Linear Programming and its Extensions

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Lecture No. # 07

Representation Theorem LPP Solution is a BFS

Assignment 1

In the last lecture, I showed you that if there is a basic feasible solution, then it was also be an extreme point, and I did try to explain to you that the two concepts - the concept of the basic feasible solution was through algebraic relationships and the concept of an extreme point was in some sense, a geometrical concept, but the 2 or 1 in the same, and I did also show you that if a solution, an extreme point is d generate, that means it has less than m components positive. Then, there will be more than one basic feasible solution which will correspond to the same extreme point.

So, this is what we had done last time at the end of the lecture. Now, let me continue with that, and so, the another important thing which helps you to, which gives you a good insight and also shows you what simplifications can be achieved by considering the structure of the problem.

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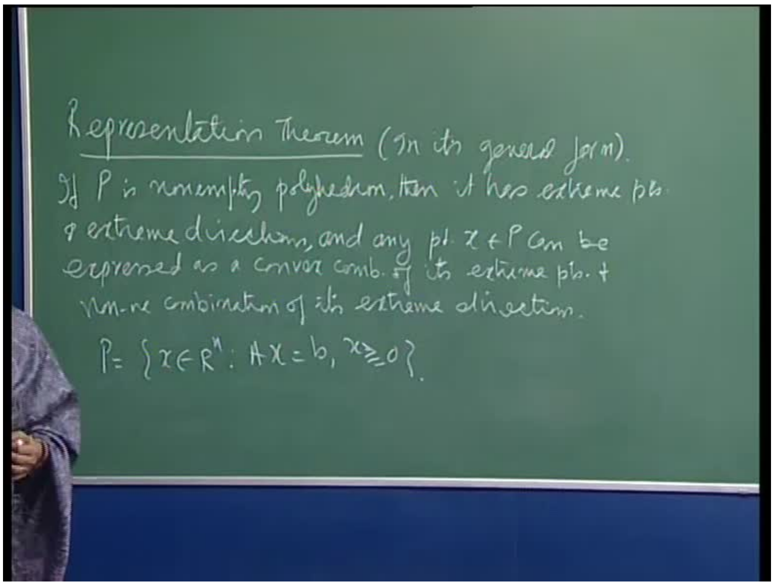

So, here, I will talk about the representation theorem and I will give the theorem first of all in its general form, in general form, in its general form. So, the idea here is that if essentially what we are saying is that, if you have a convex set, a polyhedral set, and the polyhedral set is always defined by a set of linear equations and in equalities, then we want to say that it will have extreme points, and then, any point in the set, also in the general form, when I talk about it a polyhedron may also have directions, and so, any point in the polyhedron can be expressed as a convex combination of its extreme points and non-negative combination of its extreme directions. This is what we want to say here.

So, if P is non-empty, so, I am always referring to a polyhedron by P. So, if P is non-empty polyhedron, then it has extreme points and extreme directions, and any point x belonging to P can be expressed as a convex combination of its extreme points and non-negative combination the issue of its extreme directions. I will not prove the theorem, because for a beginner, it is not very necessary to go through the rigorous proof, but the idea is very important and we will use keep using the results number of times.

So, essentially, let me just go through the theorem again. So, if P is nonempty, we showed you that P is the, say polyhedron. I am referring to the polyhedron P as x belonging to R n subject to A x equal to b x greater than or equal to 0.

So, set up equation, and in equations, they find your polyhedron, which is same as your feasible set or the linear programming problem, in the standard form that we have been discussing. We showed that if this is non-empty, then it must have a basic feasible solution. I showed you that if there is any feasible solution, then you can always reduce that to a basic feasible solution.

So, P non-empty implies there is a feasible solution, which means that there is a basic feasible solution, and from the result that I proved in the last lecture, that for a basic feasible solution is also an extreme point of the corresponding polyhedron. Therefore, there are extreme points here in P, and since number of basic feasible solutions is finite, therefore, the number of extreme points will also be finite. The number I give you a few lectures before this. I showed you there is an upper bound in the number of basic feasible solutions that a linear programming problem can have. Therefore, the number of extreme points is also finite.

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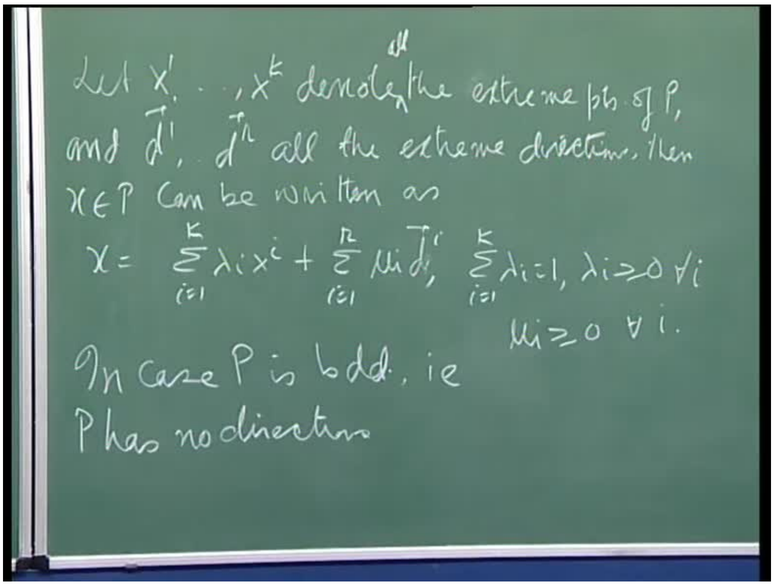

So then, what we are saying here is that, so, let X 1 to X k denote the extreme points of denote, if you want to the more specific, all the denote, all the extreme points of P and d 1 to d r all the extreme directions, then, then what we are saying is - then x belonging to P can be written as x is equal to summation lambda i X i i varying from 1 to k plus summation nu i d i i varying from 1 to R - where summation lambda i i varying from 1 to k is 1 lambda i is greater than 0 for all i and sigma, no, there is no sigma, mu i's are also greater than 0 for all i.

So, this is the general expression for… So, therefore, what essentially being said is that even though the feasible region or the polyhedron has infinity of points, feasible solutions we do not need to be concerned, because every feasible solution can be expressed as a convex combination of its extreme points and non-negative combination of its directions.

So, somehow, if we can see the behavior of the objective function at these points, that will give us a clue and that will follow, that theorem will follow after this one. So, this is the whole idea. Now, in case, in case P is bounded, that is, P has no directions. I showed you that if no directions, then your region will not extend to infinity, because having a direction means that you can go along, keep moving along that direction and stay in the feasible region.

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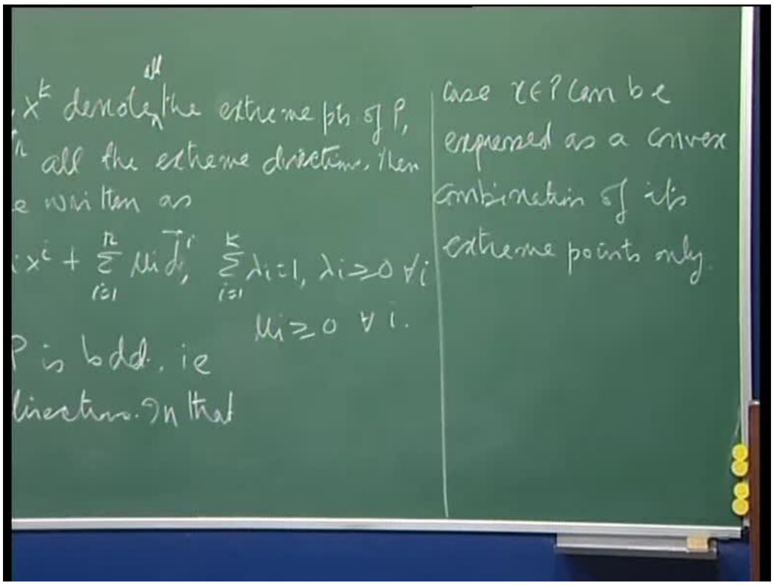

So, P bounded implies that the P has no directions. In this case, in that case, in that case, x belonging to P can be expressed as a convex combination of its extreme points. Since there is no directions present, therefore, I can write this as a convex combination of its… So, let us take up an example. So, the general form had the directions present in it, and if they are no directions present, if your set is bounded, then I just need the extreme points to express any point as a convex combination of these extreme points.

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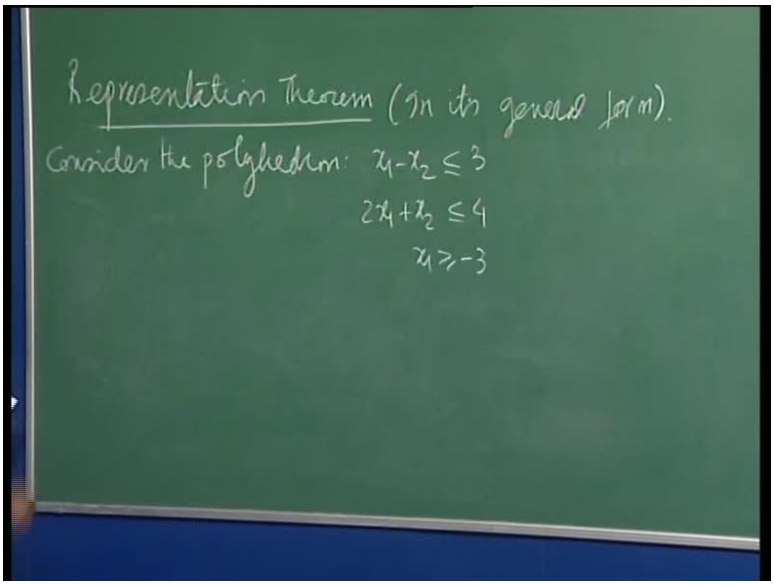

So, let me just show you and I will take the finite, the bounded case. So, suppose consider the, consider the polyhedron, consider the polyhedron x 1 minus x 2 less than or equal to 3 2x 1 plus x 2 less than or equal to 4 x 1 greater than or equal to minus 3. So, this is all, no non-negativity conditions and no equality equations to this succeed, just these set of inequalities.

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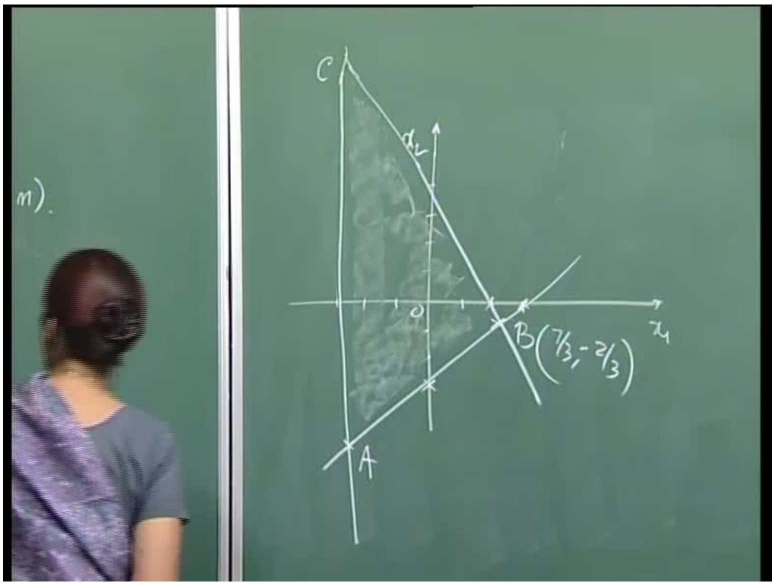

So, let us draw the… So, here, if you take this as a two axis - x 1 and x 2, the first one - when I write it as equation, this is x 1 minus x 2 equal to 3; at x 2 0, it passes through the 0.30, so, 1, 2 and 3. This is one point, and when x 1 is 0, it passes through the point 0 minus 3; so, this is one, this is one constraint and less than or equal to 3. So, this side, because 0 is on this side, so, this is the part. Then, this one gives you, x 2 0 gives you x 1 equal to 2. So, this is 2 and x 1 equal to 0 gives you x 2 4; so, this is 1, 2, 3, 4. So, this is either 1 and again 0 is on this side. So, this is the portion. Then, x 1 greater than or equal to minus 3; so, I have 1, 2, 3.

So, if you can draw this as a straight line, then, so, this will meet at somewhere here. This is considering at the point of intersect. So, this is your polyhedron. So, let me call. So, this was your o. Let me call this point as A and this point as B and this is C. So, three extreme points this is a bounded polyhedron.

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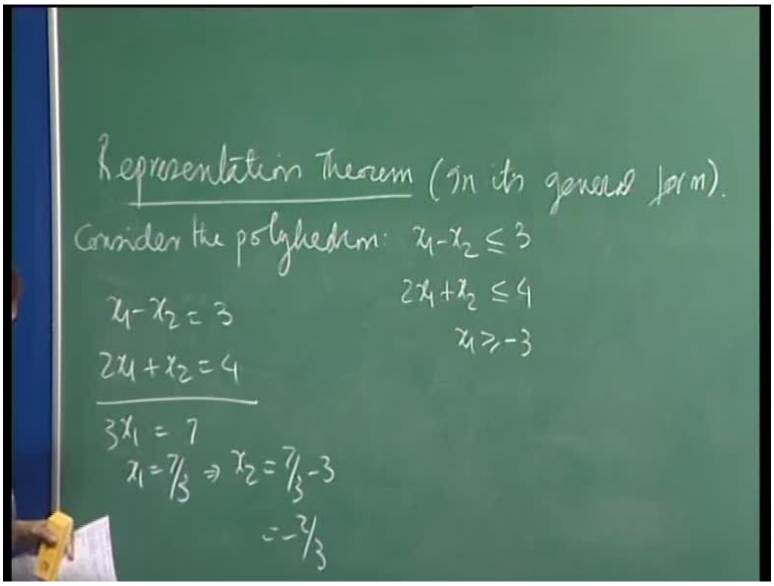

And quickly if you want to find out the points of intersection x 1 minus x 2 equal to 3 2x 1 plus x 2 equal to 4, this gives you 3x 1 equal to 7; so, x 1 is equal to 7 by 3, which implies that x 2 is equal to 7 by 3 minus 3. From here, x 2 is equal to x 1 minus 3 9, so, which is equal to minus 2 by 3.

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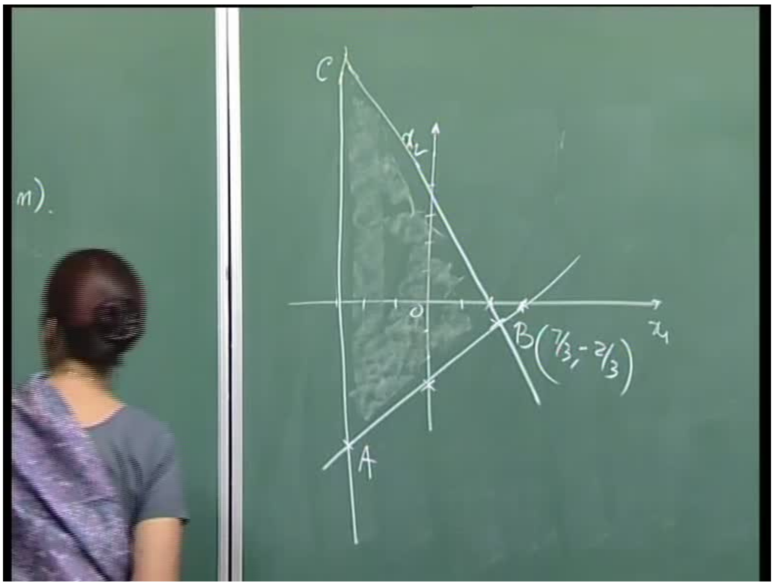

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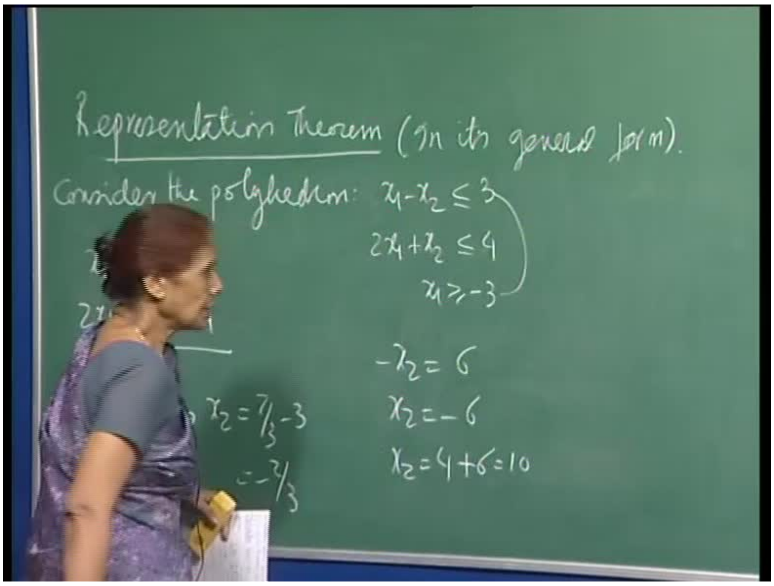

So, the point is 7 by 3. This is the point 7 by 3. So, this is 7 by 3 and minus 2 by 3. Point B is given by the intersection of these two. Then, if you want to look at this two, so, x 1 equal to minus 3 will give you what? You take it here. So, minus x 2 equal to 6, which means x 2 is equal to minus 6. So, the point A is, point A is minus 3 minus 6, and finally, if you substitute it here, that will give you x 2 equal to 4 plus 6 which is 10.

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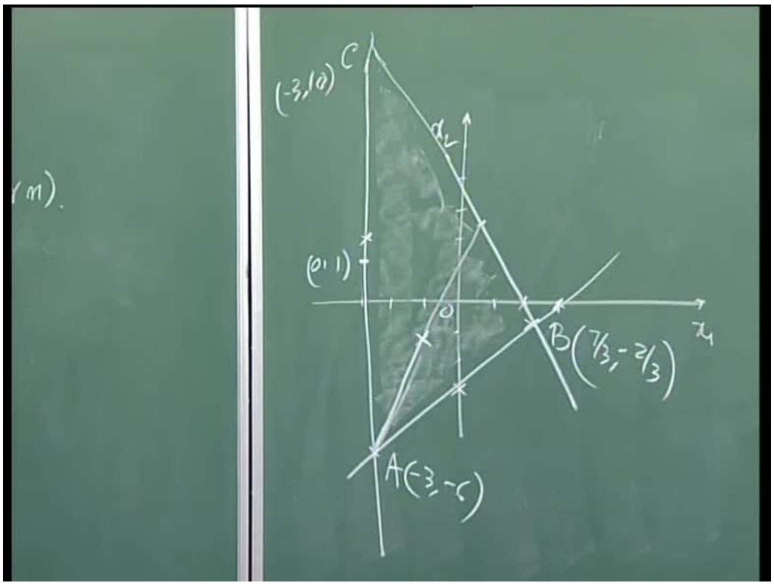

So, the, this point, point C is minus 3 and 10. So, that gives you the extreme point. Now, of course, I must point out that, for a small example, I am able to construct find out all the extreme points, but essentially, the representation term does not require you to find all the extreme points. What I am going to demonstrate is that you can have this representation, and therefore, you can simplify lot of things, but as if you had to find all the extreme points, then there will be no fun, because it could require much more effort; there may be actually solving the problem.

So, therefore, here I am just trying to show you what we mean by this representation theorem. So, I am saying that if you, for example, take a point here, then you know that it is on the join of these two points, say it will be some convex combination of A and C, and I can find out the correspond. So, if I am given this point, I can write it as a, I can find my lambda such that lambda times this plus 1 minus lambda times this is this point. If you have any point here, for example, then what do you do?

See, the thing would be, there can be many ways, we need not be… What I am showing you is the only way, so, you can have this example. You can join this to one of the extreme points; extend the line to meet it here and then you see that. So, this point can be written as a convex combination of these two. Then, this itself can be written as a convex combination of B and C. So, the final thing when you write this point, that will be a convex combination of A, B and C. So, let me demonstrate this to you, and I will take up this example, say consider the point 0 1.

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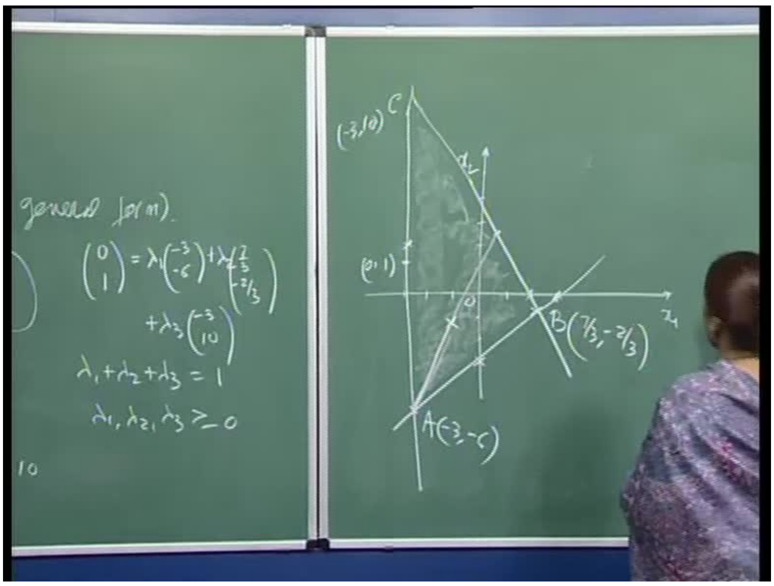

So, I am taking the point 0 1. So, let me say that it is here, this point is 0 1 and how do I… So, there can be many ways of doing it. So, here, 0 1 let me write it as lambda 1 times minus 3 minus 6 plus lambda 2 times 7 by 3 minus 2 by 3 plus lambda 3 times minus 3 and 10. I write this and I say that lambda 1 plus lambda 2 plus lambda 3 is equal to 1 lambda 1 lambda 2 lambda 3 all greater than or equal to 0.

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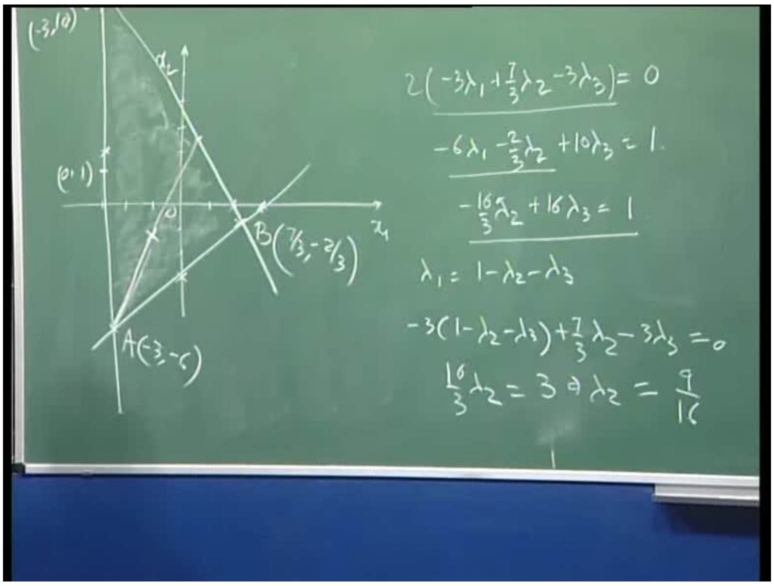

So, all I have to do is demonstrate to you that lambda 1 lambda 2 lambda 3 exist and then you have the point 1 0 1 as a convex combination of this type. So, here, what are three equations that you get? The first one is - minus 3 lambda 1 plus 7 by 3 lambda 2 and minus 3 lambda 3 is equal to 0 component wise, and the second component gives you minus 6 lambda 1 minus 2 by 3 lambda 2 plus 10 lambda 3 is equal to 1 and you add this.

So, three equations, three unknowns and you should be able to find them. Find the three unknowns lambda 1, lambda 2, lambda 3. So, maybe, we can spend just a little bit more time. So, if you multiply this by 2 and then subtract, so, this lambda 1 will cancel out, and so, you will have a 14 by 3 and I am subtracting. So, my 14 by 3 minus 2 by 3 minus 16 by 3 lambda 2, then this will be minus 6; so, plus 16 lambda 3 is equal to 2, is equal to 1, sorry.

I have taken twice this and then subtracted, so this, because they are the right hand number is 0. So then, I can find out. So, here, so, one can go and doing it. See, here lambda 2 lambda 3 I have, and also from this one, I can use from this one here. The numbers are not very good; therefore, it is wondering if…

So, the other one would be if I substitute for lambda 1, let me do it here. Let me do it in this one. I substitute for lambda. So, lambda 1 is equal to 1 minus lambda 2 minus lambda 3 from the, this one. So, let me substitute here in this equation; so, that would be minus 3 times 1 minus lambda 2 minus lambda 3 plus 7 by 3 lambda 2 minus 3 lambda 3 is 0.

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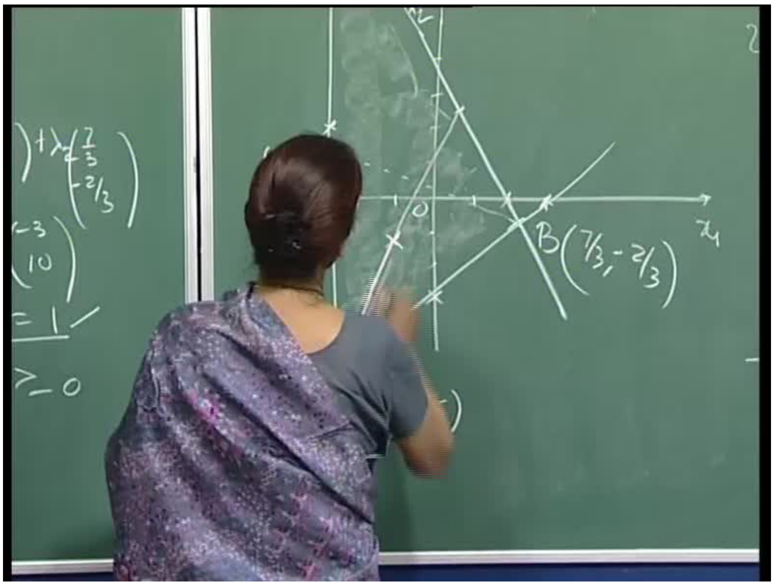

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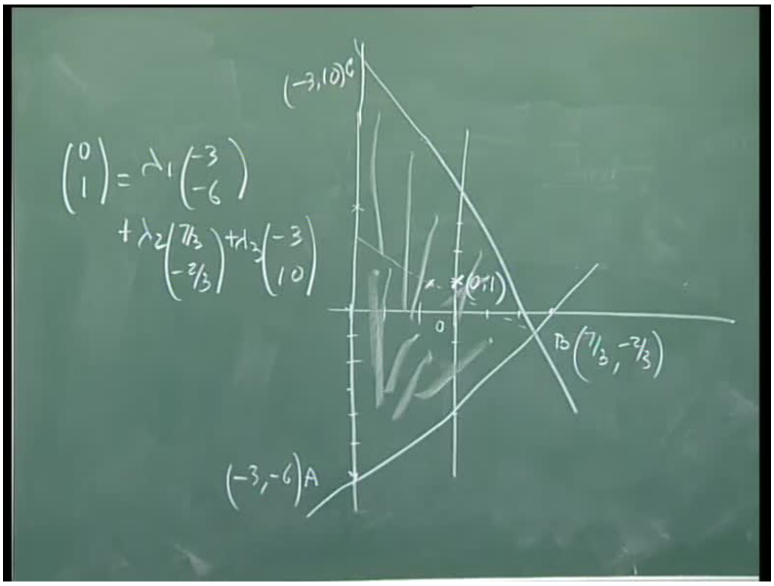

So, now, simplify, that will give you plus. See, this is plus 3 lambda 2 and this is 7 by 3, so, 9 16 by 3 lambda 2, then this is plus 3 lambda 3 and o. So, this simplifies, so, plus lambda 3 minus 3 lambda 3 is 0. So, you are left with this is equal to 3, good. Therefore, I immediately get this implies lambda 2 is 9 by 60. So, once I have lambda 2, then from here, I can get lambda 3, and once I have lambda 1 lambda 2 and lambda 3, then I get lambda 1; so, this is what you have.

The other method that I will demonstrating to you would be, see, for example, do this. I will just to revisit the polyhedron. I gave you the equations, the inequalities that define this polyhedron earlier and I was trying to show you. So, these are three extreme points. You have feasible region or set of the points that, that the constitute a polyhedron is the shaded portion, and I was trying to show you that how you can express any point in the polyhedron as a convex combination of these three points, because they are, the polyhedron is bounded; there are no directions present, so, only three extreme points. So, I said that if you took a point on this line joining A and C, then you see this can be expressed as a convex combination of A and C. If I take any point inside the polyhedron, then I can join it to one of the extreme points and then extend the line to meet one of the in equations. One of the equations defining the polyhedron, and this point will be a convex combination of these two, and since this itself can be written as a convex combination of A and C.

So, the, finally, this point will be a convex combination of A, B and C. Now, well, I was trying to show you how we will represent the point 0 1. I took it to be in here. So, actually the point is this one. So, the point is 0 1, and so, whatever calculations I did, after that, all are ok. I just, because I took the point 0 1 here. Therefore, I said that your lambda 2, because see, we try to express 0 1 as lambda 1 times minus 3 minus 6 plus lambda 2 times 7 by 3 minus 2 by 3 plus lambda 3 times minus 3 and 10.

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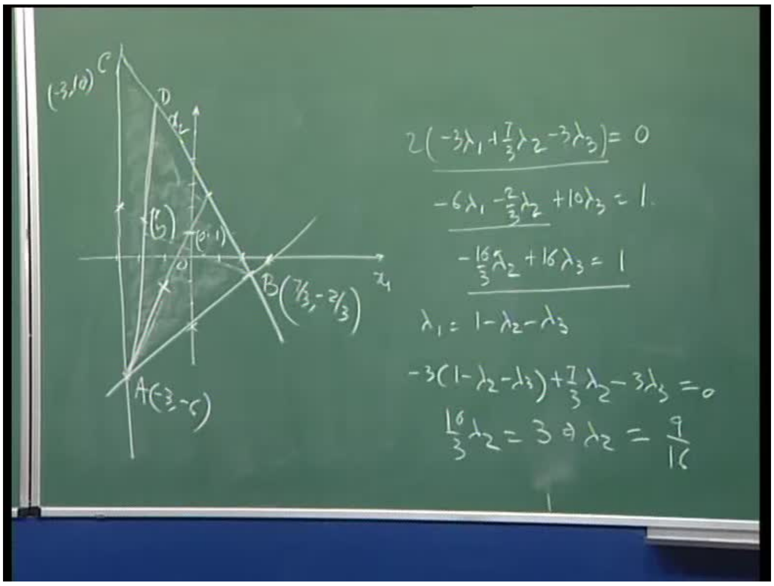

And so, since I had taken the point here, I made the remark that lambda 2 must be 0, but actually 0 1 is a point which is inside the polyhedron, and therefore, lambda 2, lambda 2 came out to be non-zero. So, the calculations were ok. It just that, so, whatever I said, afterward all follows expect that I took the position of the point 0 1 is not correctly, and therefore, with the correct position of 0 1, the calculations are ok. As some lambda times the point minus 3 minus 6 plus 1 minus lambda times the point D.

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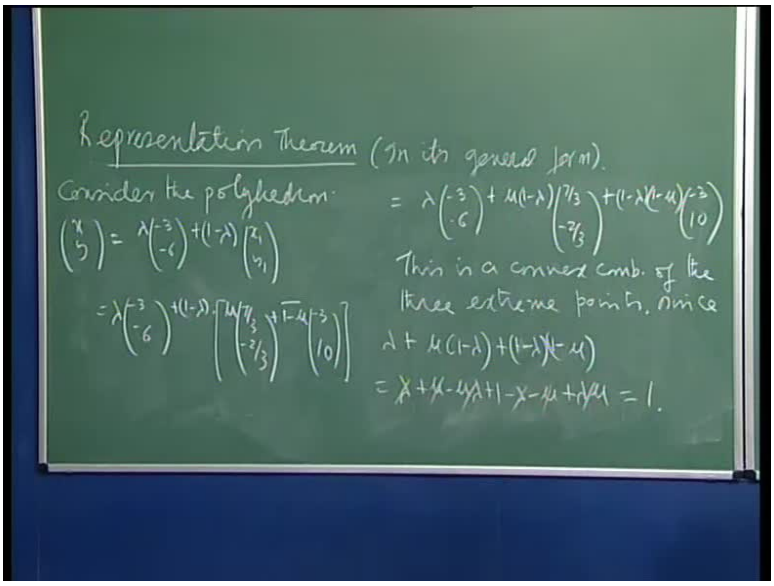

The point D which you can say the coordinates are may be, let us say x 1, y 1 the point D. Then x 1, y 1 I can write as lambda minus 3 minus 6 plus 1 minus lambda times. Now, the x 1, y 1 itself I can write as a convex combination of B and C. So, I will say that this is mu times, mu times, this is 7 by 3 minus 2 by 3 plus 1 minus mu times minus 3 and 10. Now, does it give me a convex combination of three points? Yes, because you can show that the coefficients are… So, here, lambda nu is non-negative. Otherwise, I am taking lambda 1 minus lambda n. So, you are, as, actually you are writing the point as lambda times minus 3 and minus 6 plus mu times 1 minus lambda, then 7 by 3 and minus 2 by 3 plus and this is 1 minus lambda 1 minus mu, so, 1 minus lambda 1 minus mu minus 3 10.

Now, it is, is it a convex combination of a three points? Yes, because all the components are non-negative; the coefficients are non-negative, and you see that this is a convex combination of the three extreme points, since lambda plus mu 1 minus lambda plus 1 minus lambda 1 minus mu. If you open up the expressions, write this is the lambda plus mu minus mu lambda here and you have plus 1 minus lambda minus mu plus lambda. That you see the lambda cancels with lambda; mu cancels with minus mu; minus mu lambda this; so, this is equal to 1. So, this is a convex combination. Since this is what I want to show you.

So, now, the next results that I will give you will show you the actual simplification in the presentation of the linear programming problem, and hence, the algorithm, very efficient algorithm can be efficient in the practical sense. I am not talking in terms of practical, in terms of technical sense, because there is a proper definition when you say algorithm is efficient.

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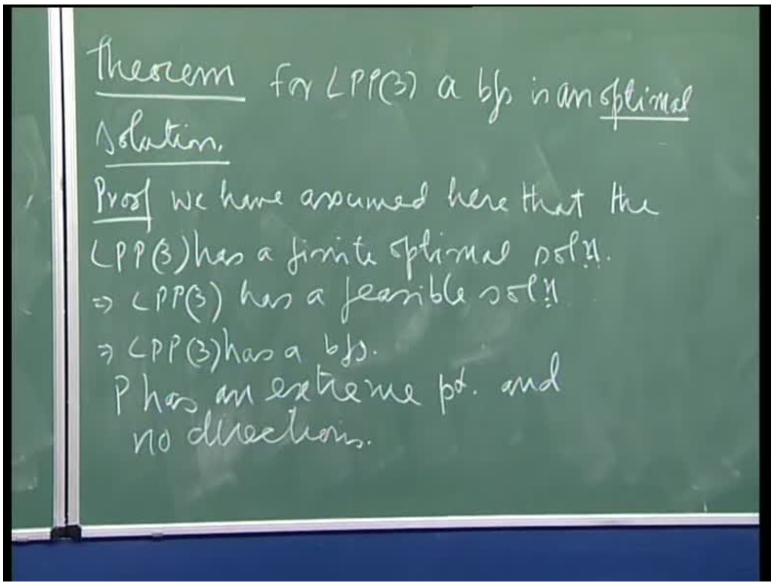

So, I do not mean that way, but certainly, there is lot of simplification. So, the calculation here needs to be checked and I will give you the corrected calculation next time. So, resulted we are going to talk about now is that the, so, this is a theorem for LPP 3 a basic feasible solution is an optimal solution, is an optimal solution. This is the whole idea behind it that. Now, I can say that I do not have to worry with the above with the whole feasible set which has infinity of feasible solutions. If I can prove this result, then it means that I just have to check out a finite number of basic feasible solutions and pick up the one which is the best.

So, but here, LPP basic feasible solution is an optimal solution. I am assuming here the assumption is that the LPP 3 has an optimal solution. So, I will state that first proof. So, we have assumed here that, that the LPP 3 has a finite optimal solution, which I mean to say that the optimal solution exists possible that the optimal solution may not exist. So, I am at working at the assumption right now, and then, we will come back and show you that assumption is not really needed.

So, we have assume here that the LPP 3 has a finite optimal solution. This implies that LPP 3 has a feasible solution. If there is an optimal solution, obviously, that has to be first a feasible solution. If a feasible solution with this implies that LPP 3 has a basic feasible solution. Remember, I showed you quite some time ago that you can always reduce a feasible solution to with, and now, with our result that I proved in a last lecture, the corresponding PP has extreme point. So, P has an extreme point, and so, let us assume that it has a finite number of extreme points and we will use our representation theorem now.

So, existence of a finite optimal solution implies that a basic feasible solution exists, and therefore, extreme point exist, and since there are no, since it has a finite optimal solution, I can also assume that there are no directions in P, why? Because the efficient optimal solution is finite, so, it is clear to me that even if the region is not bounded, I can bounded, because my optimal solution is finite.

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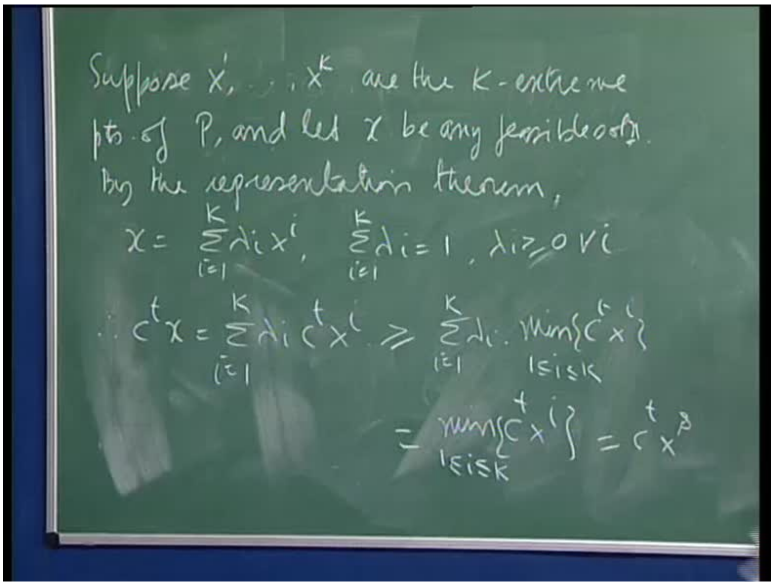

So, I am not disturb with the reason is unbounded and I will talk about it later on again. So, P has an extreme point. Therefore, now, suppose and maybe I add here. P has an extreme point and no directions; n has no directions. See, suppose X 1 to X K are the K extreme points of P.

So, here, P is the polyhedron corresponding to the feasible set for LPP 3 points of the… Then, so these are the extreme points. Therefore, by the representation theorem, suppose this and let x be any feasible solution by the representation theorem, x can be written as summation lambda i X i i varying from 1 to K summation lambda i equal to 1 i varying from 1 to K lambda i greater than or equal to 0 for all i by the representation theorem, which we just talked about and you have this. So, therefore, c transpose x is equal to summation lambda i c transpose X i i varying from 1 to K.

Now, remember, lambda i is add up to 1, and so, here, this you can say is therefore greater than or equal to sigma lambda i i varying from 1 to K times minimum of C transpose X i 1 less than or equal to i less than or equal to K. See, if I replace each C transpose X i by the minimum, so, therefore I have this K numbers; so, I have this K values C transpose X i i varying from 1 to K. I have this K numbers; I choose a smallest one. Whichever, so, I call it the minimum of all the C transpose X i. Then, if I replace each C transpose X i by the smallest 1, then since lambda i's are non-negative, I have this inequality and all these lambda i's will get added up together, this is equal to 1 by the condition. Therefore, this is this reduces to minimum of C transpose X i 1 less than or equal to i less than or equal to K and suppose the minima occurs for which is C transpose X sum s, for sum s.

So, therefore, for any feasible solution, what I have shown you? That the value of the objective function will be greater than or equal to the value of the objective function at a point, at one of the extreme points which corresponds to, which corresponds to the extreme points for which this value smallest. So, out of these K extreme points, I have computed the value of the objective function at these X points. Chosen the minimum one, and suppose that corresponds to C transpose X s, then every feasible solution will have its objective function value higher than the value C transpose X s, of course, there can be more than one, extreme points for which this minima occurs.

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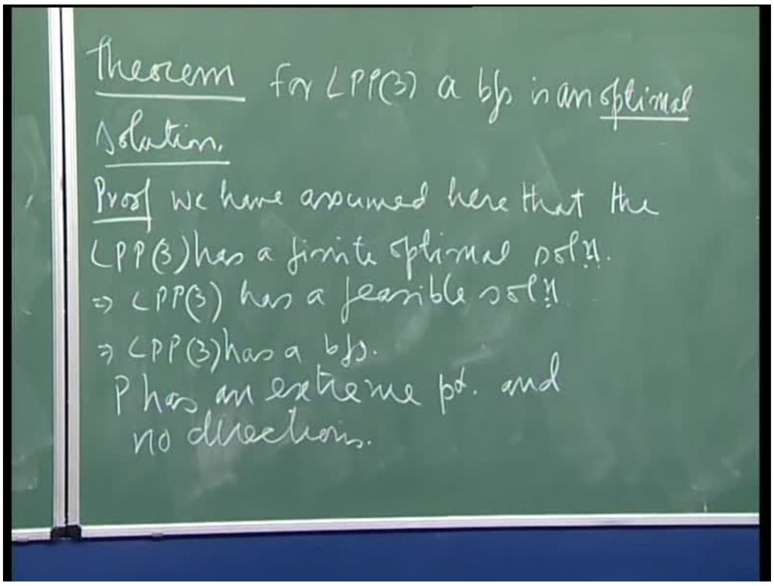

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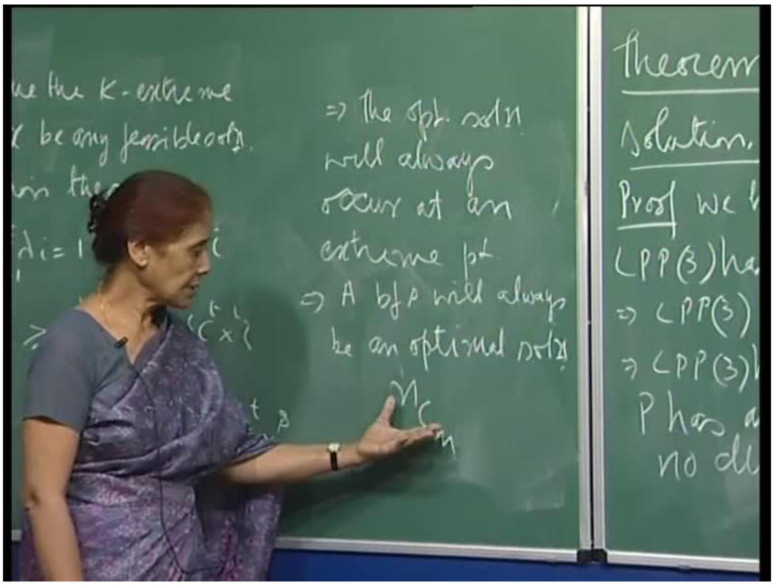

So, that shows that this must always, that means there is always be an extreme point, which will correspond to this smallest value of the objective function, of course, with the assumption that there is a finite optimal solution to the standard linear programming problem, and so, for this implies that the optimal solution will always occur at an extreme point.

So, I am not denied here that there may be many other points at which the optimal solution will occur but what I want you to show is that an extreme, the, it will always occur at an extreme points, which implies that a basic feasible solution will always be an optimal solution. Now, let me write, wording it for a basic feasible solution always be an optimal solution.

So, essentially, what it boils down to is that our search for an optimal solution can only be now contained to finding basic feasible solution, because after I have shown you that this a basic feasible, an extreme point it is also a basic feasible solution, then if optimal solution occurs that in extreme point; that means it occurs that a basic feasible solution, which means that I do I only have to search among the finite basic feasible solutions and find the optimal one, but again, this does not tell you that you can do away with regular algorithm, because this number of basic feasible solutions is an exponential one; it can be very high. Remember, the upper bound that I gave you was n c m. So, with n large and m also recently large, then this number can be very big.

And so, the quested of finding out all the extreme points of the all basic feasible solutions and then computing the value of the objective function will not be very efficient way of solving this linear programming problem.

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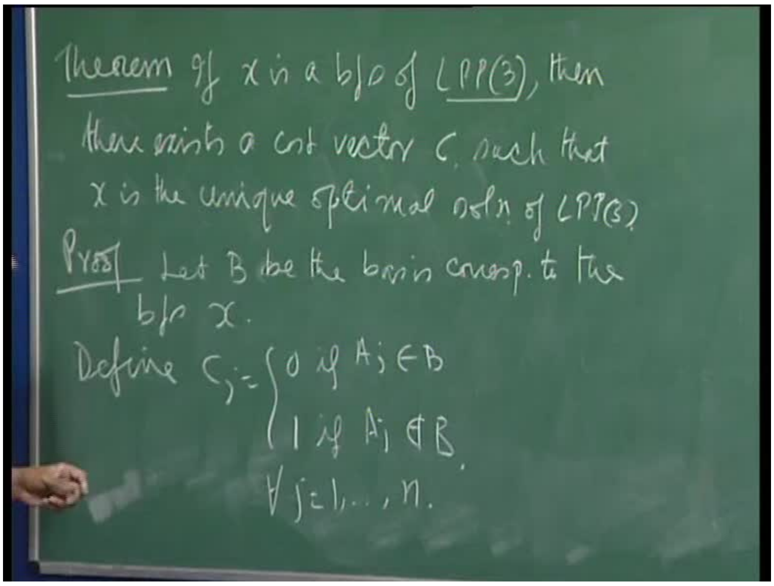

So, we have to judiciously search among the basic feasible solutions and find the best one. Now, see, in the last result, I showed you that they will always be a basic feasible solution which is an optimal solution for a linear programming problem. Provided the linear programming problem has it finite optimal solution. I will like to show you may be as a matter of theoretical interest or the result will be used at some other times, but now, I want to show you that its, see, if x is a basic feasible solution of, again let me keep referring to a linear programming problem in the standard form.

So, if x is a basic feasible solution of LPP 3, then there exists a cost vector c such that x is the unique optimal solution of LPP 3. So, I hope this significance is clear here. Earlier I showed you given a linear programming problem, there will always be a basic feasible solution which is an optimal solution. Now, here, I am saying that if you give me a basic feasible solution of a linear programming problem, I can find a cost vector. If I say LPP 3, so, here I am saying that C is not given.

So, now, I will specify the c. So, given a basic feasible solution, I will give you a C so that this basic feasible solution is a unique optimal solution of the corresponding linear programming problem. So, let me define. So, I just have to give you, so, let B be the basis corresponding to the basic feasible solution x. Define C j as 0 if A j belongs to B and 1 if A j does not belongs to b. So, this is a clear definition of my vector C j of the vector C, the component, this j th component is 0. If A j is in B and if A j is not in B, then the value of C j is 1. So, if I define this, define C j like this for all j 1 to n, then you see what is the value of….

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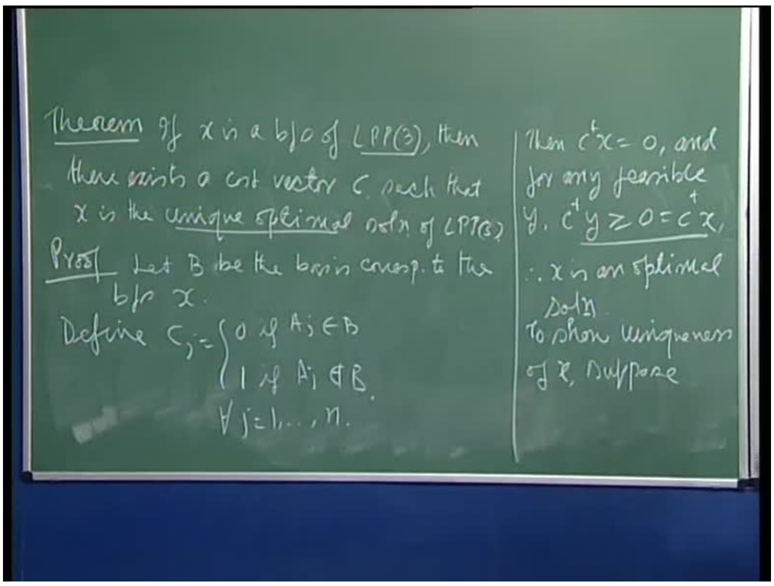

So then, c transpose x is 0, yes, because B is the basis corresponding to the basic feasible solution x. So, for all the C j's corresponding to A j in B are 0; that means here, when x is positive and x is a basic variable, the corresponding c is 0; C j is 0, and when x is non-basic, then x j is 0. Hence, the corresponding C j is 1.

So, this value, then c transpose x is 0, and for any feasible y, for any feasible y, c transpose y greater than or equal to 0, which is c transpose x, because any other feasible solution will have at least one component different from x and so the corresponding C j is 1. So, for any feasible solution, the value of c transpose y will be or it could be 0. So, we are saying that c transpose y is non-negative, but c transpose x is 0; so, that means for any feasible solution, c transpose y will always be greater than or equal to c transpose x. Therefore, x is an optimal solution.

Now, we want to show that it is a unique optimal solution. So, to show uniqueness of x, suppose. So, that means there is no other I want to show you that if there is any other solution - feasible solution - which has the value of the objective function is 0, then it must coincide with x.

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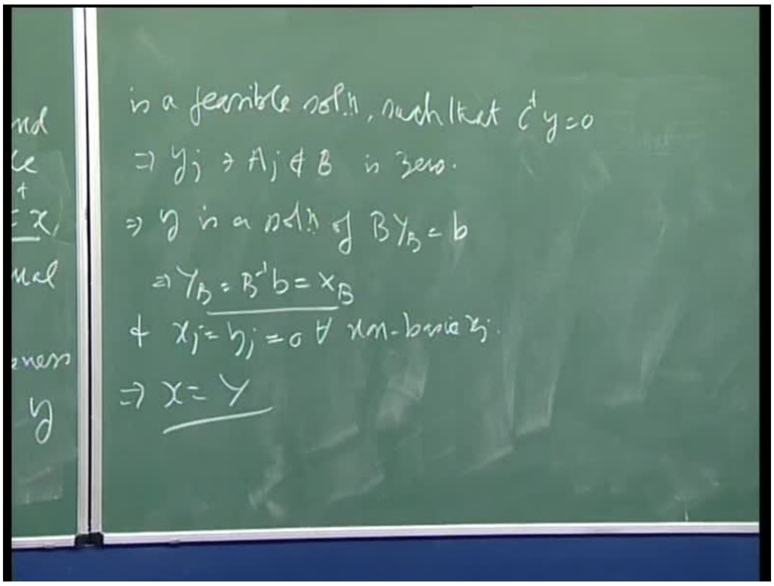

So, to show uniqueness of x, suppose y is a feasible solution, such that c transpose y 0, because the optimal value is 0. So, therefore, if there is any other feasible solution which is also optimal, then the objective function value must be 0. So, c transpose y is 0, which implies that y j such that A j does not belong to B, y j is 0, because you see the value of the objective function is the cost coefficients are such that this is 1 if A j does not belong to B.

So, if any feasible solution has the objective function value as 0, then the y, the y j value corresponding to the A j such that A j is not in B has to be 0. Otherwise, I will add corresponding one here. So, the value of the objective function will not be 0. Remember, your y's are all non-negative. So, and if you are c's are also non-negative, the, all the objective function value will always be non-negative. You will say that that it is 0. I must have that y j such that a j does not belong to is 0, which implies that y is a solution of BY B equal to 0, because for all j said that A j is not in B, the corresponding y j is 0.

So, the system A j y, the system A y reduces to this - where Y B are the components corresponding to the A j’s which are in B. So, B y b is 0, sorry, this is equal to B, which implies that Y B must be B inverse b and this is equal to your X B. Here, again, I am referring to X B is the part of the basic variables, because X is a basic feasible solution. Therefore, this is and x j is equal to y j is 0 for all non-basic, non-basic x j.

See, the value of the objective function is 0 and we already said in the beginning that the y j must be 0 n so that the corresponding column is not in B. So, the non-basic variables anyway agree the non-basic values agree, and now, the basic values also agree, which implies that x is equal to Y. So, that shows the uniqueness of the solution. Therefore, I can always construct the cost vector such that any basic feasible solution will be the unique optimal solution corresponding to that.

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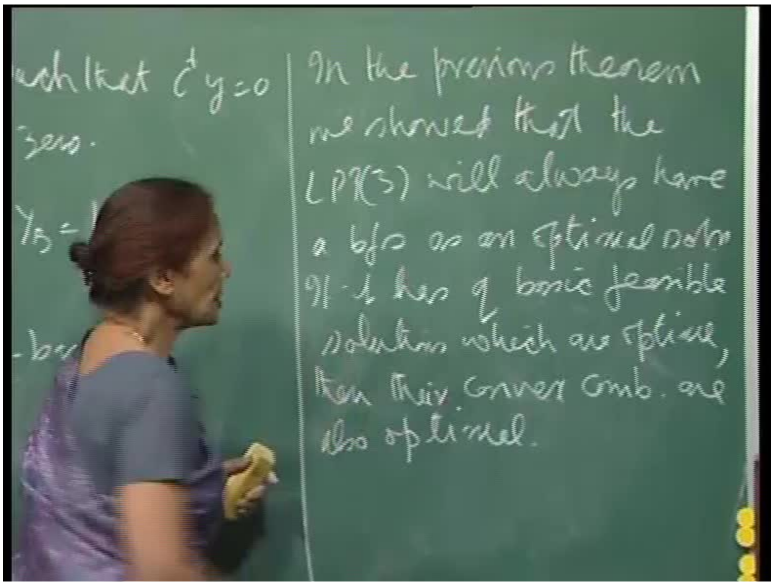

Now, in the previous theorem, let me just revisit. In the previous theorem, we showed that the LPP 3 will always have, will always have a basic feasible solution as an optimal solution. Now, I show that add it to is another part and, that is, now, if it has q basic feasible solutions, basic feasible solution which are optimal, which are optimal, then their convex combinations, t h e i r, then their convex combination are also optimal. So, in that case are also optimal.

See, I was reminded by the words, because here we have shown uniqueness. So, what we are saying now here is that if in case the basic feasible solution is not a unique optimal solution, then there will be more than one. So, if you have q basic feasible solutions which are optimal, then their convex combinations will also be optimal; that means they will infinity of optimal solutions. So, either you have a unique optimal solution or you have infinity of them.

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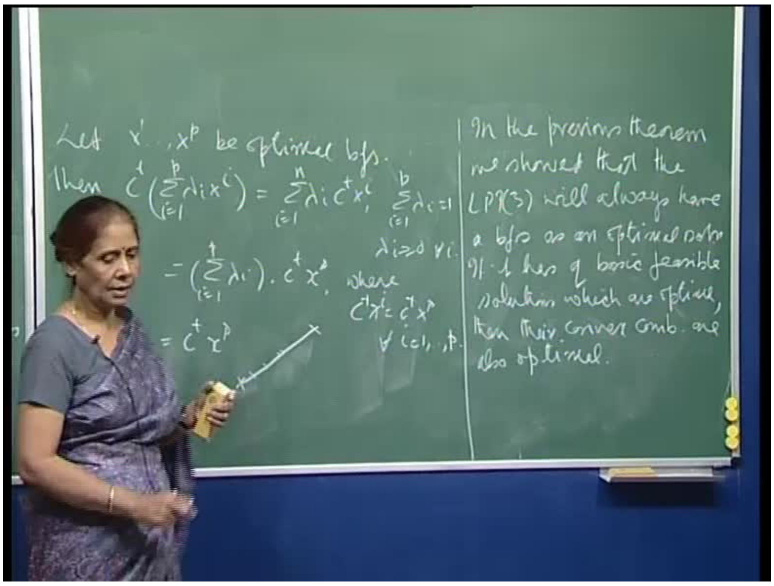

And this part is not difficult to prove. Let me do it here, it here. So, suppose, so, let us x 1 to x P be optimal basic feasible solutions. Then, C transpose summation lambda i x i i varying from 1 to P is equal to sigma lambda i C transpose x i i varying from 1 to P, where your sigma lambda i i varying from 1 to P is 1 and lambda i greater than 0 for all I, but since all are optimal; they are equal.

So, this implies this is equal to summation lambda i i is (( )) 1 to P times c transpose x s, whatever it is, because all are equal. So, where C transpose x i is c transpose x s for all i 1 to P. Now, this is equal to 1. So, this is c transpose x s. So, we know that because it is a feasible region is convex, so, this is a feasible solution and we are showing that the value of the objective function at this convex combination of the optimal a basic feasible solutions is also give you the same objective function value, and therefore, this is also optimal.

So, that means if you have, essentially what we are saying? If you have two points, two basic feasible solutions and so corresponding to extreme points, if they are optimal, then any point, every point on the line joining the two optimal basic feasible solutions is also optimal.

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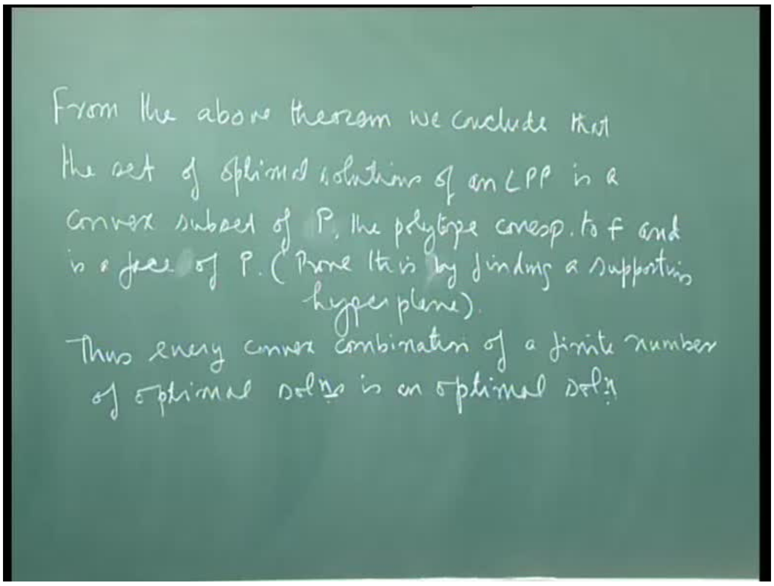

From the above theorem, we conclude that the set of optimal solutions of an LPP is a convex subset of P, the polytope which corresponds to the feasible region F for the LPP that we have been considering so far which is in its standard form and it is a face of p.

Now, this I would like you to prove by finding a supporting hyper plane just that the exercise that I did obtaining faces for or supporting hyper planes for different faces of a polytope. I want you to here find out the supporting hyper plane and show that the collection of all optimal solutions of a linear programming problem is a face of the polytope P, which corresponds to the feasible region, and thus, therefore, we can then concluded every convex combination of a finite number of optimal solutions is an optimal solution.

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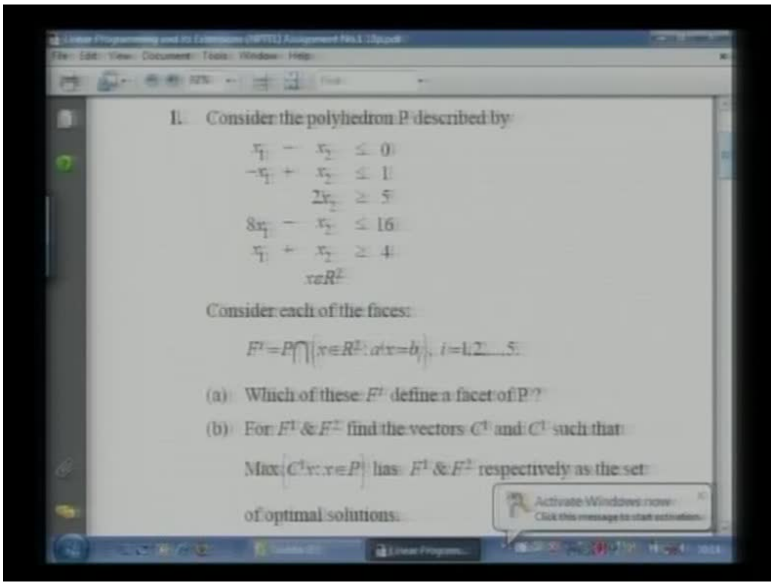

So, let me just consider I show you the, I have collected a few assignment problems, which it would be worthwhile for you to go through and I will give you some comments here go for each problem so that it will help you to work them out yourself. So, in the first, the first problem is the, is consider the polyhedron P described by certain in equalities of both kinds less or greater. Then I am saying that consider each of the faces F i, which is given by making one of the in equality constraint as inequality constraint.

So, you have five faces, because they are five constraints; so, you have five faces, and then, I am saying which of these F i define a facet; that means you have to now tell me that if you take one of the in equations as inequality, then are there other inequalities also satisfied as equality or not. So, you have to, because remember the dimension of facet is one less than the dimension of the polyhedron.

So, here, the polyhedron is of dimension 3. So, you have to show me that which are the facets, which are of dimension one less than the dimension of P, and then, I am saying that for F 1 and F 2; that means F 1 corresponds to the first in equality being taken as equality and F 2 corresponds to the second in equality being taken as equality. So, find the vector C 1 and C 2 such that max. Now, here it is a maximization change from minimization so that the maximum has F 1 and F 2 respectively as the set of optimal solutions.

So, you have to find now I have given you this construction here. So, please try it will be very interesting if you can really construct your C 1 and C 2 such that all points in F 1 are maxima points are the object function with cost vector C 1 and all points on F 2 are maxima points for the cost vector C 2.

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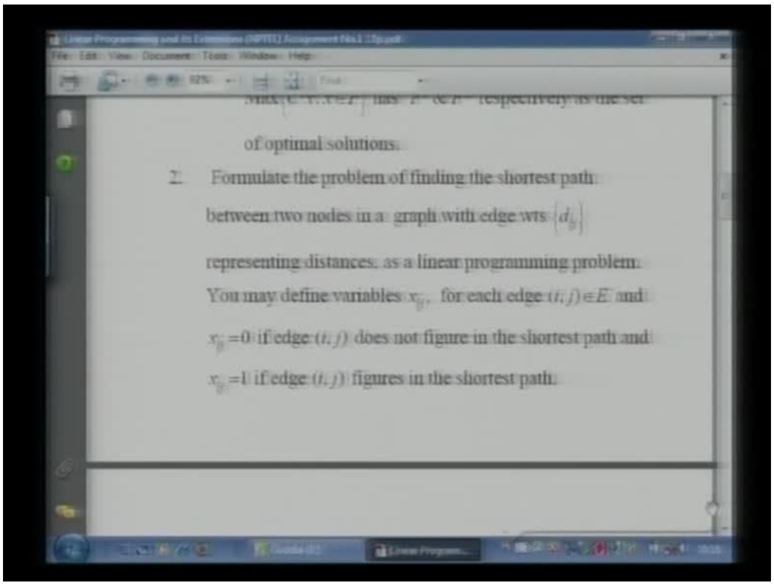

So, now, let us go to problem 2 - formulate the problem of finding the shortest path between two notes in a graph with edge weights d i j; so, that means the if you join the node i and j, traversing that edge, the weight or the cost of that edge is d i j or the distance representing distances as a linear programming problem, you may define variables x i j for belong each edge i j. We are saying that the capital E represents the collection of all the edges and i and j of or i refers to a node. So, you have a graph here in which you have these nodes and your joints some of the node not all are joint.

So, i, j within brackets denotes in h; so, e is the collection of all the edges. So, we say that x i j is 0 if edge i j does not figure in the shortest path and x i j is one. If edge i j figures in the shortest path, so, this will be, because we have to compute the shortest path; that means you take a note. Now, you might say that I have not given you the nodes from which the path has to be found.

So, just fix a node. Let say fix node 1 and node n as the two nodes with between which you have to find a path and said that when you add up the weights or the distances of the edges, they must be the smallest; that means there be many path from 1 to n, and you find the path for which this weight is shortest. This is the problem, and what you have to take here is that you see at every node, if path is entering a node, then it must also leave that node. So, you have these conservation constraints.

So, you have objective function would be summation d i j x i j over all i j in E, the, because x i j will be 1, then you will add the weight; if x i j is 0, you will not add the weight. So, your objective function is fine. Now, your constraint and the variables are x i j equal to 0 or 1. So, here, again I choose this problem, because I want to show you that it is not enough to always describe a problem by saying that your decision variables are non-negative. You might also have here discreteness; that means you only allow x i j to take the value 0 or 1, and then, the node conservation constraints are necessary here to formulate the problem correctly and you have to say that summation x i j over j; that means from i you are going to j node; so, there may be many such nodes. So, all this summation must be equal to the edges which are leaving the node; so, that means the minus this thing here.

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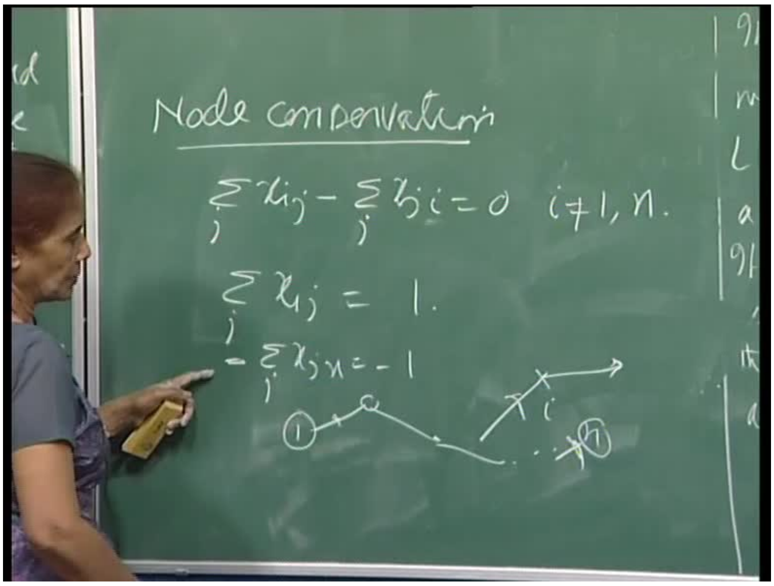

So, node conservation constraint I am saying, node conservation. This is summation x i j j. So, here, I will not say anything, because whatever edges are nodes are being joined by this minus summation x j i again summation over j here or from this is sigma x i j minus these are going out and these are coming in. So, this should be 0; this should be 0. When i is not equal to 1 or n that means I am saying you have to determine in the path from 1 to n.

So, what would be the node conservation for? So, here, it could be x 1 j summation over j. This you want as 1, because we have a node i here then you have arrow like this and you have arrow like this. So, it depends sometimes some people this as a minus 1 or this as a minus 1 does not matter, that will not be important, because ones you stick to one convention then it is ok. So, when you take the node 1, then you are saying that all, because I am wanting to find a path from 1, and here, you have n here. So, from node 1, you only want the arrow to go out. You are (( )) the path edge to go out, nothing should come in, because you wanted to them in the path from 1 to n.

So, therefore, here I will say sigma x 1 j is 1, only one edge will be chosen for the shortest path, and similarly, here, when you write this, this is summation x j n summation over j; this will be minus 1, because when you are saying that the sign for the edge entering, so, here also for the sign for the edge entering, so, x j i is minus right. So, here, minus x j n; this is minus 1. So, set for these two; otherwise, for node conservation, you are going to have zeros.

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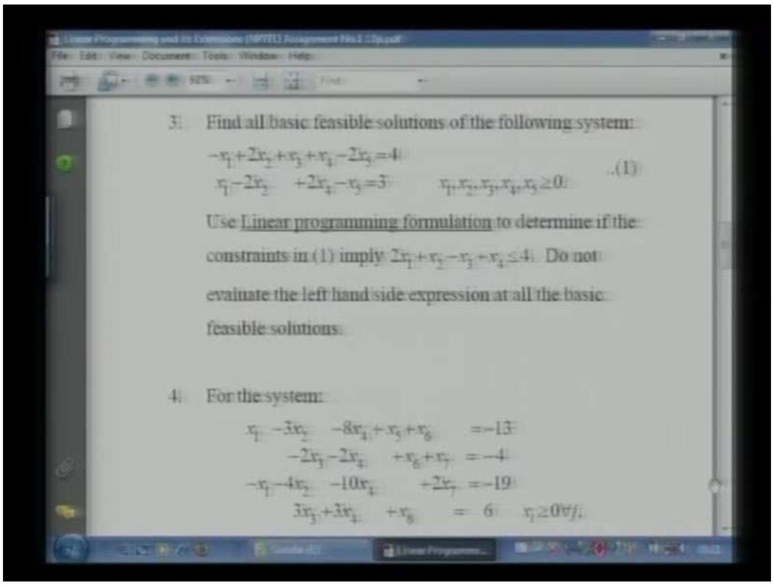

So, now, I come to problem 3. I have just ask you to find all feasible solutions of the following system all the basic feasible solutions. So, remember, you have to, because though two constraints, so, pair two columns at a time and find out which of them give you a basic feasible solution. Then, for problem 4, ok, fine.

So, again, I have given you a system of linear equations and I have given you a feasible solution 1 2 1 1 0 0. I am saying is it a basic feasible solution, and you remember that because four variables are positive here; that means if the, if the corresponding columns are linearly independent, it will be a basic feasible solution. If the corresponding columns are linearly dependent, it will not be a basic feasible solution. So, I have asked you to construct the basic feasible solution from it.

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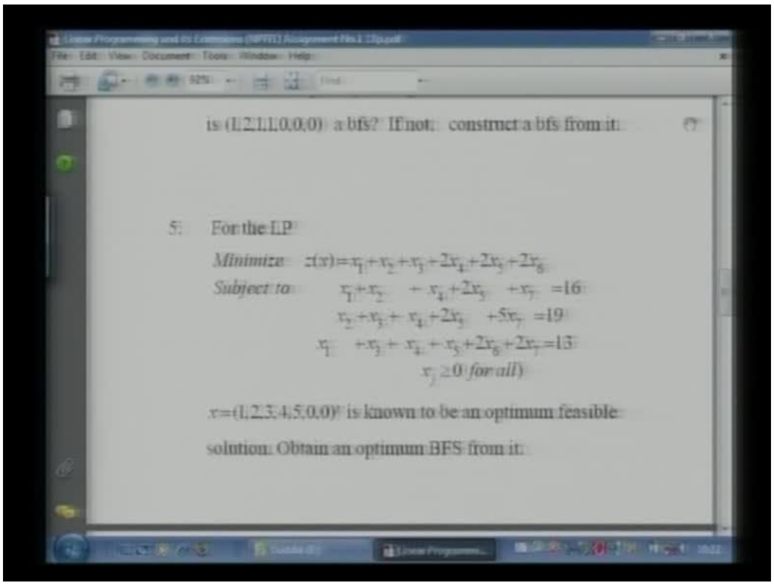

Problem 5 - I have given you a solution which is known to be an optimal. So, there is a linear programming problem. I have given you a solution which is known to be an optimal feasible solution, but which is not a basic feasible solution.

So, here, again, because I have discussed enough constructions how you move from a basic feasible solution to another also improving the value of the objective function. Now, here, there is a twist here. I do not want you to improve the value of objective function, I want you to maintain the value of the objective function the same, because the given solution is optimal. So, whatever the value of the objective function here, so, you have to go through the proof in which I showed you how to move from one basic feasible solution to another while improving the value. So, just go through that proof and you will be able to construct. Find out how you should move over. I have given you note here also.

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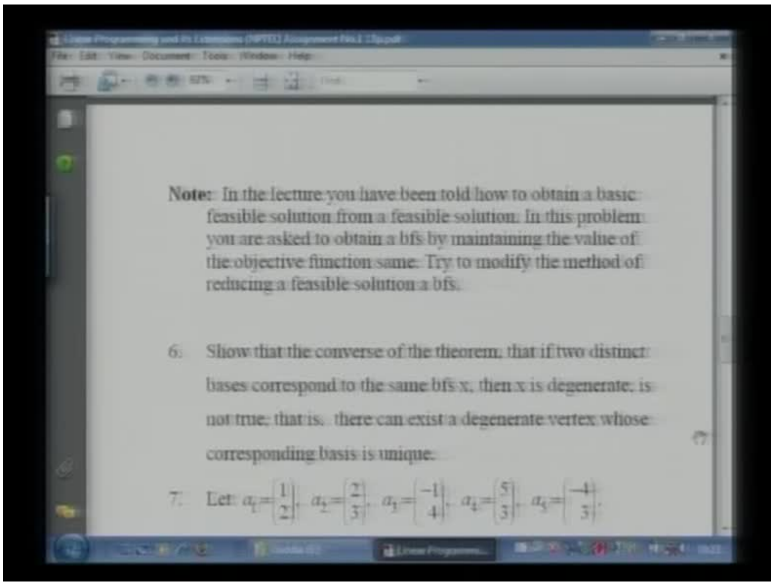

Now, in the 6th problem, show that the converse of the theorem that if two distinct bases correspond to the same basic feasible solution x, then x is degenerate, is not true the converse, that is, there can exist a degenerate vertex whose corresponding basis is unique. So, you can construct a system of linear equations and then try to show me that how a degenerate vertex may correspond to only one basis.

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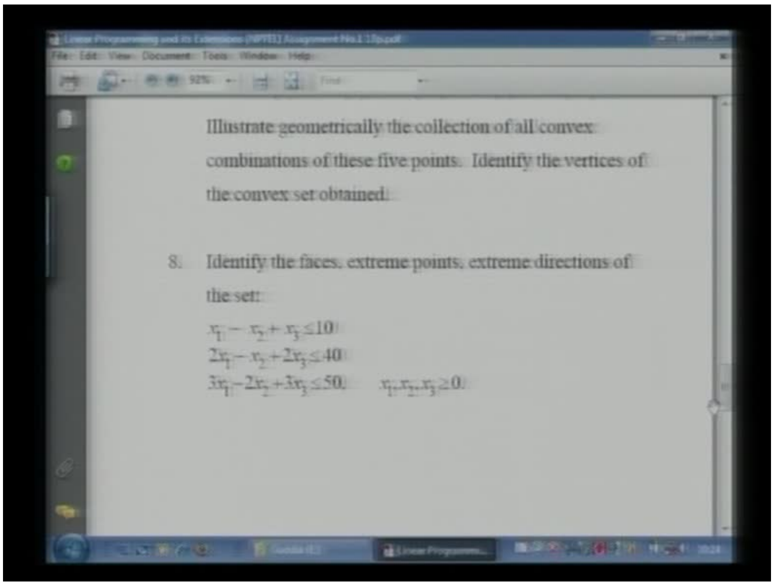

In the 7th problem, I had given you five vectors and I am asking you construct the convex hull. Illustrate geometrically the collection of all convex combinations of these five points. Identify the vertices of the convex set obtained. Then, 8 and 9 again, I had given you two polyhedrons and I am asking you to identify the basis extreme points, extreme directions of this. So, I hope you enjoy going through this as I mentioned.