set c. This problem linear programming problem was first studied during the time, and during that time it was a air force which had given certain problems to a team lead by George Danzig in the U S in the rand corporation, and they modeled those problems as linear programming problems, the type of problems that we had seen, so Dantzig was yet not sure what to call this problems, what name to be given to this problem.

So, one day he was walking with the famous economist T C Koopmans, and then he said you know all this problems have come out, they have all problems have to minimize near function under some linear or affine constants, and Koopmans said you see this is what you are doing, this is the program of air force and you are trying to solve their problems, you solved a part of their programs, so why do not you call as the linear programming.

So, it became this term became came to invoke linear programming which had, then translated into in general optimization finite dimension is also known as mathematical programming, which lately has been now called mathematical optimization.

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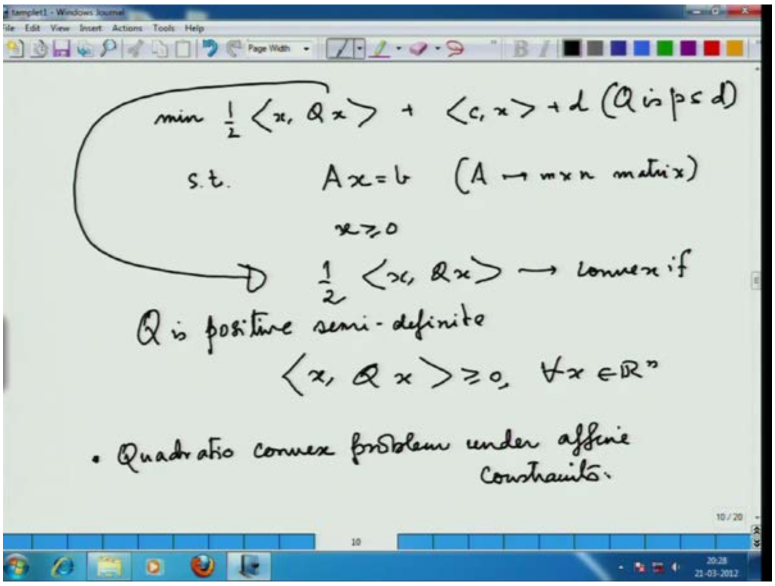

So, another important class of convex optimization problems is a class of quadratic optimization problem under affine constraints that is minimized. of course a is again just before like in the last one, this a you can easy understand this in m cross in matrix, when we were studying linear programming it is taken to be a full rank, but that is really not necessary when you discuss a theory. So, A is again in m cross m matrix and b is; obviously, R m which I do not have to tell you. Now, this problem is a Quadratic problem, you see there is a quadratic form here where this is the matrix defining the Quadratic from. This function, this part only, is a convex function that is if you take just this part. Then this is convex if Q is positive semi definite, positive semi definite matrices are true generalizations of non negative real numbers, that is if Q satisfies this condition and this is convex function.

Now, if you take two convex functions; f 1 and f 2, and if you add them, they remain to be a convex function. So, this is a convex function, because this point is affine function and you have added it to a convex function. So, this becomes convex. So, if you want to have a convex programming problem, then you have to assume that Q is positive semi definite p s d is the short form of our positive semi definite used everywhere in the world. So, this problem is called a convex Quadratic optimization problem under linear constraints. It is a very important class of problems, because for example, in the sequential Quadratic programming method, this problem is the class of sub problems that is solved, and these classes of problems are repeatedly solved. So, trying to solve this problem is a very important thing to do.

So, our problem is called Quadratic problem and affine constant or linear constants, Quadratic convex problem with the affine constants. Now, these problems where always studied in the 60s and early seventies, and convex optimization was people was thinking that was almost going to end. It was the time of eighties or nineties for the rise of non convex optimization with people talking about (( )) function and trying to handle them, but they soon realize possibly that it is not so easy game to handle on convexity. Then came in the horizon in a class of optimization problem which completely change the phase of optimization, and till date they remain it to be a thriving area of research, and it has again brought back convex optimization to the central and core or rather into heart of optimization theory and current optimization research.

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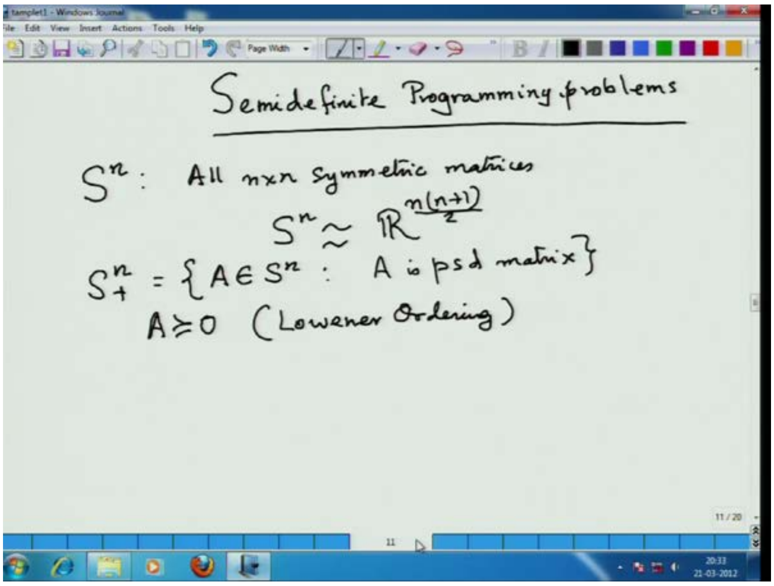

So, what I am going to now talk to you is some class of problems, called Semi definite Programming problems. On the phase of it when I write it down, it would look as if I have just copied the linear programming problem into a scenario, where my decision variables are no longer vectors, but matrices. So, we will start by considering the space S n of all n cross n symmetric matrices. Of course, you know of symmetric matrices of the one whose cross the elements A i j is equal to A j i. Now, if you take the class of all square matrix, n cross n square matrix. If you stack up, if you take the first column and then stack up the second column below, third column below the second column and so on, if you stack up all the n columns. So, you will get vector which is having n square components.

So, which means every matrix; n cross n matrix corresponds to some element in R n cross n. Now, this class where you have this matching between a i j and a j i, is actually isomorphic not to R n cross n, but to a sub space of that, which is R n into n plus 1 by 2. I will not take of the fun; I would let you try to figure this out. Now, S n plus is a set of all matrix A in S n. See, this is also finite dimension space, because it is isomorphic to this R n into n plus 1 by 2, this is actually the dimension of S n actually. So, A element of S n such that a is a positive Semi definite matrix is a p s d matrix. So, S n plus this set is the collection of positive semi definite matrices. So, if a is positive Semi definite, it is often written like this. This is called the lowered ordering. Now, let me go back again to this linear programming part.

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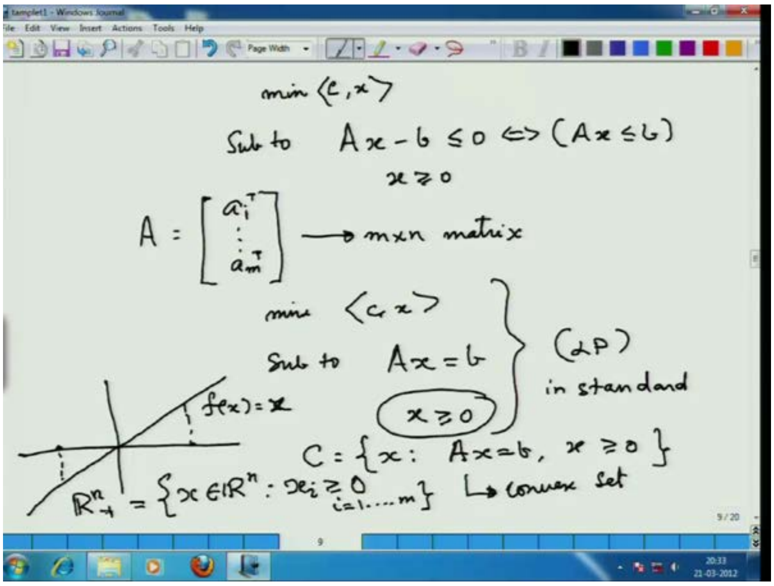

Now, when I am writing x greater than 0, what is the set it is representing. It is the set of all x in R n, such that all the corresponding components excise, each of the components is greater than equal to 0. This set is called R n plus, and you might observe, because positive Semi definite matrices are actually generalizing non negative real numbers. We have given the symbol S n n plus quite in harmony with R n plus. Now, once I know this, I am in a finite dimensional space you might ask me what is the inner product between two symmetric matrices.

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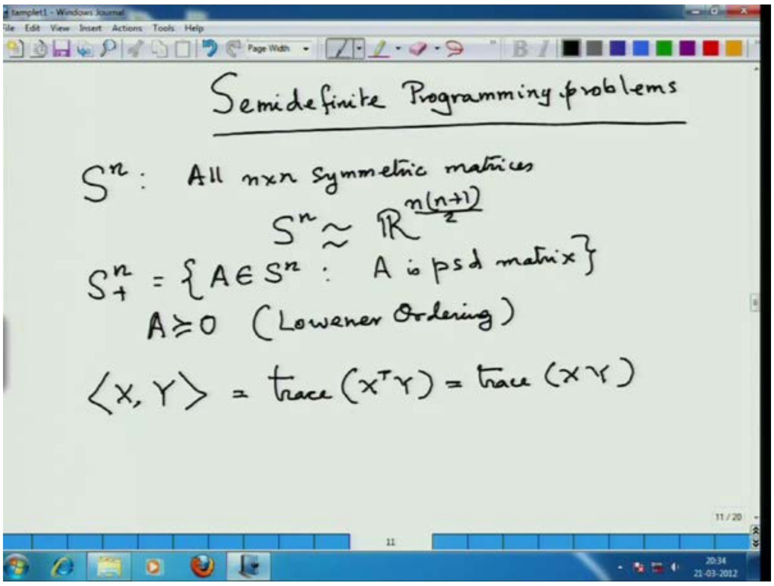

So, if I take a symmetric matrix x, another the symmetric matrix y, what is the inner product between two symmetric matrices. This is trace of x transpose y, but for symmetric matrix x transpose is equal to x. So, this is equal to trace of x y.

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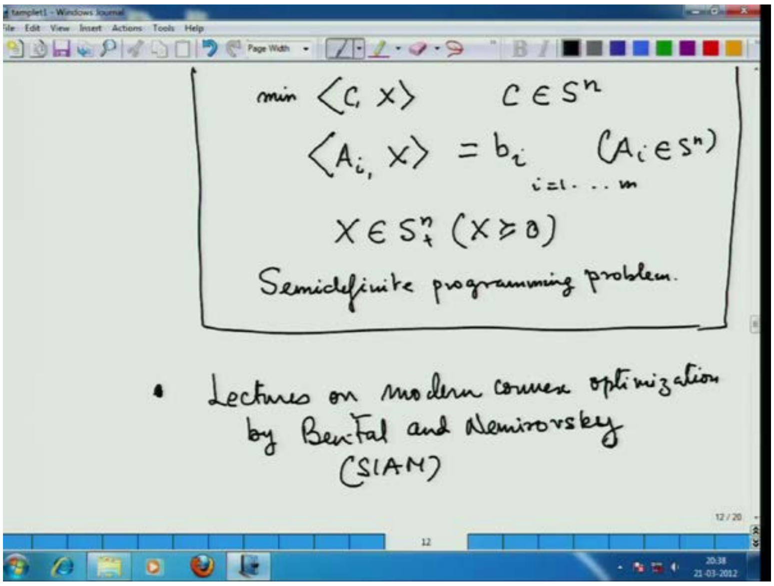

Now, what would I do, you will see I will write down a problem like this minimize C X, where C is; obviously, in S n and then take all the a is in S n and X is obviously in S n. So, all this a is are in S n and i is obviously from 1 to m. So, I am just (()) the linear program, and x is either I write it like this or in this lowered ordering form, just I have written x greater than 0 in the linear case I have written like this. I can write it like this or this whatever same thing. So, this problem is called a Semi definite programming problem, ultimately the decision variable has to be a p s d matrix Semi definite programming problem. You might ask the question what a big deal, we have just changed the space from the space of vectors we are in spaces matrixes. Is this is a linear program in the problem in the space of matrix, but symmetric matrixes.

But, the answer is no, in general a Semi definite programming problem is not a linear programming problem. Why it is not a linear programming problem is a question that we can only give and learn something more about convex sets that we will start doing tomorrow, and we will show that this is not a linear programming problem, but a convex programming problem in general. So, thus this class of problems cannot be handle by the methods of linear programming like simplest method, and it new set of methods has to be developed for them, and the Semi definite programming problems are now showing great power in solving a class of problems called polynomial optimization problems, which are actually very difficult n p R non convex optimization problems, and they are showing great power in very far in order to solve such problem, these things are showing great power in order to get approximate solution very fast.

The powers of these problems are very lately coming up and they are coming up in many applications, there are commercial software's now to solve this class of problems. So, it is very important for us that in this set of lectures in convex optimization, we will spend a little part with Semi definite programming problem. So, with this basic introduction about convex optimization, about the very basic facts, about convex functions minimize over convex set, that every local minimum has to be global, and telling you some three important classes of convex optimization problem; first is the linear programming problem, second is the quadratic optimization problem with linear constraints, and third is a very important modern class of convex optimization problem; the Semi definite programming problem.

So, if you want to know more about these classes of problems, I would suggest you a book called lectures on modern convex optimization by Bental and Nemirovsky, it is probably by SIAM society for industrial and applied mathematics. So, with this very basic introduction to convex optimization, that is what the title of the talk says today what is convex optimization. I stop here, but let me tell you this is the mathematicians point of view or lecturing, I have not given concrete examples, I cannot claim to be expert in each and every different discipline, but if you look into these books, this book in particular you will see the lot of important engineering applications in this book, which can be modeled as either linear problem or a quadratic problem or Semi definite programming problem. So, than you very much. Goodnight.