Linear Programming and its Extensions

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Lecture No. # 05

Convex sets Dimension of a Polyhedron Faces.

Example of a Polytope

So, I will continue with my discussion of Mathematical Concepts, that we need to build up, the, this is a simplex algorithm, and let me, I had in the last lecture, I had defined, I had defined in linear independents dependents and then concept of a basis and so on.

So, let me just go back to one of the things, which I sort of left out and we can come back to it. So, this is augmentation theorem; so, the idea here is that, if you have a subset of linearly independent vectors which do not form a basis, then I can always add some more vectors to it and get a basis formed, that means, if the number of linearly independent vectors is less than the dimension of this vector space, then I can always add some more vectors to it, so that it becomes a basis.

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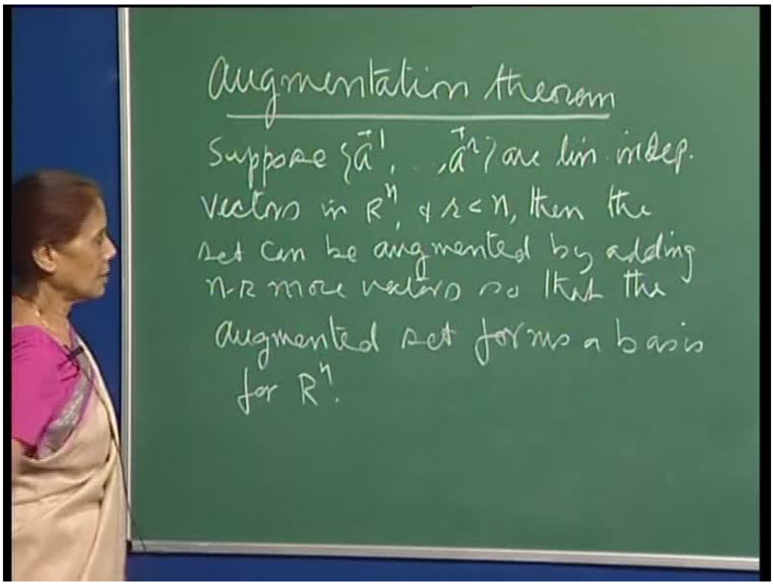

This is the idea of augmentation theorem and what it says, is that, suppose a 1 to a r are linearly independent linearly independent vectors in R n and r is less than n, then the set can be augmented by adding n minus R n minus r more vectors.

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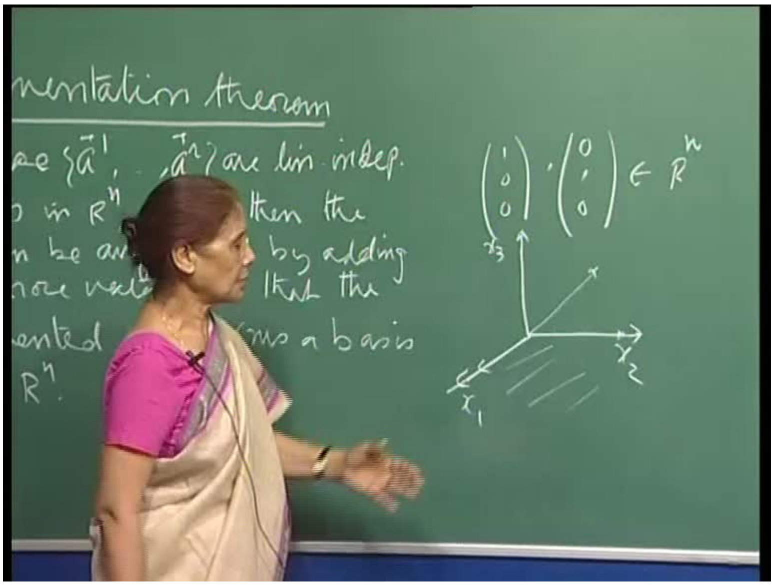

So that, the augmented set, augmented set, forms a basis for, basis for, R n; so, this is the whole idea, I will just give the idea behind it without actually giving you the proof. So, what is being said, is that, suppose you take two vectors, let me take the vector 1 0 0 and 0 10 to take a simple example belonging to R n; we know that, these, these vectors are linearly independent, because if you take a linear combination then and it is equal to 0, then immediately see that your c 1 will be 0 and c 2 will be 0; if I take c 1 times, this plus c 2 times, this vector equal to 0 vector and it will always imply that c 1 and c 2 both have to be 0. So, therefore, this is a linearly independent set, I can augment it and what is the idea behind it the idea here, is that, you see, these two vectors, if you take this, see let me take three-dimensional things, so this is x 1or x 2 and x 3; so, the three axis and here 1 0 0 1, so 1 0 0 is this, is the unit vector in this direction, 0 1 0 is the unit vector in this direction. So, actually any linear combination of these two vectors represents your x x 1 x 2 plane fine all over.

So, if I take any vector, any vector which is not in the plane, then certainly it cannot be expressed as a linear combination of these two vectors, figure it out, it is very simple; so, if I take a vector, what does it meant to have a vector which is not in this plane, that means, a vector which is not in this plane will have its third component non zero, that means, any vector of that kind 0 0 x 3 this also belongs to R 3 and 0 0 x 3, I am sorry, this should be 3, these both belongs to R 3.

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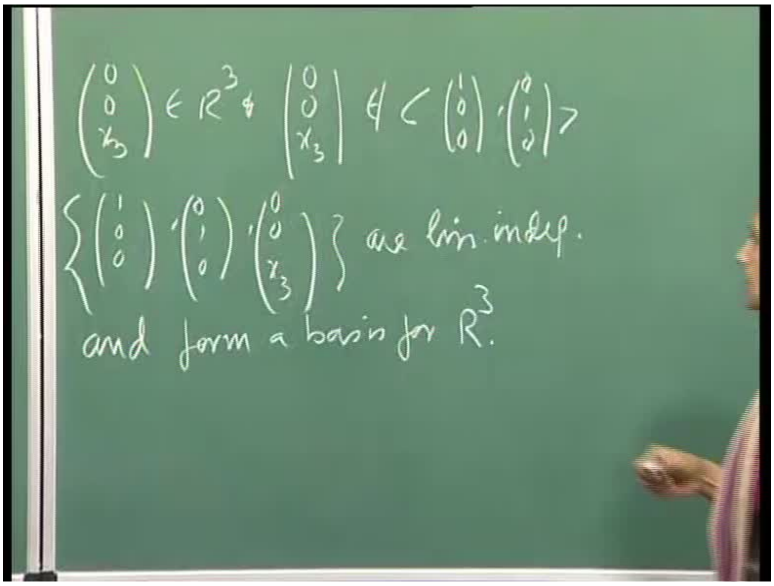

So, this does not belong to, does not belong to the linear combination of the vectors 1 0 0 and 0 1 0, so I have shown you this; so, if this is how one get go on augmenting a set, that means, if you have a set of linearly independent vectors, find a vector and since r is less than n, obviously this cannot be a basis, that means, it cannot be spanning set for R n.

So, therefore, there will be vectors, which are not in the linear combination of these all vectors; therefore, I can find 1 add it to this set, that will be, that will remain linearly independent, again if r plus 1 is still less than n, I can find a vector which is outside linear combination of the augmented set and add that vector to it; so, that process move on goes on like this and the proof requires lot of time; therefore, we assume, the since we need the result, I am just giving it to you. So, to say that linearly independent subset, which does not form a basis for R n will always, it will be always possible to add vectors to it and a formal basis out of it.

So, therefore, you see that, this is this and so what I am saying is that, 1 0 0, 0 1 0 and 0 0 x 3 are linearly independent and since there are three, there is a form and forms a basis and form a basis for R 3; so, this is the idea behind augmentation theorem and later on we will show you, how we can extend basis for our degenerate basic visible solution or we can demonstrate, that they can be more than one basis correspond, but in more than one basis corresponding to one degenerate basic visible solution and so on.

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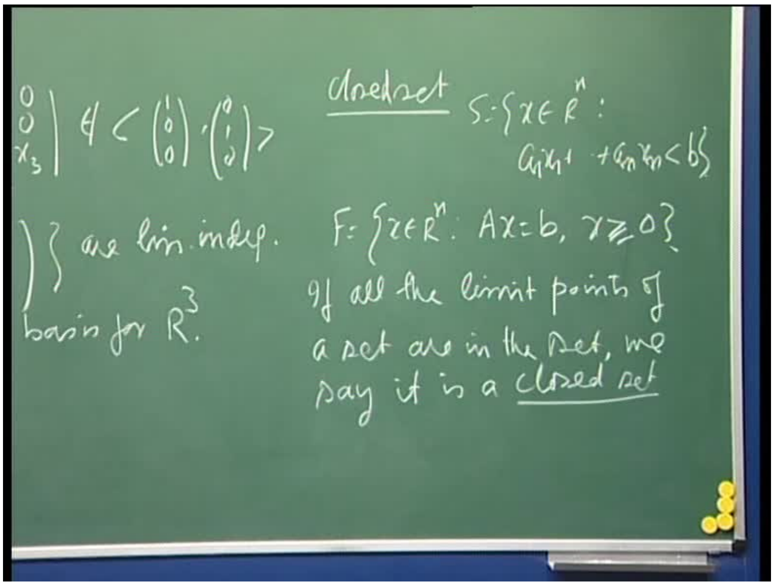

So, this was your augmentation theorem; let us get to you some more concepts here; so I want to just quickly revise, because you have already, say for example, what do we say a close set, now they can be a closed set. What we mean by closed set, is that, all its limit points or in the set or we can say that, the you can also define the boundary of a set and you can say that, if the boundary is all in the set, then it is a closed set, whatever it is say for example, a very easy way of say if I have defined, if I define a set by saying that, s is equal to let say x belonging to R n, such that some a 1 x 1 plus a n x n is strictly less than b; suppose, I have split inequality, then you can see that, this is not a closed set its a open set, because the bound, that means, I can take a sequence here in its possible that the sequential converge to b, when I take this summation here add it up little converged b and so that, limit point of that sequence will not be in the set, because I am taking only all those points in s, for which this is satisfied is strict inequality.

So, in other words, when you, when you take the set, when you take the feasible region, feasible region for the linear programming problem, we say x belonging to R n such that A x is equal to B x greater than or equal to 0; so, we have equality sign you have n equations here and you have n inequality equality zeros are also allowed value will be available can take the value 0.

So, this will be a closed set, because I have equality, I am allowing equality constraints also, whereas here I am not, so I am not writing down rigorous definition for a closed set, but we can say the definition of a closed set would be, if all the limit point so f a set are in the set, we say it is a closed set, we say it is a closed set and the, and if all the limit points are not in the set, then it will be a opened set, a bounded set.

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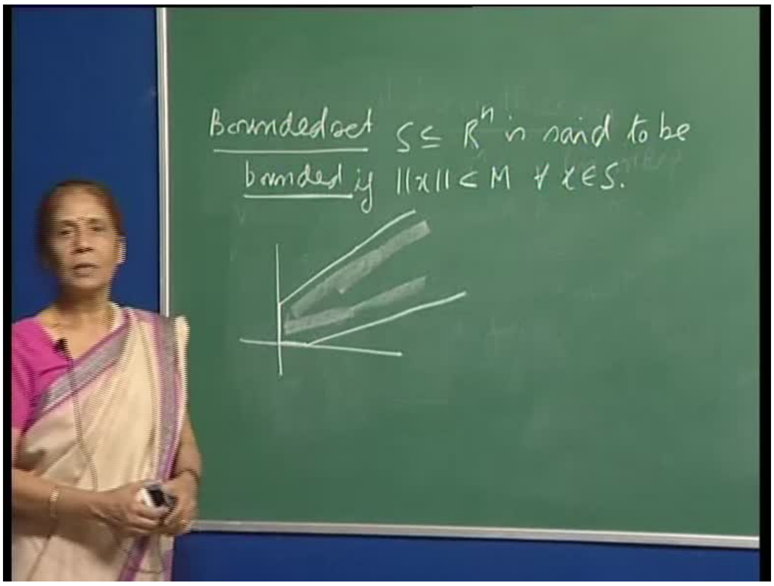

So, s contained in R n is said to be bounded, I said to be bounded, if norm x is less than sum m for all x belonging into x. We have already introduced the concept my R n, that was in my earlier lecture, when I was talking about the mathematical concepts I looked upon, I define the certain operations on R n and through inner product, we also defined a norm of a vector on a r n and so we are saying that, if a norm or the length of a vector is less than some a pre-specified number n, then or I can find an n, such that norm x is strictly less than m for all x and s and we say that the set is bounded.

So, for example, if you, if you take the set, yes something like this, again I will draw it through that, the, suppose you have this. So, if I, my set is something like this, when you see this is extending to the infinity; so, therefore, if you give me any number, I will always be able to find another number, I will able to find a vector in this set, whose length is or whose norm is more than the number that you have given me.

So, I will always be able to extend it. So, this is this is an unbounded set and now, let me go on to… So, the next point that we want to make is, look at, the, because again remember all my definitions and results are such that they relate somehow to mind in a programming problem. So, for example, if you look at this, now, let me defined a dimension of a subset, dimension of a subset s of R n.

So, the idea here is see, so far when I defined the dimension of a sub of a vector which was of a vector space, that means, we had a notion of linear independent and so on and it was, we define the dimension of vector space or a sub space, but it had to be, it had to satisfied the certain criteria for or the vector space.

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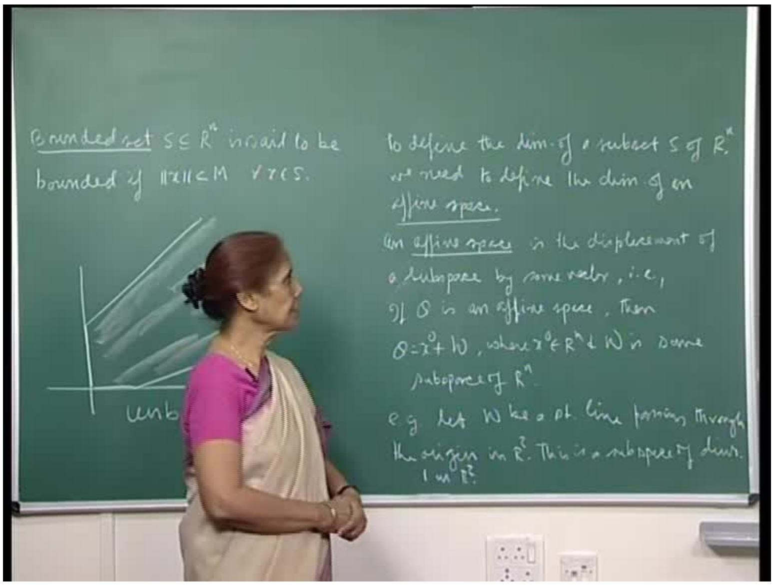

So, we want to defined the dimension of a subset s of R n; so, to do that, we need to defined the dimension of an affine space, because I can only define the dimension of a subset, through the dimension of a affine space, which can, now what is an affine space, this is the displacement of a subspace by some vector, that is, if q is an affine space, then q can be written as x naught plus w, where x naught in R n is some vector, a fixed vector and w is some subspace of R n, so, any affine space, that means, you are actually taking a subspace w displacing it by the vector x naught and then you get an affine space.

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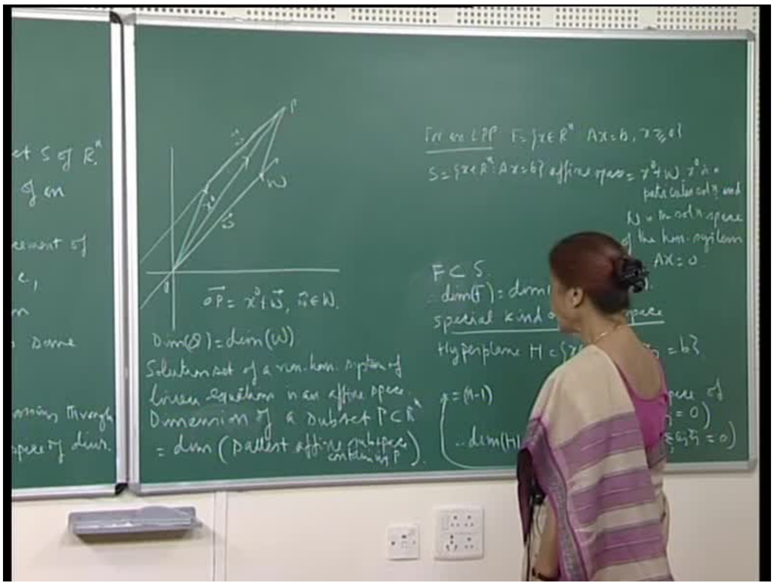

So, this, this stops of being a subspace, because obviously 0 vector does not belong it. So, then, now for example, you can see that if you take w to be straight line passing through the origin in r 2, then this is a subspace of dimension 1 and r 2; see, you take this straight line, which is passing through the origin; so, this is the subspace in r 2, w is the subspace; now I take a vector x naught a fixed vector and so I had displaced w by through the vector x naught.

So, therefore, what we are trying to say here is that, any, if you take, now I just take some vector small w here and then, you want look at the vector x naught plus w. So, then see a complete this parallelogram, so this vector would be also w; so, then the sum would be given by the vector o p which is x naught plus w so w belonging to this. So, any point on this, displaced subspace, displaced straight line can be obtained as x naught plus some w in capital w; so, this is the concept of an affine space.

And then, we say that the dimension of a, q of the affine subspace, q is the same as the dimension of the subspace w which is defining it and then, so here is an example of a affine space, which you already have come across is solution set of a non-homogenous system of linear equations is an affine space, because you know, you know, when you take the non-homogeneous system, you have a particular solution and then the homogenous part.

So that, solution space is a subspace and so, you get the solution space of a non-homogenous system, which now becomes an affine space. And now, you come back to the defining the dimension of a subset p of R n and that would be the smallest dimension of, the smallest affine subspace containing P; so, this is the concept which we have to also and of course, like which a smallest, I have not written it properly smallest. And let us, we will have to spend time here, actually telling you how to get this smallest, but when you come across the concept of defining a subset, dimension of a subset, I have, suppose, this will be clear and mainly, we would be dealing with a dimension of a, of the a, of these feasible region corresponding to a linear programming problem; so, there should be no problem and I will try to explain that here.

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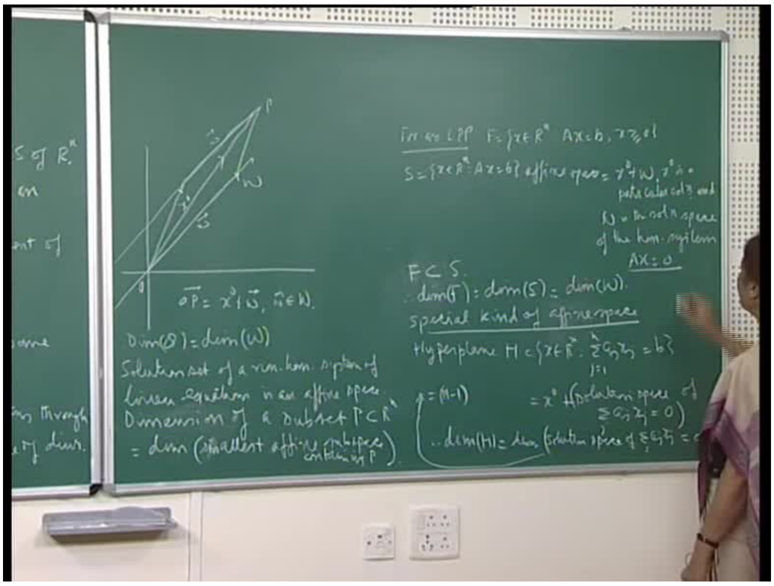

So, therefore, we want to find out the affine space which contains a subset, the smallest affine space and then the dimension of that affine space would be the dimension of that subset.

Now, so, let us look at the feasible region for an l P P and the standard form this is your l PP then you will first look at the subset or the x equal to R n a x is b; now, this is the affine space, I told you, because solution space of a solution set of a non-homogenous system of linear equations is an affine space. So, this can be written as x naught plus w, where x naught is a particular solution; a x naught is B and w is the solution space of the homogenous system A x equal to 0, this is the subspace.

So, the solution, space solution set here, can be obtained as a particular solution plus the subspace, which is the solution space of the homogenous corresponding homogenous system.

Now, you can immediately see, that f is in s and s is the smallest affine space containing, because in F, now, you require the solution should be all non-negative t. And so, this will be the smallest subspace contain affine space containing F and therefore, dimensionof F will be dimension S, this affine space containing F and the dimension of S is nothing but dimension W, where we are saying, W is the subspace the containing of the solution for the homogenous system.

So, this is what we, therefore, you know how to define the dimension of a, of the feasible region corresponding to a linear programming problem. A special kind of affine space which we will be using very often is a hyperplane and a hyperplane is defined by one non-homogenous equation, so for all x in R n which satisfy sigma A j X j equal to B j varying from 1 to n. So, this will be my hyperplane and we can immediately see that this will again be some x naught, which is, that means, which satisfies this, it is a particular solution a particular point in h and then plus solution space of homogenous a linear equation sigma A j X j equal to 0; so, this is only one linear equation norm 0, because our A j's are not all zeros.

So, therefore, the dimension of the, this dimension of the solution space would be what here; the dimension the solution space will be n minus 1 because only one linear equation, homogenous equation is describing the set. So, therefore, the dimension of the solution space is n minus 1 and so, dimension h is nothing but dimension solutions space of sigma A j X j equal to 0 which should be n minus 1. So, hyper planes are special kind of affine spaces, which have dimension 1 less than the dimension of the, of the space in which you are working.

So, we would be talking of supporting hyper planes and so on; so, this would be idea and it helps to understand the geometry and of course, or the dimensions helps you to visualize the things better and so, the idea now you will go on to defining your three-dimensional polytopes and so on and then, we will restrict ourselves to three-dimensional, here of course, but you can have and then I will talk of different face sets and so on of the polytopes.

So, now let us continue with the some more definitions; I am trying to give plus straight and so on, so break a just do not want to go on feeding you with definitions of the time, but it helps to collect together all the theory that we need; so that, we keep in referring that to it and then, when we develop the algorithm, we do not have to call the time go back and say this is what we have to define.

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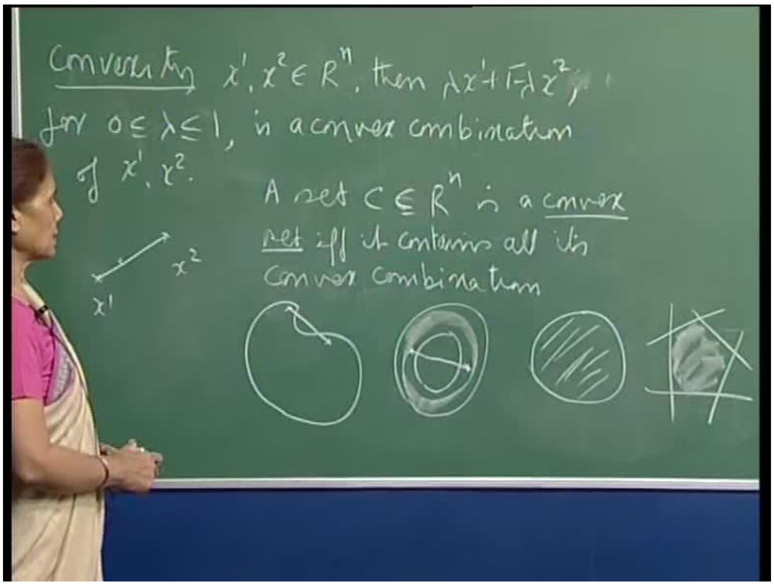

So, it is better, that I do it all in the beginning; so, now let me define another concept and that is the convexity; this is very important and you see how much this concept helps to simplify the concept of the whole theory that we are going to develop. So, first of all, I will say that. if you have let x 1, x 2 belongs to R n, then lambda x 1 plus 1 minus lambda x 2 is lambda x 1 plus minus 1 minus lambda x 2, for lambda greater than or equal to, need I say yes, I need to say, it is between 1 and 0, not just so this for 0 less than or equal to lambda less than equal to 1, is a convex combination, convex combination of the two points of x 1, x 2 and as you vary lambda you get all possible.

So, in other words, geometrically it will see what it means, it means that, if this is a point x 1 and this is a point x 2 and as you vary lambda, you get all the points on the join of these two points is not it; the line joining these two points you get other points. Now, if lambda is 0, if lambda is 0, then you get the point x 2 that means, you get this end and if lambda is 1, then this is 0 and you get the point x 1 and for all values of lambda between 0 and 1, you will get points on their join.

So, this is called a convex combination of two points. Now, we immediately, we can defined a set to convex set and that is when, a set c R n is a convex set, is a convex set, if then only if and only if it contains all its convex combinations, all its convex combinations, so it is a convex set; if it contains all its convex combination, that means, I take any two points in c, I take the line joining the two points, that the whole line should be in c and this should hold for all points in c, then we say that the set is convex. Now, for example, if you take this set, then I take a point here, I take a point here and I join the two points, then see part of the line is not in the set.

So, it will not satisfied this definition right, because there are holes indentations, similarly, if you take a ring, then and if you say that, this is your considering this set ring, then I take a point here, I take a point here, the whole line is not in the set; so, this is not a convex set, that means, a convex set has to be without any pose indentations, so a circle, that means, inside of the circle and if you include this, then this is a convex set, then you can also have, yes, so if you take this set you see, because its enclosed by the straight lines; therefore, there are no holes and I take any two points on the set the joint will be; so, this is the idea of a convex set. Now, you can also extend this notion to a finite combination and I said if and only if see here, that means, if a set is convex.

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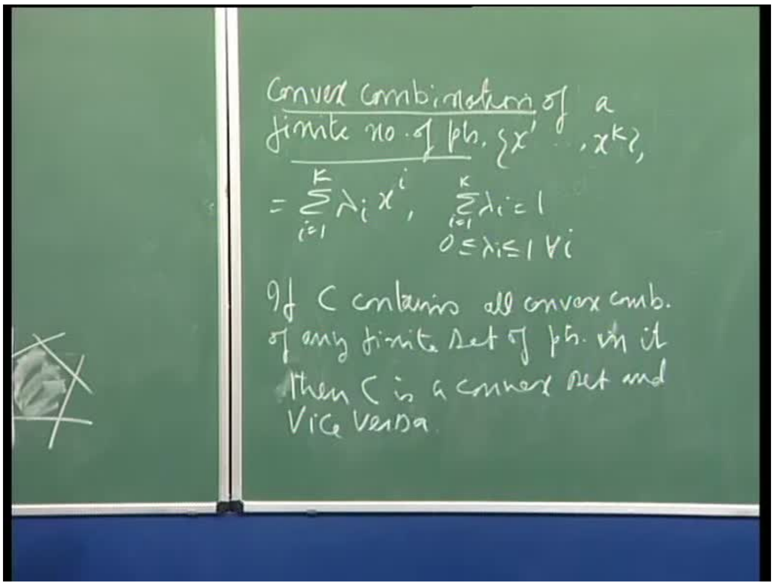

Then all convex combinations any two points in the set, must be in the set and, the, if the set is convex, then this and if all convex combinations of any pair of points in the set or in the set, the set is convex, so it both ways, I can define. One can have a more general concept here also, you can extend this notion, you can also say that, convex combination of a finite number of points, of a points say x 1 x k, i have stop using the word, this sign of a vector, because it is now understood.

So, is convex combination of a finite number of points, this is something like this lambda i x i, i varying from 1 to k, where summation lambda i is 1 and 0, i varying from 1 to k and this is for all, it is just extend the notion; for example, when k is 2, I get convex combination of two points and if k is more than 2, then I get this with the lambdas must add up to 1 and they must be both, they must all be lying between 0 and 1; so, this is a convex finite and you can see that, this is more general and if a set contains all possible convex combinations of a finite set of points, in the set, its convex and vice verse and by iteratively using this definition, you can show that, the two things or two concepts are the same.

So, this gives you another way of defining convex set; now, the thing is that, therefore, I can state more general theorem and say that, if c contains, if c contains all it all convex combinations, so actually convex combination implies that you are taking convex combination of a finite number of points, but may be one can state it also if c contains all convex combinations of any finite set of points in it, then c is a convex set, convex set and vice versa same thing if and only if, for that is, if it c is convex, it must contain all convex combinations of any finite set of points in it and if it contains all finite combinations of convex combinations of finite set of points in it is convex.

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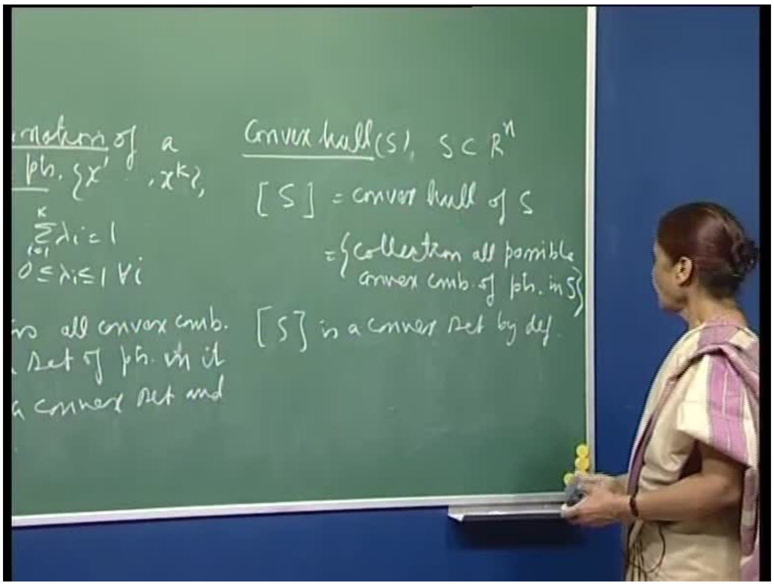

Now, let me define another concept in which is convex hull; so, convex hull would be, it is a convex hull of a set S; so, S is in R n, then we say that this is the convex hull. Actually the notation, let me make the correction, because remember I have used this for all possible linear combination, so here I want to make, so maybe this, this would, so this a convex hull convex hull of S and this is collection of all possible convex combination of points in s, I have stop drop the word finite because its understood.

So, if you take all possible combination convex combinations of points in s, that becomes the convex hull and by definition and this is a convex set by definition; see, remember I gave this definition for a convex set, then I said that at this is sort of a lemma or a theorem which says that, you, if you can also say that finite, if you take a convex combination of finite points in S and this convex combination is an S for all possible such combinations then c is convex.

So, actually, I use this to prove this and now, because of this is or we can say that S is a convex set by definition or by theorem or whatever it is. This, is a, this is a convex set and convex hull of a finite number of points finite number of points, we would want that all the convex all possible convex combinations should be within the set, with within the convex hull.

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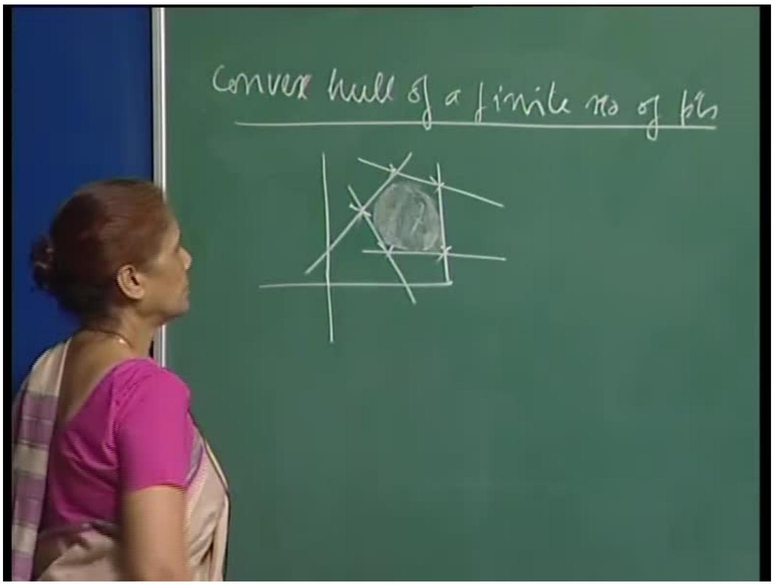

So, here, if you take for example, say these are some finite number of points and like this, but we want all possible convex combinations of these points to be in the set; so, then, I will draw these lines, so to make sure that no convex combination of these two points is outside the set and this will be this and this is, so then this will be my convex hull of this given set of finite number of points. For example, for this one, for this one, I would, if I want to the convex hull, then I will have to include this also, so the whole thing, then it becomes a circle and this. Is a convex set; so, convex hull, what, when you take the convex hull, it actually fills in the gaps and all the holes and all the indentations that are there in the set they get filled up and it becomes a convex set.

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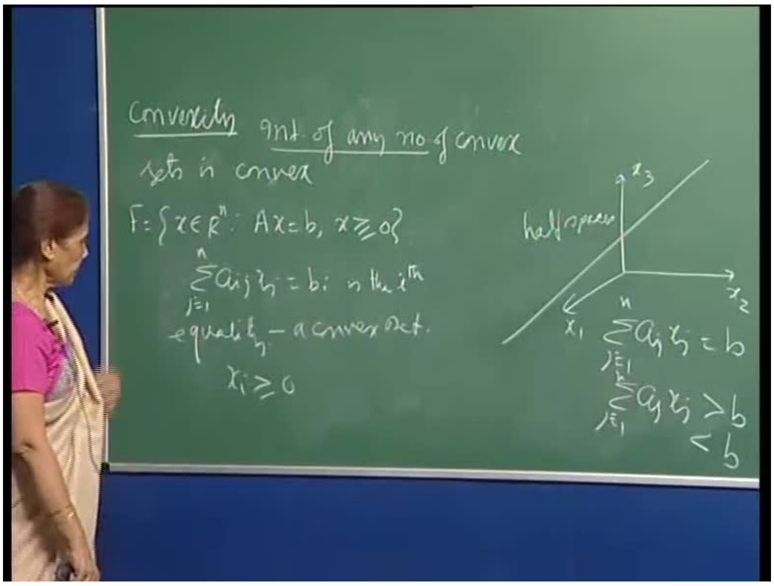

Now, we can also define intersection of, intersection of any number of convex sets is convex; so, intersection of any number of convex sets is convex, it is very, by using the definition when immediately prove it and so by virtue of that, it will look at the feasible region for l p p and you have A x equal to b x greater than or equal to 0; so, how many convex sets you have. Each hyper plane is a convex set vice, so here you see each of the equations summation a i j x j, k varying from 1 to n equal to b i is the i th strain is the i th, is the i th, is the i th equality. So, this constitutes a hyper plane and it is a convex set, because if you take any two points satisfying this, then they join the, that is a concept of hyper plane.

So, this is a convex set or it is the convex set so f and then when you take x i greater than or equal to 0, what do you get see here, I should have may be spilt it, this out that if you have let say taken this is x 1, x 2, x 3; so, when you take a plane hyper plane like this, it is a two-dimensional thing, you can imagine it sending like this; then this hyper plane actually gives you to half spaces also, half spaces.

So, when you have that summation a j x j, j varying from 1 to n is equal to b here, then they will be set of points which satisfy the constraints as a j x j, j varying from 1 to n greater than b and they will be a set of points which satisfy it with less than b; so, this is these are the two half spaces with a hyperplane defines and we can say that this is convex, this is convex, they are no poles, there no breaks and even analytically you can show that these are convex subsets.

So, here you have m such a convex sets here, then each of the non-negativity constraints also gives you a half space, x i because x i equal to 0 will be a hyper plane and then x i greater than 0 is half other one half of the hyper plane and so, here you have, that means, f can be looked upon as intersection of m plus n convex sets and that gives you the, so that gives you convex set, so this what we mean intersection of any number of convex sets, is a convex sets; so, I showed you that the feasible region for linear programming problem is a convex set and in fact, we and this is a special kind of a convex set which we want to give it a name in fact f is a polyhedron, f is a polyhedron, so we give it a name because this is a special kind of a convex set, because it is a define by linear equations and inequalities.

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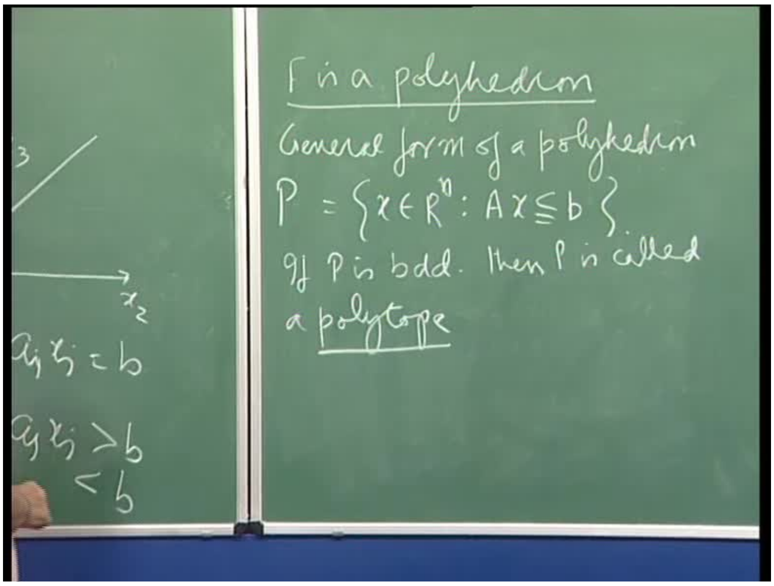

So, for such convex sets, we give a special name and in general, you can, you can also say, that I mean, a general form of a polyhedron; so, general form of a polyhedron can be defined as, so I am, let me call it p. So, I will distinguish between a general polyhedron and the polyhedron which corresponds to a linear programming problem; so, p x belongs to R n, such that a x is less than or equal to b, so what we are saying here, is that, even if there are some in equal, non-equal, non, non-greater than or equal to chi, this is the non-negativity constraints, they can be part of this here, because and I will try to show you or you can write, because I can, I can multiply this by minus sign.

So, this constraint, for example, can also be written as minus x i less than or equal to 0, so it is all the big deal, I mean, I can take these as the equations and inequalities put them together and whether the more concise way, you say that, this is your description of polyhedron.

So, and we will mostly be dealing with polyhedrons, if a linear programming fine. And then if P is bounded, remember I define, that means, if I can say that, there are no this thing here, the, you can, you can bound the length of a norm of every vector in P; if P is bounded, then P is set is called a polytope and we will see the advantages of having a polytope also, because if a set is bounded, it gives you some special leverage to talk about the corresponding linear programming problem, then you can also define.

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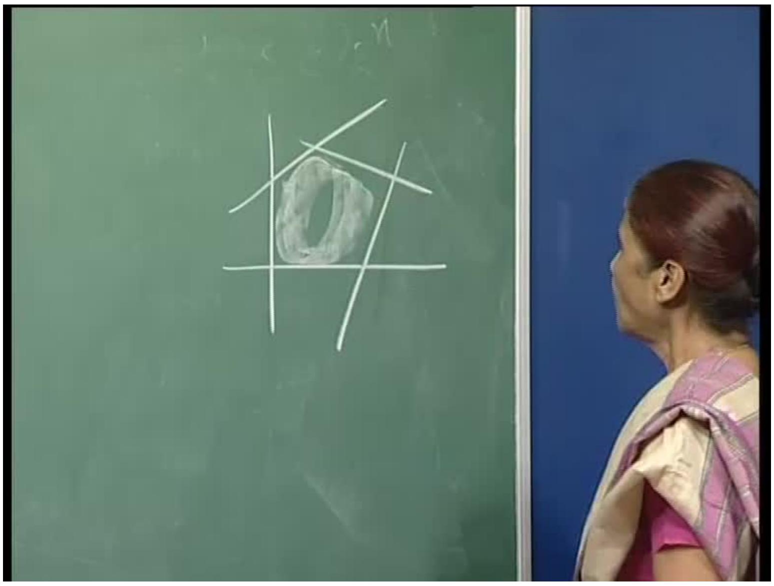

So, now, you have a polyhedron, that means, essentially because it so much convenient to draw this thing in, so a polyhedron what we are saying is something like this; so, it is a bounded by linear constraints straights lines in two-dimensions, in three-dimensions that will be hyper planes and so on.

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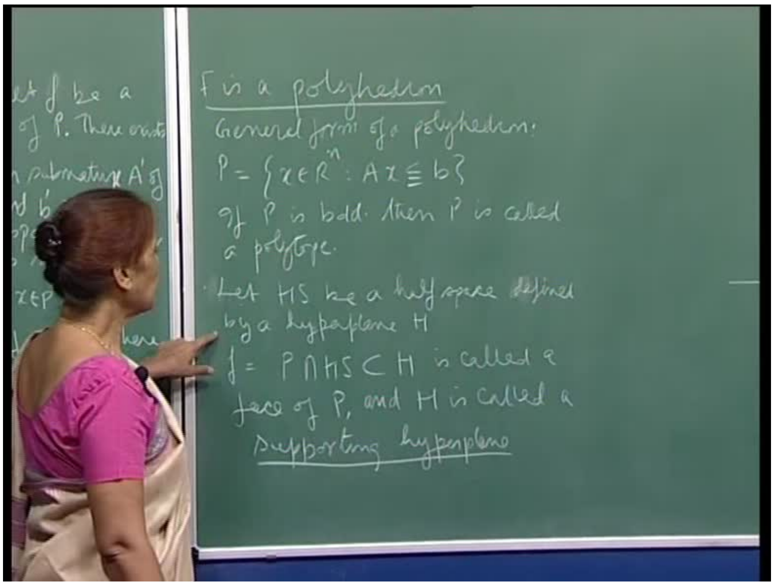

So, this is the polyhedron; so, h now after having defined a polyhedron, let me now talk off subsets of polyhedron also and some of, some of them have a lot of significance in the development of the simplex algorithm. So, H S be a half space defined by hyper plane H and I talk to you about the half spaces and high and every hyper plane will define two half spaces and so, if h S is a half space.

Then the set f which is P intersection H S; let me the intersection of P with the half space and it is a subset of H, then it is called a face of P and H is called the supporting hyper plane; let me give you an example here, see for example, if you have a hyper plane like this, suppose this is a hyper plane h naught, then you see the hyper plane has to half spaces on this side and this side.

So, the whole of the polyhedron lies on this, in this half space and the intersection of the intersection or you can say that, I will look at this half space, so half space and intersection P is this point a and this point a is on the hyper plane H naught; therefore, I will say that, H naught is a supporting hyper plane for the polyhedron P and the face f just consists of the point a, this is the idea.

Now, for example, if I take a hyper plane like this, suppose I take this hyper plane H 1 then you see the half spaces are this and this and the intersection of, if you look at this half space, then the intersection of the half space with the polyhedron is all of this and not all of it lies on H1.

Similarly, if you take this half space, the intersection of P with this half space is all of this polyhedron and again it is not a subset of H1; so, therefore, H1 is not a supporting hyper plane. Then, similarly, if I take a hyper plane like this, so I am just trying to show you different faces, that a polyhedron can have and of course, will give you more examples by better examples by a three-dimensional polytope and so on.

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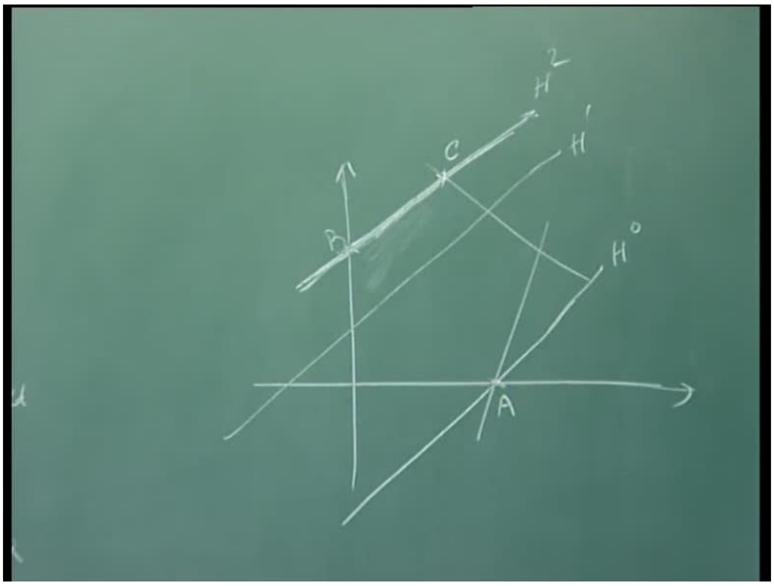

So, here you see, this is you take this hyper plane, then the intersection of the hyper plane with this half space, is only this part right, this line segment and this is part of H 2. So, again H 2 is a supporting hyper plane and B C will be A space of the polyhedron that I true.

So, this is the idea of a supporting hyper plane and a face and then I will talk to you about, but before I start this more about special kinds of faces, let me first define the dimension of a polyhedron which is important and again here there are a little few a, but any way what we are saying, is that, the dimension of a polyhedron P and remember the general definition that I gave you was A x less than or equal to B, x in R n and then I have just defining a polyhedron by and so, I am calling these as the defining inequalities and I had told you that the defining inequalities also.

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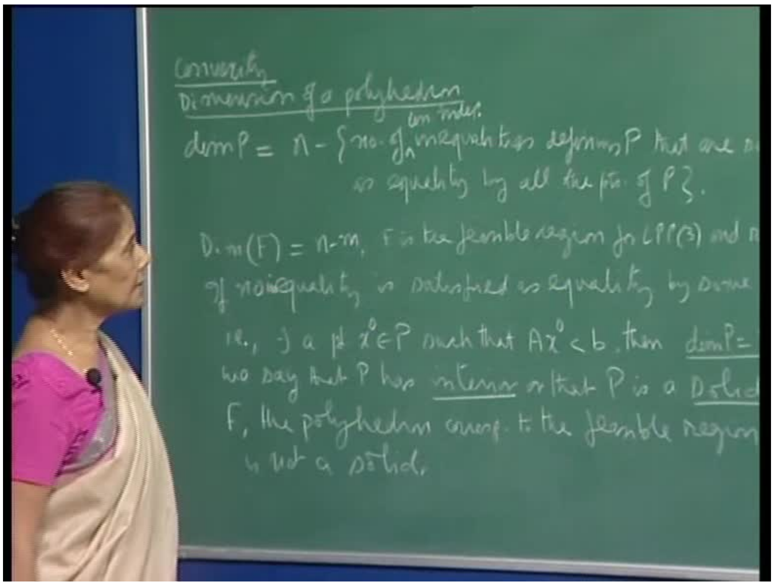

You know, there is a concept of a minimal number of inequalities that are defining polyhedron and those things are there, but anyways, so let us and hope that, we have a minimal representation and then I am talking of the dimension of a polyhedron.

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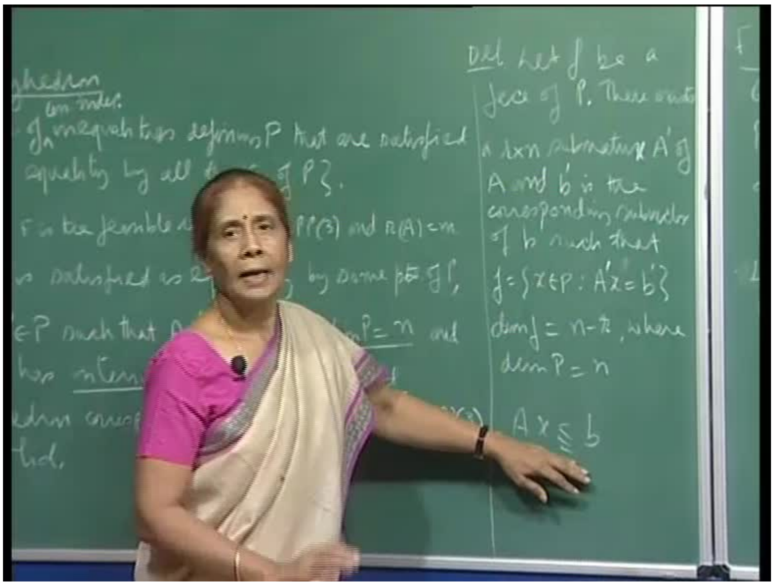

So, if n is the number of variables, then the number of linearly independent inequalities defining P that are satisfied as equality by all the points of b, then I say that, the dimension of P is n minus this number, let me say all there are all points of P satisfy certain inequalities as equalities and I take them to be linearly independent, then I take that number and subtracted from n and then that gives me the dimension of P, because again if you can go back to I have talked about degrees of freedom and so on.

Let me talk of a system of equations and inequalities; so, here, for example, for the, for the, if f is your feasible region for l P 3 and remember I am also referring to it as a polyhedron is a convex set; so, then we undertaking rank of the coefficients matrix to be m, then dimension of that will be n minus m because I am assuming that all.

So, all the, see the feasible region, the first standard l P3 is an standard form; so, m equations are there and rank of a is m, so there all linearly independent and so, if the rank dimension of f for the l P3 is n minus m, now, if no inequality, if no inequality is satisfied as equality by some points of P, that means, that is there exists a point X naught in P, such that a x naught is strictly less than b; if at least 1 point is there, then we will say that dimension of P is n or that it is P has an interior or that P is a solid.

So, these are different concepts which mean the same thing that we will say that P is a solid, because it has dimension n or it has in interior. Now, for the, for the, for the l P P 3 you are for feasible region is not solid right, because its dimension is n minus m and so there are points which satisfy the first m equations; so, therefore, it cannot be a solid fine. Now, let me define a definition of a face also and then we will look at the examples here; so, f is A face of P, when there exists in R by n sub matrix of a prime of A, that means, you take some r rows and then A and B prime is a corresponding sub matrix of B, that means, here if I take the, suppose I take the first r rows, then I will take the first r components of B or whichever r rows, I take from here the corresponding components of B I will take; so that, that comprises of B prime. So, then f can be describe as set of all points which satisfy these are inequalities as equalities, so, a prime x is equal to B prime; so, every face will be satisfying some inequalities as equalities and then, in that case dimension of S will be set to be n minus r, where dimension of P is n.

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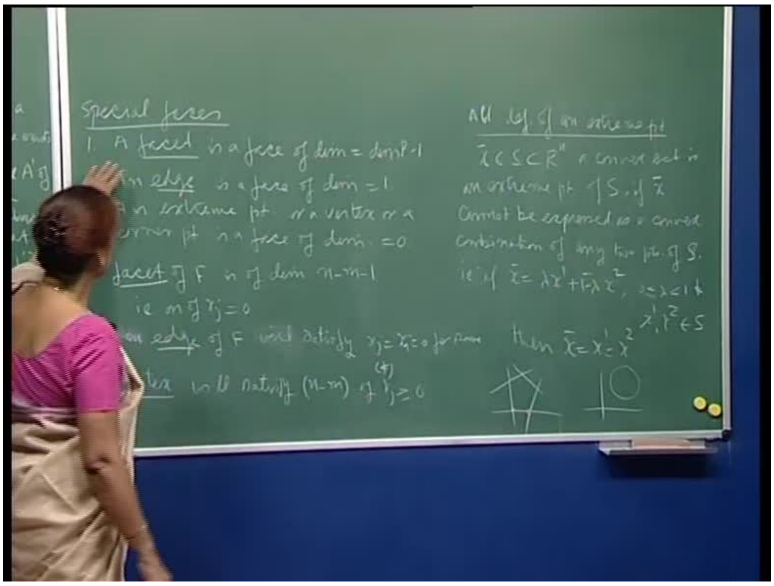

So, for example, when we define faces of f, we will have to take in, in place of n, we will write n minus m assuming the rank of a is m. Now, let us quickly just give you definitions of some particular faces; so, the special kind of faces, that I was mentioning is a face t which has I mention 1 less than the dimension of the polyhedron.

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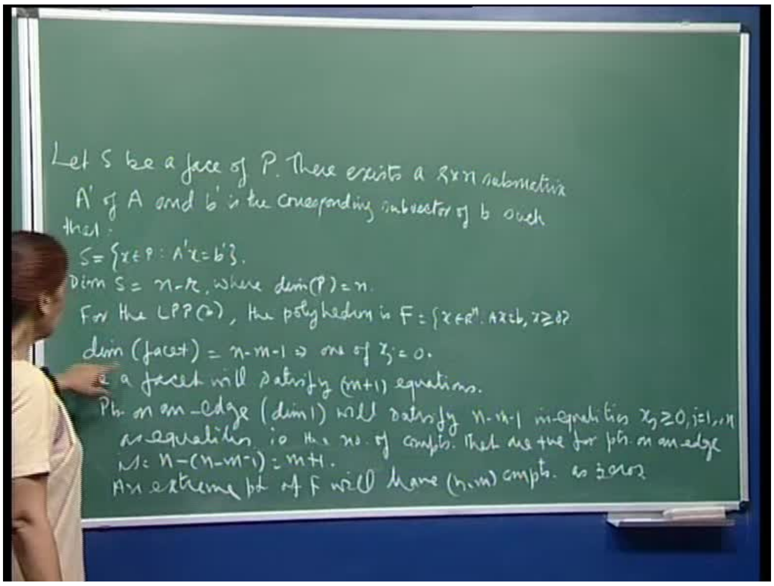

An edge will be a face of dimension one and an extreme point or a vertex or a corner point is a face of dimension 0; now, I want to just translate this concept to the faces of the polyhedron of the feasible set for the linear programming three problems. We define so the, we know that the polyhedron f is define by A x equal to B x and greater or equal to 0 and we are assuming that the dimension of A is M.

Therefore, this polyhedron has a dimension of n minus m, because the number of variables is n and then A facet has to have dimension 1less than the dimension of f; therefore, it must satisfy the dimension is n minus m minus 1 which implies that one of the x j must be 0, because for f, you see though inequalities or here so these are n inequalities, all these m equations have to be satisfied by all points of f.

So, this is the n th, these are the n inequalities, so your facet must satisfied at least 1 of them as equality; so, this is it and they, that means, a facet will satisfy m plus 1 equations m here and one of them, one of these inequalities as equalities, then points for an edge, because the dimension of an edge is one, that means, it must satisfy n minus m minus 1 inequalities from here as equalities, which means that, the number of components positive components, number of component that are positive for points on an edge will be equal to n minus of n minus mminus 1 which means m plus 1.

So, for points on an edge the number of component that can be positive would be m plus 1, because the dimension of an edge is 1; so, that means, it must satisfy the dimension of f is n minus m, so it must satisfy n minus m minus 1 inequalities as equalities and for an extreme point, because that dimension of an extreme point is 0, so it must satisfy n minus m, this number is, this is n minus m; so, n minus m components from here must be 0, because the dimension of an extreme point is 0, so n minus m minus of n minus m should be 0. So, therefore, the for every component, for every extreme point of f, the number of components that will be 0 has to be at least n minus m, of course, if it is a degenerate point, then the number of zeros can be more than n minus m extreme point of P of A polyhedron P.

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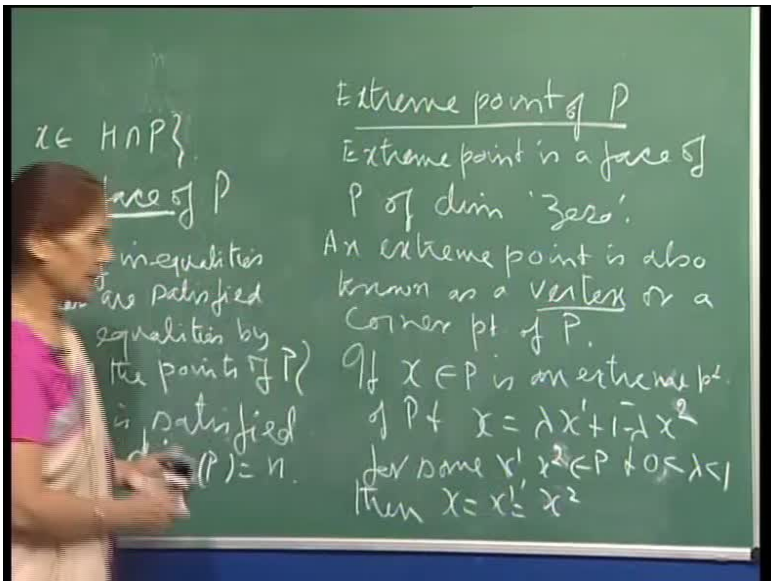

Now, they can be, one way of course, actually who is that the extreme point is the face of P of dimension 0 or the geometrically also we can define, because you see, it is in a sense, since the hyper plane intersects it only at a one point; therefore, he can also have this concept of a vertex or a corner point; so, these are the other names x and so an extreme point is also known as a vertex or a corner point of P and we can give a geometrical definition of an extreme point, that is, if x belonging to P is an extreme point, extreme point P and x can be written as lambda x 1 plus 1 minus lambda x 2 if and x is this, for some x 1, x 2 belonging to P and 0 less than lambda less than 1, then x is equal to x 1, is equal to x 2, I hope you can read this, I will try to rewrite it x 2, say in the other words, x if it is an extreme point of P, it cannot be written as a convex combination of any two other points in P; so, this was the concept of a vertex and I showed you in the picture theta and of course, the other definition was that, it is a face of dimension 0 of the polytop .

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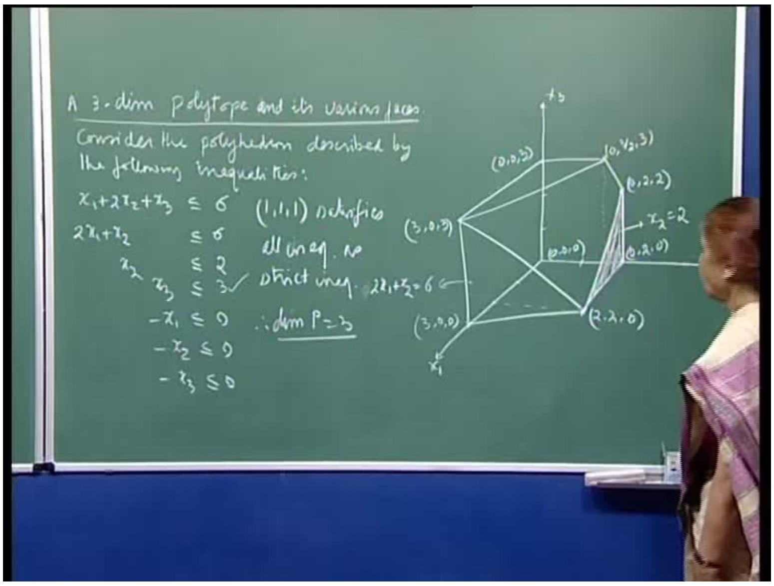

And so, let us look at this example, here there is a polyhedron described by these inequalities and I have drawn the diagram here in the figure, you can see that for example, the this case on which the points 0 to 2, if you look at 0 to 2, it satisfy this as equality when 0 3 by 2, 3 also lies on this phase and then this point 3 0 3, because 3 plus 3 add up to 6; so, this here is the first constraint as a equality. And now, so according to our concept you see, this is, if I take this equal to 6, this will be a hyperplane and the whole of the polyhedron lies in those half space, which is because here this is less than 6 so all points 0, 0 and everything this is in the half space which is less than 6 less than or equal to 6.

This is the picture; then you can see that this face of the polyhedron is because x 3 is 3 for all 3 points l; so this, this is the hyper plane. If you take the hyperplane x 3 equal to 3, then it needs the polyhedron in this face and similarly, this one corresponds to 2 x 1 plus x 2 equal to 6.

So, this face, if you look at then I has shown you and this, this is another face which is x 2 equal to 2, it was all the three points lying on it, have this is the second coordinate is equal to 2. So, this is your three-dimensional polytope and you can see that, if you look at the point, for example, 1 1 look at the point 1 1, then it satisfies all the constraints as strict inequality.

2 plus 4 less than 6, this is less than 6 everything; so, this satisfies all constraint, all inequalities as strict inequalities, as strict inequalities, as strict, as strict inequalities and therefore, the first thing is therefore, dimension of dimension P is 3, because now P has an interior, so I talk to you about it, that if in case you can find a point in a polyhedron which has all which satisfies all constraints defining it as or inequalities as strict inequalities, then it is a solid and we say that, it is a same dimension as the number of variables defining it.

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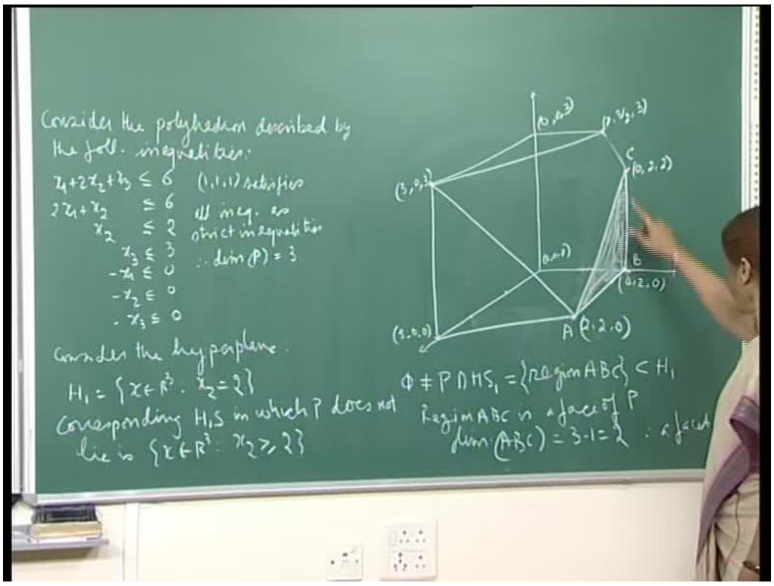

So, dimension P is 3 and we continue with this example, so I am now saying that you consider the hyper plane h 1, which is all points x in R 3, so that, x 2 is 2, if you take this hyper plane, then obviously we get 2 half spaces; one would be x x 2 greater than or equal 2 and the other would be x 2 less than or equal to 2; let me consider the half space x 2 greater than or equal to 2, then you see that x 2 greater than 2 is this side and P does not lie on it. And then, I am saying that now you consider the intersection of P with this half space and this comes out to be region A, B, C , because the half space contains x 2 equal to 2 also and this is your hyper plane h 1 and x 2 equal to 2is satisfied by points on the polytope.

So, when you take this intersection, of course, so here I am saying a non-empty and then this comes out to be the region A, B, C; you see that this is the region, because every point on this triangle has x 2 equal to 2, you can check that yeah convex combination of every point here of these three points will again have x 2 equal to 2, because every all the extreme points have the second coordinate equal to 2.

So, you can see that very well and so this is the region A, B, C and this is a part of h 1. So, from our definition of a face of a polytope, if I can find a hyperplane, such that you know this definition is , then this is a face of P and by of definition by the dimension of A, B, C this would be 3 minus 1, because this is h is an hyper plane h 1or you can say that x 2 equal to 2, is the, this is the inequality which is satisfied as equality by all points on the region A, B, C and therefore, this is a facet of dimension, it is a face of dimension 3 minus 1 which is 2; hence, so it is a facet.

Now, in my definition earlier, when I had talked about a supporting hyper plane and so here you will say that x 2 equal to 2 h 1 is a supporting hyper plane 2, the polytope P. Now, in my definition, if I take the half space to the x 2 less than or equal to 2, then P would be in that half space and therefore, I do not have to write this.

So, please check that in the when, when I was defining concept of a face and face of a polyhedron and the supporting hyper plane, I main, I may have not mention not equal to but see the understanding is at if you are taking the half space in which p lies, then obviously this will not be empty P intersection H S1. So, the whole idea I am trying to show you, is that, you take a half space and you take a polytope then the two just meet in this face, A, B, C and that is the whole idea.

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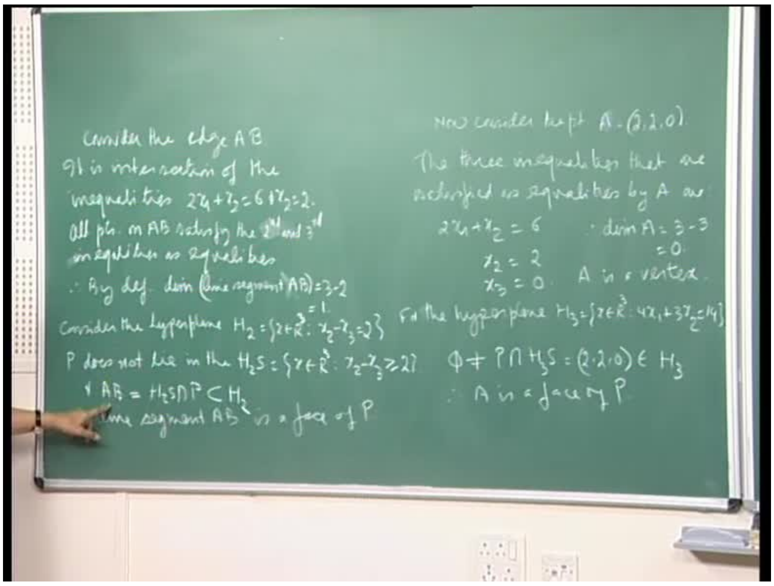

So, whether you take the half space and which lies or which P does not lie does not matter, but basically they will intersect at that face of the polytope. Now, similarly, let us take another dimensional face of the polytope P; so, if you consider the edge A B, so I am taking two different definitions, I am trying to show you, now it is a intersection of the inequalities 2 x 1 plus x 2 equal to 6 and x 2 equal to 2 ,you can see from the diagram also; that means, every point on the, a line segment A B satisfies the second and third, the second and the third inequalities as equalities and so, by definition, we had two definitions of the, so by the dimension of the line segment A B is 3 minus 2 because two linearly independent inequalities, there is being satisfied as equalities and the dimension of the polytope, we had said is 3, so it is3 minus 2 which is 1; therefore, this will be an h of the polytope by our , but now here I also want to show you, that you can construct the supporting hyper plane and therefore, it will be a face; so, this supporting hyper plane I am saying consider the hyper plane x 2 minus x 3 equal to 2. Then you can again see that 4 x 2 minus x 3 greater than or equal to 2, for all point I mean, if you take the half space this, then P does not lie in this half space and therefore, and the intersection of this half space with P, is this line segment A B which lies in h 2 and so, again A B is a face of P by our definition and the dimension as we said because two inequalities have been satisfied as equality.

So, therefore, the dimension of this face is one. And now, you consider the point a, which is 2, 2, 0, this is the point of the polytope and this is a vertex and so we must have three inequalities that are satisfied as equalities by the point a, see look at these three and therefore, the dimension of this vertex is 0 which is what it should be that is why we call it a vertex and again just for the fun, see you can also try to construct some other hyper plane which is a supporting hyper plane to the polytope at the edge A B, then or for some other edge A B and then here for the hyper plane h 3 4 x 1 plus 3 x 2 equal to 1 4, you see that the point a satisfies this as equality and intersection of the half space P intersection h 3 3, here, of course, here again I am taking the half space to be greater than or equal to 1 4, then the intersection isthe point 2 2 0 and this would be this is contain in h 3; therefore, a is a face of P and the dimension is 1.

So, therefore, you know, you can do the same exercise for all other edges like, for example, this is an edge, this is an edge, so try to for construct supporting hyper planes at these vertices as well as for the edges and see the show that, for example, x 3 equal to 3 is a facet of this polytope which you can sit down and show that, you know you consider the hyper plane x 3 equal to 3 and so on.

So, I think this should give you a good idea about the how to determine the dimensions of a faces or different faces of a polytope and the important kind of a faces that we will be a using when we want to give a geometrical picture of this simplex algorithm, which are the extreme points of the vertices and the facets supporting hyper planes.