Linear Programming and its Extensions

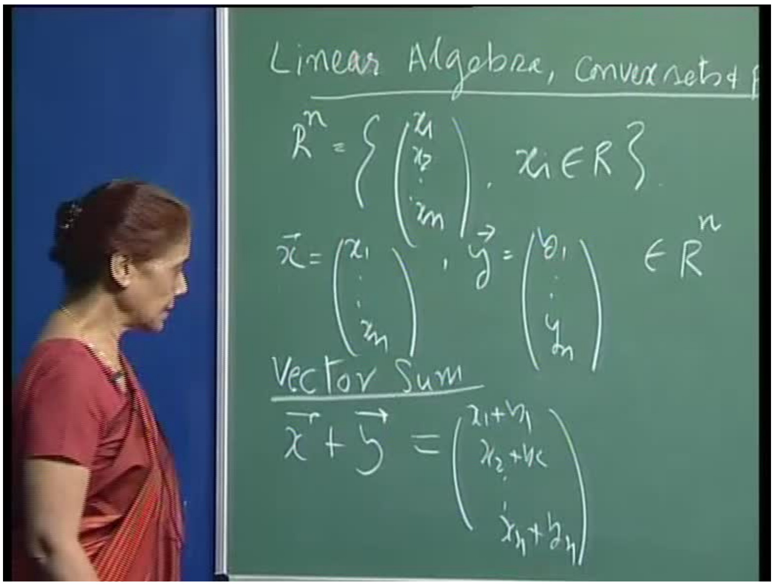
Prof. Prabha Sharma
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture No. # 04

Basic Feasible Solutions Existence and Derivation.

So, today I will talk about linear algebra convex sets and polyhedral, because this is the basic mathematical concepts that we will need, to develop the theory for the simplex algorithm and related topics. So, let me begin with, you may find it a little boring, because we will be giving lot of definitions and the few lemmas and theorems, but I will try to give you examples, so that, the monotony is broken any way. So, let me just begin with, and most of these concepts, may be you have come across somewhere, sometime, but let me just collide them together. So that, then we know what all the results we need and I can keep referring back to them, whenever I need them; so, I begin with, see we all understand this symbol.

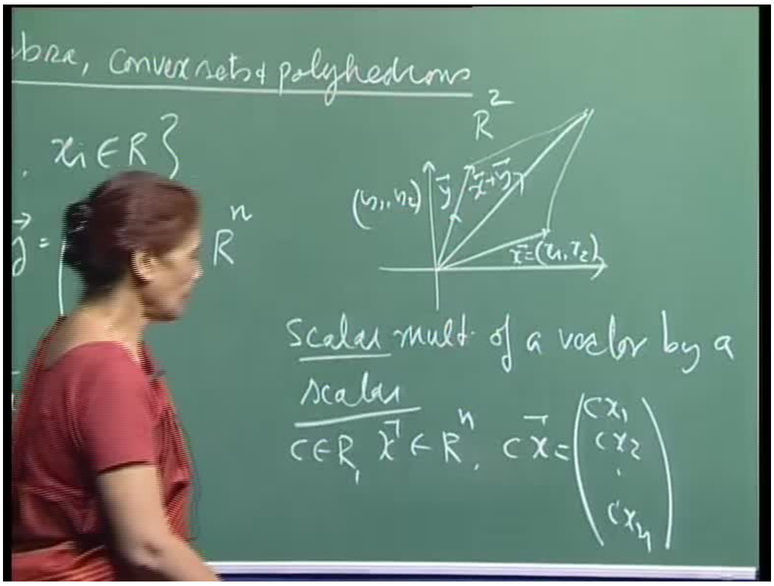
(Refer Slide Time: 01:01)



This is R n and this is collection of all n tuples x 1 x 2 x n, where your xis are all real numbers. So, if you take collection of all these n tuples, you get this R n and let us give it some structure, we already know, for example, that if I take two vectors and for short, I can call it this tuple is x 1 to x n; so I will sometimes refer to the vector as, so this, this will be I will call it n tuple or a vector and I will sometimes refer to the vector will like this or may be like an n tuple and later on I will also give you another shortcut. So, this will be let us, say another n tuple and they both belong to R n.

So, let us first assign operation on these n tuples or in these vectors we say vector sum, so, I am defining an operation vector sum on R n; so, I will say that, x this some summation of the two vectors is nothing but component wise some of the vectors; so, this will be x 1 plus y 1 x 2 plus y 2 x n plus y n.

(Refer Slide Time: 02:35)

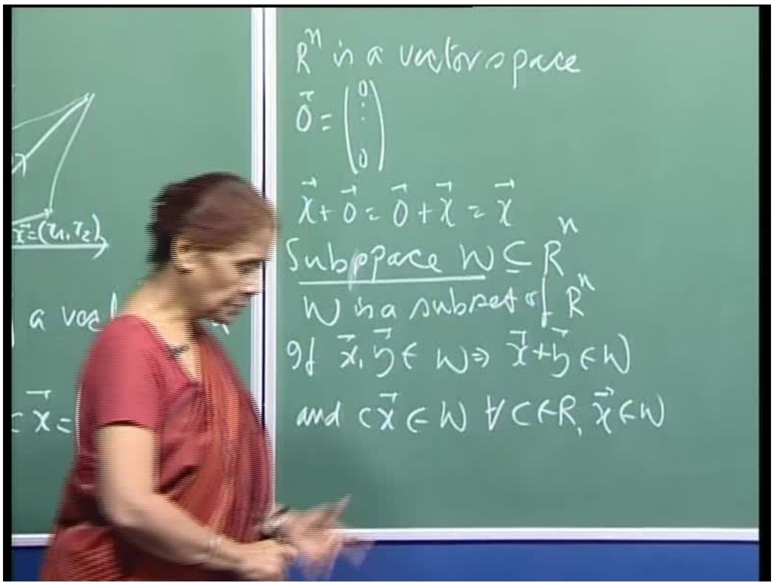


So, this defines the summation of two vectors and in geometrically you can see, that if you have a two-dimensional or that means, I am talking of r 2, you had been handling r 2 and r 3. So, if I have a vector like this, here let say this is my vector x and this is my vector y, then you all know that by completing the parallelogram this vector denotes x plus y. So, vector sum you already have handled it, you know it, you have been using it. So, now, if I extend this notion to an arbitrary n tuple, that means, a vector having n components, then I will just add them up vector wise and here also you can see that the vector x, if this is written as x 1 x 2 and y is written as y 1 y 2, then the vector x plus y will have components x 1 plus y 1 x 2 plus y 2. Another operation that we defined on this collection of n tuples or vector as we are calling them is scalar multiplication.

A scalar multiplication of vector by a scalar and let me also make this clear a scalar, when you say a scalar multiplication, what I mean is that, the number is again a real number is coming from r, but it is not part of a tuple, n tuple. So, I will say that for c belonging to r and x belonging to R n, We say that c x is another vector, which, in which all the components get multiplied by c, that means, this will be c x 1 c x 2 c x n.

So, this is, these are the two things and even though I will not use this concept of vector space very often, still we will say that, when you, when you define this two operations vector sum and scalar multiplication on this collection of n tuples, when r n becomes a vector space.

(Refer Slide Time: 04:53)



So, R n is the vector space and you see that 0 vector belongs to this, because 0 vector will have all components 0. And when you have this concept, that if you add the 0 vector, it does not matter, here this is this and this is equal to x, also when I defined the vector sum, you see, it does not matter, whether I write x plus y or I write y plus x

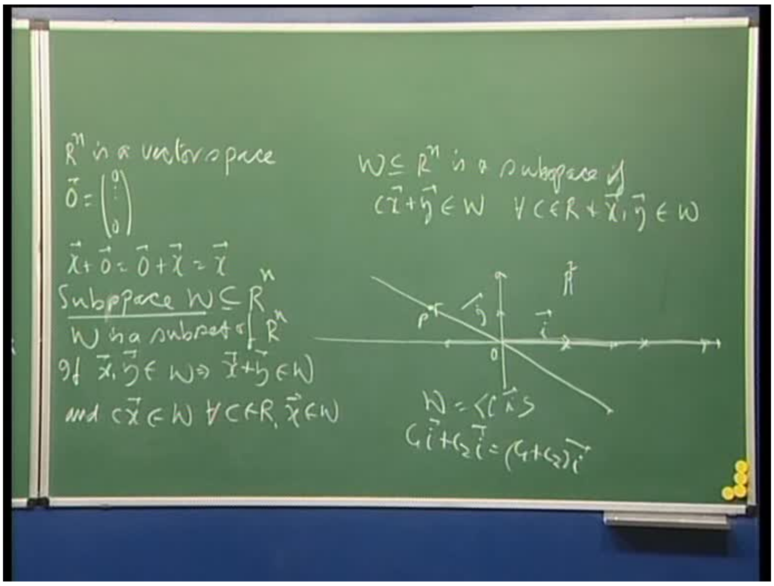
So, this, the concept that, that means, it is said that the vector sum is commutative, because the order in which you add the vectors does not matter here, because it is all component wise, whether I write x plus y or y plus x, it will be the same fine and so on.

You can see all these things, because we already are using them, it is all that we are now simply writing them in a axiomatic way, so that, what we are saying is that R n and you can have more general concept of a vector space also, but we will not bother about it. Now, our concerned is with R n and so, we have these two, we equip R n with these two operations that we have been using them.

Now, you can have a concept of a subspace W of R n, so this as the name suggests, it is a subset W is the subset of R n, which is, which also satisfies the concept of which is closed under vector addition and scalar multiplication, what I mean to say, is that, you take w is the subset, is a subset of R n, that means, the vectors in W R n, R n.

Now, if x n y belonging to W, the vectors implies x plus y also belongs to w, then we say that W is closed under vector addition and also if this and c x belongs to W for all c belonging to r and this belonging to W, then we say that W is closed under scalar multiplication. So, a subspace a subset of R n, if it is closed under vector addition and scalar multiplication becomes the subspace, because it is also a vector space and in fact, can is a very concise way, say that, w a subset of R n, is a subspace if c x plus y belongs to W for all c belonging to r and x and y belonging to W.

(Refer Slide Time: 07:54)



So, same to both the things I have said in a concise way; so, this is that, it is closed under vector addition and scalar multiplication, because I can put y as 0; if I put y as 0, then c x y belongs to W, that means, the scalar multiplication is satisfied and if I put c as 1 then x plus y belongs to W for all x y.

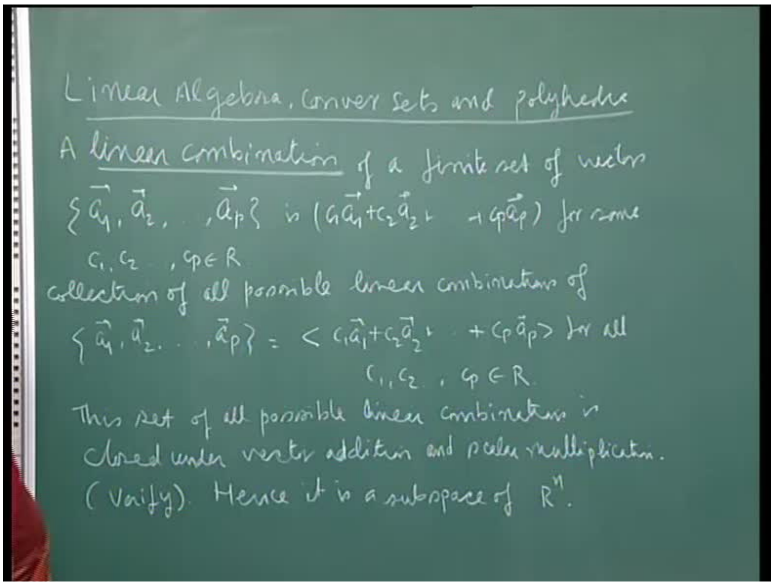
So, this is the concept, now here you see that, for example, what we are saying is that, in R 2, so this is your R 2, this is your R 2, so that, this the two collection of all two tuples, so the whole space, that means, you can say that the extend the black board, in either direction and this is what you get is the a whole of R 2.

Then, if I simply say that, my W and here I now want to you introduce the concept of linear combination. So, W for example, can be written as, if I write this vector here, we are use to writing this as i and this as j, you have all come across vector algebra, you have done it in your school and so on. So, what I am saying is that, if I take the vector i, this is my o, so the direction of i is this, then if I can describe the vector this subspace W or the subset W as all scalar multiples of this. So, that means, of this direction, I, so here for example, I will take the number here which is equal to this length and then multiplied by psi as i get this vector i multiplied by i by negative vector, I will get things on this direction.

So, you see that, the subspace here is the subset of vectors lying on this axis and similarly, this could be one subspace, then I could also have a, if you, if you take a direction like this, just take a vector here and you take the, you can say this all scalar multiples of this vector, whatever you want to call it o p, when you take all scalar multiples of this a vector, then you get a subspace, because it will be closed under you; see, if you take vector here and you take a vector here, when you add up you will get a corresponding vector here only, because c 1 i c 1 i plus c 2 i, it will give you c 1 plus c 2 i and so on; so, I will continue to remain.

So, therefore, the collection of the vectors here on this line on this axis is closed under scalar multiplication and vector addition; so, this is the subspace. I hope, see actually we do not have time to dwell over too much on this thing, but just want to introduce you to the concepts and you can will be suggesting a books for your supplement reading and you can then look up there also this definition. So, this gives us, now since I have allowed scalar multiplication, therefore, I can talk of linear combination of vectors. So, let me define a linear combination of a finite set of vectors a 1, a 2, a t and as we are assuming where it working in R n; so these p vectors are there and then a linear combination will be defined as c 1 a 1 plus c 2 a 2 so on up to c p a p, for some c 1 c 2 c p will belonging to r; so, just take a set of p real numbers and then you take this linear combination, this is what we meant by a linear combination of a finite set of vector.

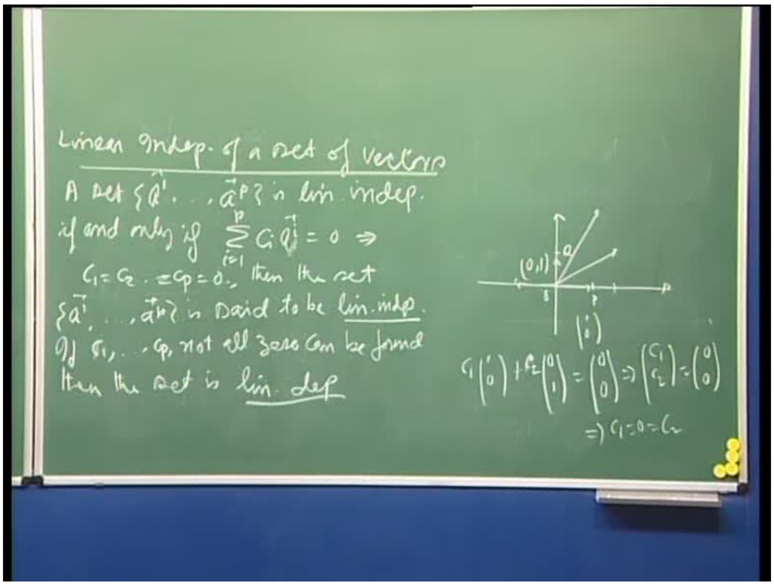
(Refer Slide Time: 11:34)



Now, when you take collection of all possible linear combinations of these p vectors, that would be expressions of this kind, where c 1, c 2, c p can be any set of p real numbers. So, this could be possible collection of, now, you can immediately verified I want you to do this, that this set of all possible linear combinations is closed under vector addition; when you take two linear combinations, add them up, they, it will again become a linear combination of a 1, a 2, a p and similarly, it is closed under scalar multiplication also by definition. And then, if you remember these are the actions that you require for a set of vectors to be a subspace and so hence it is a subspace of R n. So, this is the idea where n you can easily see, that this is how you can also given a set of p vectors a finite set of vectors, when we take all possible linear combinations you actually generate a subspace of R n.

So, this is the idea and we will continue using this concept on later on also. And this is also you have been using somewhere in some way without actually calling the name these names vector space are subspace scalar, multiplication vector, addition vector addition, you must have already talked about, because you use the parallelogram law.

(Refer Slide Time: 12:53)



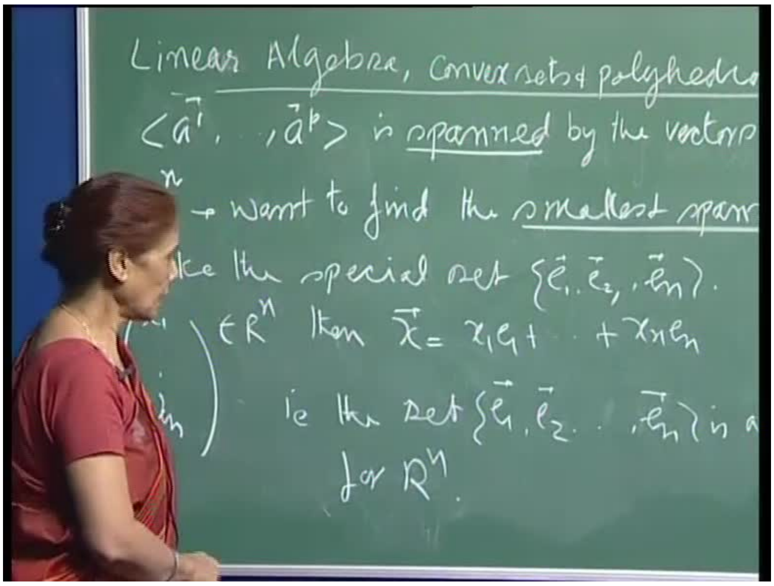
So, now I have this concept of linear independence of vectors independence of set of vectors. So, the idea here is that, in some way, I cannot make this, if I have a set of vectors and I want to, I want to find out whether there is a possible linear combination of this set of vectors which is equal to zero vector. So, that must here linear independence of set of vectors; so, we say that a set a 1 to a p is linearly independent, I, if and only if, this, as you can use this small a i a i bar, i being from 1to p, this equal to 0 implies c 1 equal to c 2 equal to c p is 0.

So, I was saying that, if you have a set of vectors and if you can and if some linear combination of the set of vectors equal to 0, implies that the scalars must all would be 0; So, we say if and only if, this c 1 c 2 c p are 0, then the set and the set a 1 to a p is said to be linearly independent. So, it said to be linearly independent, but if I can find certain scalars not all 0, you see the converse could be immediately, if I can find some scalars not all 0, so if that some that, so the corresponding linear combination is 0, then the vectors will be linearly dependent. If some, if c 1 to c p, not all zeroes can be found, not can be found, then the set is linearly dependent, this is linearly dependent. And again, the concept is very simple to understand, because what we are saying is, for example, here we are saying that, if say for example, here again let me go back to this thing, I can describe this tuples, as for example, I can take let say a vector o p here and let me say that the components of 1 0 and let me take at the vector here q and let me say that the components are 0 1. Then if I write a linear combination of these two vectors, it will be c

1 times 10 plus c 2, c 2 times 0 1 and say this is 0, then if you compare, because obviously 2 n tuples will be equal if they are component wise equal. So, here you see the vector that you get, so this implies, this implies that c 1, so I am using scalar multiplication plus vector addition.

So, this vector becomes c 1 0 this vector become 0 c 2, then I add component wise, so I get the vector this and this is 0 0; so, which implies let c 1 is 0 is equal to c 2 and therefore, the two vectors are linearly independent, but if I take a vector here and then I take a vector here, then you see this will be simply, you multiplying it by some negative scalar and so the two vectors would be linearly independent and of course, you can have so many concepts here, I have a vector here and I have a in two-dimensions, you can immediately say that, two vectors will be linearly independent if and only if 1 is not a scalar multiple of the other, because here if I take these two vectors, then obviously there linearly independent and so. So, once you define this concept, then we want to because we now have linear combinations, we have concept of linear independence, let me describe the situation. So, I will now talk of a spanning set, see these preliminaries you would I will take time to explain, so that later on when we keep referring to them there is no problem.

(Refer Slide Time: 18:06)



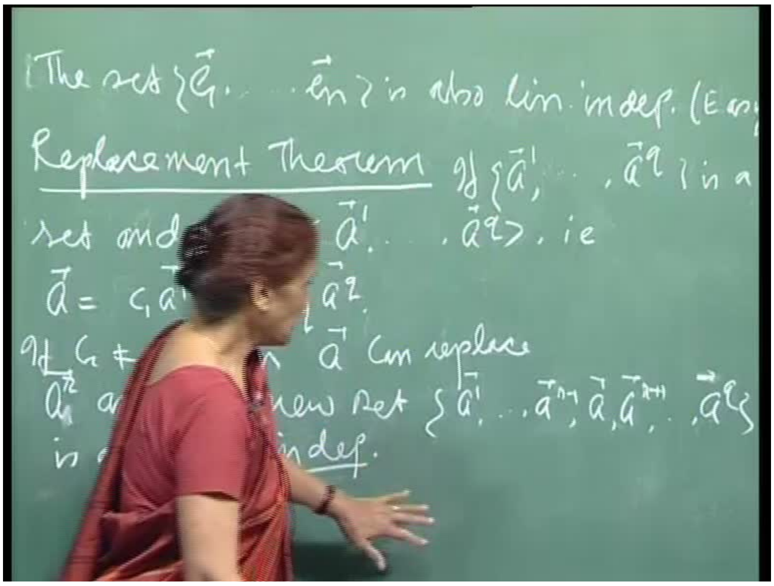
So, now what I will says that, if see for example, or this set, this is what I said is collection of all possible linear combinations of the vectors a 1 to a p, then this set this subspace, the set of all linear combinations is spanned by the vectors; so, spanned is the word which is underlined by the vectors a 1, a p span, means that, any linear combination of this set of vectors is here; so, that is what we say that, it, if a set is spanned by a set of vectors, when that means, that set has all possible linear combinations of the of that set of vectors.

So, this is spanned by the vectors a 1 to a p and now, what we are looking for, for example, for R n you want to find the smallest spanning set, the smallest spanning set; so, for R n we want to find, because it will very convenient, when I want to refer to R n, I do not have to write the, I keep saying that, the, all collect collection of all possible n tuples. So, I would like to have a nice representation for R n through a spanning set, because once I know the spanning set, it is going to be finite set and then any vector in R n can be obtained as a as a linear combination of this a spanning set. So, this the idea, so we want to be able to represent the vector space in asprecise of way as possible and of course, so, want to find a smallest spanning set and this is what we are trying to now, say that for example, for example, you look at the set, so you take the vectors, take the special set, take the special set e 1, e n, some of you may be familiar with this notation, but let me say that e 1 is the vector is the n tuple which has 1 a unit in the entry in the first place and 0 the elsewhere and similarly, e 2, e 2 will be a vector which has n tuple, which has a unit entry in the second place and 0 all over and similarly, e n will have this.

If you have this collection, then if you look at any n tuple x n belonging to R n, then this x can be easily written as x 1 e 1 plus x n e n, because again you see scalar multiplication is valid, then vector addition I am using and so from here, you see the, you are only get the component x 1, because all of this will have 0 entries, here in the first place. Then from the second one, I will get x 2 e 2, I will get x 2 in the second place and all other vectors have 0 in the second place and so on. So, this is what you have, therefore, I have found for you quickly spanning set and you can also show that, so then, this is this, that is the set e 1, I did not, but it is under student sometimes, I, me forget, but this is the vector.

So, this is a spanning set for in the spanning set for R n; now, is it the smallest one, it is the question. So, i will show you that since or what I will try to say is that, this is a linearly independent set and therefore, it has to be the smallest or maybe that is not the good enough answer; I will have to show you in another way thatthis will be the smallest set.

(Refer Slide Time: 22:50)



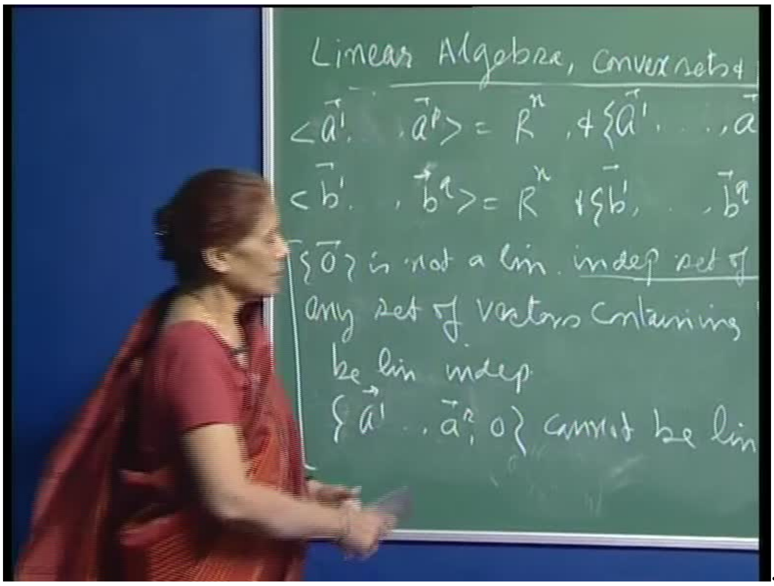
So, that is the question and maybe I will just not prove the result, but so, is a spanning set, the set e 1 to e n is also linearly independent and that is easy to see, I got the same reasoning, because if I say that some linear combination of these vector, suppose is 0, then since this will become an n tuple, x 1, x 2, x n, so this has to be equal to 0 vector; so, there, the, that will imply that each of the x i's are 0. So, the set is also linearly independent easy to prove, I will say it is easy to prove, because if you take any linear combination of these vectors, you put it equal to 0, then in a component wise, these scalars will turn out to be zeros and so you have done. So, therefore, this becomes a linearly independent set.

Now, of course, again, if we are depend upon mathematical rigor, then it is not enough to say that, this is the smallest set, what I will show you is that, suppose, before that let me talk of the and then I will give you some example replacement theorem. So, replacement theorem says that, if let say a one or let say q is a linearly independent set, is a linearly independent set and some a belongs to this linearly combination; a 1 a q, that means, a is a vector which can be expressed as a linear combination of the q vectors and a is this, that is, a can be written as some c 1 a 1 plus c q a q, because a is a linear is the, is a part of this, therefore, I can find scalar c 1 to c q, such that a 1 can be written in this way.

Now, if c r is not 0, c 2 is not necessary that all the c i's are non 0, some will be negative positive zeros, see if c r is not 0, then a can replace and that is why the replacement theorem, a can replace, a can replace a r, a r and the new set, the new set which is a 1 a r minus 1 a r plus 1 a q, so this new set, is also linearly independent; so, the new set is again linearly independent.

So, now using this concept, that means, what we are saying here is that, if I express a vector in terms of a linearly independent set of vectors, then I can always replace the vector here in the base, in, in the, in the set or which the corresponding coefficient is non 0; so, all I need is that the, if for example, c 1 is non 0, then I am saying that a can replace a 1 and the new set, that I get will again be linearly independent. So, would you will have to expect this without proof, because in any case, the proving part is not so important, it is a results that will be using.

(Refer Slide Time: 27:35)



So, this is it, now using this idea we what we can says that, if there are two spanning sets, so I just now told you that, we can have, we have a spanning set for R n and it is also linearly independent; now, what we want to say is that, this is the precise representation for R n, which means that, I want to say that, if say for example, a 1 to a p, so this is, this is R n, which means that, there are three vectors which form a spanning set for R n and the vectors a p are linearly independent are linearly independent.

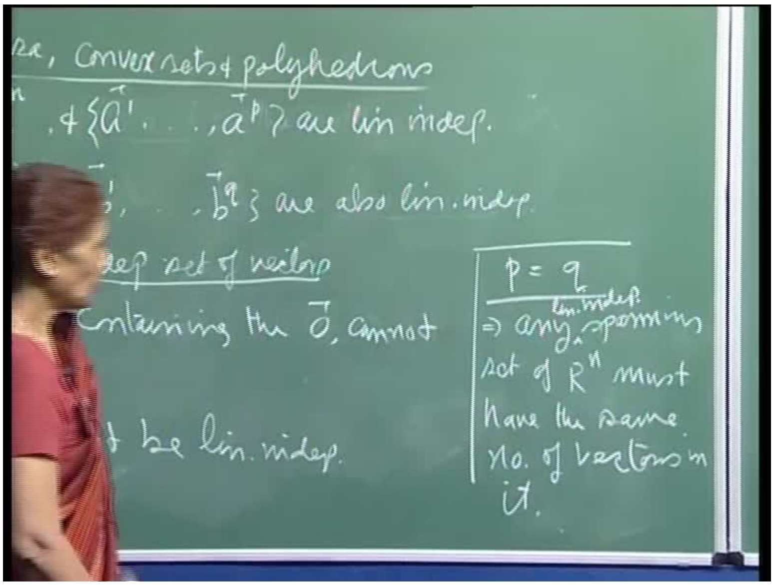
Similarly, suppose there is the another set, which is also a spanning set for R n and the vectors b 1 to b q are also linearly independent. Then using the replacement theorem 1, at a time find, what I can show you is that, again here we will not go into the a proof exactly, what we can show is that, you see for example, take a 1, I will just give you a little idea about the proof, that I take a 1 since this is the spanning set, a 1 can be expressed as a linear combination of these vectors, because this is a spanning set for R n; so, any vector in R n can be written as a linear combination of these vectors and then they will be certainly some scalars which should be a non 0, because a 1 is a non 0 vector why, because remember, when I, when I introduce the concept of linear independence, maybe I should I appointed out, that 0 alone is not a linearly independent vector set, vector set of vectors.

By itself 0 cannot form a linearly independent set of vectors and in fact, any set of vectors, containing the 0 vector cannot be linearly independent, cannot be linearly independent, yes clear, because if you have the 0 vector, if you have the 0 vector, then what I can say, you take any set, let say a 1 to a r and you take the 0 vector. This is not, I am not do not want the, this is, this cannot be linearly independent why, because you see it is remember we said that, they should be, if there is any linear combination of the set which is equal to 0, it must imply that all scalars are non 0, but here you see what can I do, I can, I can choose c 1 to c r as 0, but the coefficient for 0, I can choose as 1.

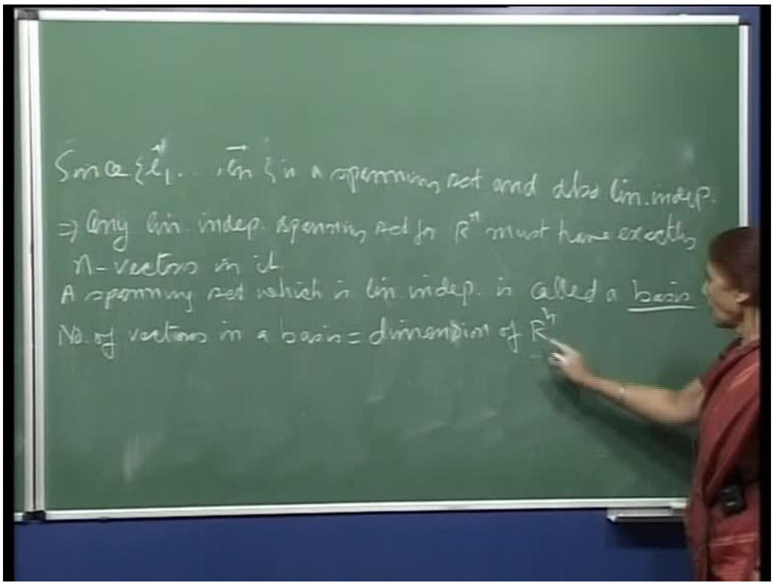
So, then I have a collection of non 0 scalars, that means, I have a collection of scalars in which at least 1 of the scalars is non 0 and the linear combination will give you the 0 vector; so, this cannot be linearly independent. So, i am using this concept, so this was, you now just a dig ration I should have mention this earlier fine anyway. So, what we are saying is that, here if you take the vector a 1 and this, since this is a spanning set, a 1 can be written as a linear combination of these vectors, they will be at least 1 scalar which is non-zero and so, a 1 can replace the corresponding vector here.

So, similarly, I can go on and with this process; I take the second vector, I can replace 1 of the basis vectors here and so, if p is less than q, I am not going to write it down, because I just want to explain this, if p is less than q then you see, you will run short of because you will have a set of linearly independent vectors here and then you will have some extra work ones, which must by definition be a linearly independent set; therefore, p cannot be less than q and similarly, you can start the replacement process from here and you will show that q cannot be less than p. So, what we have shown is that, p is equal to q; so, by verbal argument, I have shown that p is equal to q, using the replacement theorem we have been able to show that, p must be equal to q, which implies that any spanning set, any linearly independent, let me say here any linearly independent spanning set of R n must have the same number of vectors in it, vectors in it. So, right now may be the argument has been shown to you through lot of hand waving but does not matter, this can be found in any standard text. So, what we have said is that, if there are two different spanning sets and the number of vectors in this different, then you, it will contradict the definition of linear independence of one of the sets.

(Refer Slide Time: 33:04)



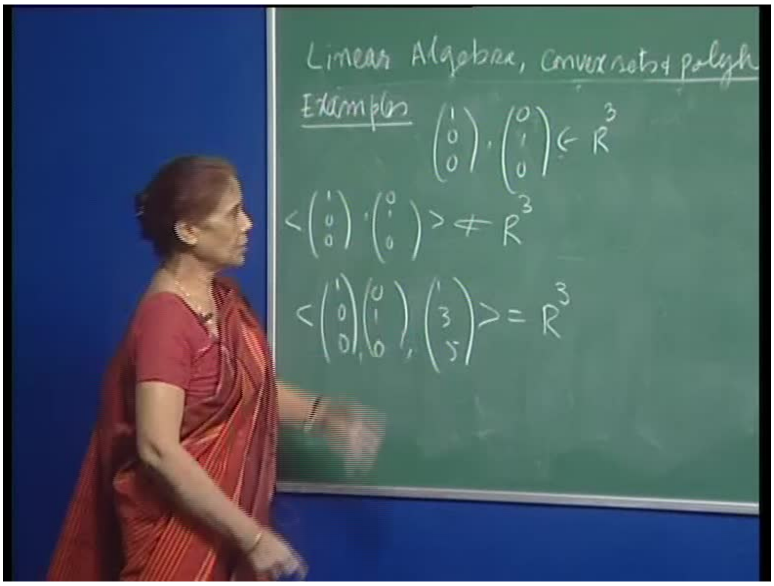
(Refer Slide Time: 33:57)



So, the number of vectors in any two spanning sets, which are linearly independent must be the same and so, this is what is it and I already showed you, I already showed you a spanning set for R n, which is also linearly independent and which has n vectors in it, e 1, I have been using this, e 1 to e n is a spanning set and also linearly independent.

This implies any linearly independent spanning set for R n must have exactly n vectors in it, exactly, because I already have 1 with n vectors, therefore, any other spanning set which is linearly independent must have n vectors in it. And we say that spanning set which is linearly independent is called a basis, called a basis and the number of vectors in a basis, is the, is the dimension. I mention, so I will just take off from, where we number of vectors in a basis will be the dimension of R n; R n is arbitrary, therefore, what we are saying is that, whatever n will a l, therefore, the number n now has important meaning and that is, it will always refer to the dimension of the vector space that we have, we have taken R n with this structure of vector sum and scalar multiplication R n becomes a vector space of dimension n. And we have said that, term any basis, any spanning set which is linearly independent for R n will have the same number of vectors in it.

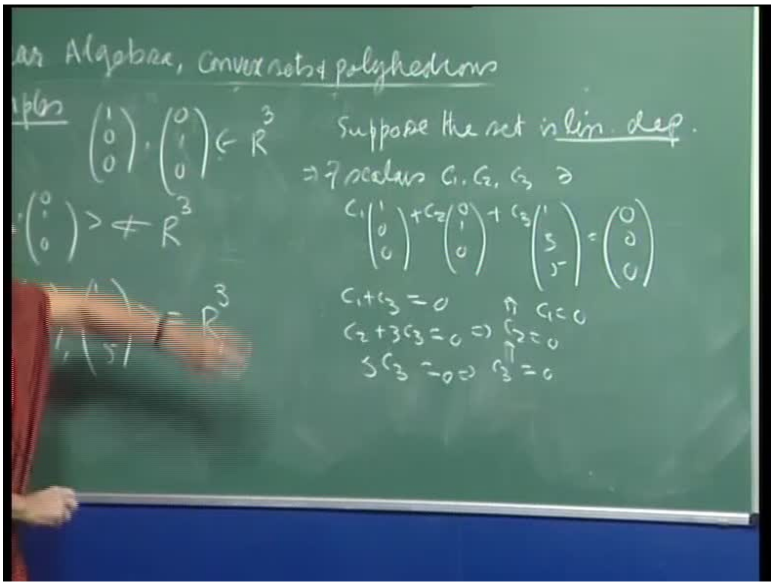
(Refer Slide Time: 36:25)



So, let us just look at the examples to illustrate the concept, that I have talked about here and say see if you can immediately, see that if you take, let say these two vectors, the this belong to R3, then obviously these two vectors cannot form a spanning set, that is the vectors 1 0 0 0 1 0, if you take all possible linear combinations, this is not equal to R 3, because what is happening, if I take a linear combination here of these two vectors and add them up, then the second third component of the vectors in this subspace will be 0 which is not exactly R 3, because the third component in R 3 has to be any arbitrary number; so, this is not equal to R3.

If I take say this and any arbitrary vector here, that these takes the 3, since 5 is present here and so, this combination, you can show is R 3 and what do we need to show here, we need to show three things, two things, that this is the spanning set for R 3 and it is linearly independent.

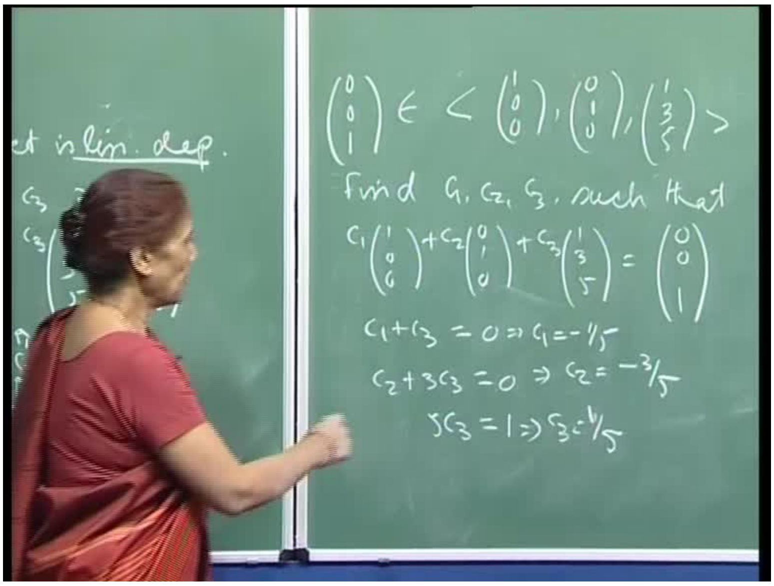
(Refer Slide Time: 37:52)



Now, you can immediately check, here for example, how will you check the independent? So, suppose the set is linearly dependent, this implies there exists scalars c 1 c 2 c 3, such that c 1 1 0 plus c 2 0 1 0 plus c 3 1 3 5 is the 0 vector, let us see, we will try to find the solutions here and see if c 1, c 2, c 3 are non zero or not.

So, the first, if you look at the first component and compare, then you get that c 1 plus c 3 is 0 and from here you get c 2 plus 3 c 3 is 0 and from here you get 5 c 3 is 0; so, this implies c 3 is 0, if c 3 is 0 from here, you get c 2 is 0, which, so if you go back upwards then, this implies that c 2 is 0 and this implies that c equal to 0, therefore, the three vectors are linearly independent. Now, if the three are linearly independent, they have to form a spanning set figure it out yourself.

(Refer Slide Time: 39:55)

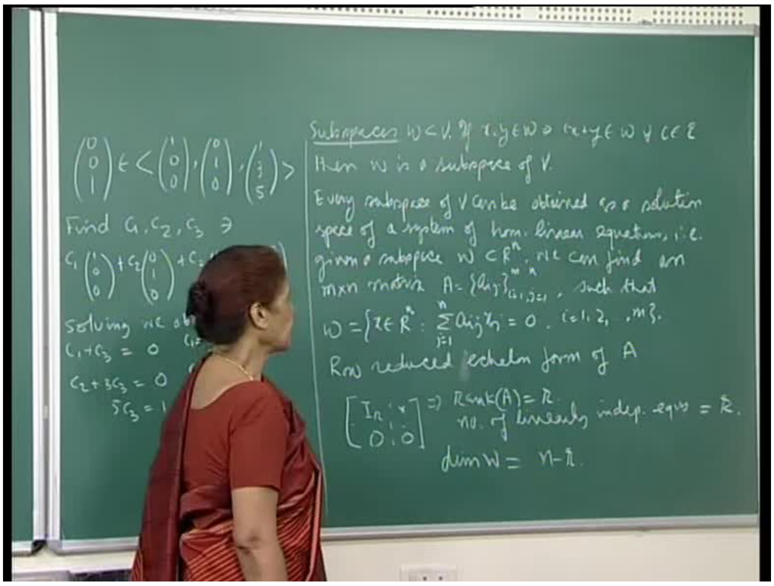


So, is a simple to you can argue it out, that if I have, because I know that dimension of R3 is 3. So, any spanning any linearly independent spanning set must have three vectors; so, here, if I have found already three vectors in R 3 which are linearly independent, they must also form the spanning set, that means, this is the basis for R 3. And now, if you want to look at the concept of replacement theorem, let me give it to you and I will revisit the concept of subspace all, now, for example, look at this vector 0 0 1, therefore, this belongs to the span of; so, how do we find out that, sure I mean, how it, how do we find out the linear combination as we know, because this is the basis for R 3, so any vector in R 3 should be expressible as a linear combination of the basis vectors.

So, we want to say that, find c 1, c 2, c 3, such that, yes such that c 1 1 0 0 plus c 2 01 0 plus c 3 1 3 5 is equal to 0 0 1, I, what to express this vector as a linear combination of these three; so, again if you compare component wise you get that c 1plus c 3 is 0 and from here c 2 plus c 3 is 0 and 5 c 3 is equal to 1, because the contribution from here is 0; so, 5 c 3 is 1 this implies, that c 3 is 1 by 5. So, go back, this will imply that c 2 is minus 3 by 5, yes and from here give this implies that c 1 is minus 1 by 5; so, just verify that this, what will get c 1 is because the c 1 plus 3 is 0 c 2 plus 3 c 3 is 0, by choosing these values and this.

So, you have this linear combination c 1, c 2, c 3's are non-zeros. So, then by our replacement theorem, we have that, this vector can replace any of the three vectors and you get a new basis, to get a new spanning set which is linearly independent; so, in fact here I can get because all c 1, c 2, c 3 are non-zero; therefore, I can get three different basis by replacing, for example, I can replace 1 0 0 by this, I can replace 0 1 0 by this and this.

(Refer Slide Time: 42:50)



So, this is what you give, you feeling of and of course, we are just need to be very familiar with the concept of linear independence and dependence and the concept of basis. The few lectures, before I had find subspace for you and the idea was that, if the two vectors in the subspace W, when c x plus y is also in W for all c belonging to r, then W is the subspace of v, this was the definition I had given to you some time ago.

Now, I would like to give you an alternate definition of a subspace and which is the very convenient way of looking at a subspace and that and that is, that every subspace of v can be obtained as a solution space of a system of homogenous linear equations; so, that is, that is given a subspace W, suppose I am talking of a subspace in R n and we can find an m by m matrix in a, such that you know the components are i, j, such that W can be written as all x which satisfy these linear equation, linear homogeneous equations m of them and so, the solutions space now, of course, we have also seen that when you take a system of homogenous linear equations, then this forms a subspace in R n, if I am talking of in terms of n variables then, because the sum of two subspace a two solutions will also be a solution then in scalar multiple of a solution, will also be a solution, so that means, that condition will be satisfied, that for all x y and w c x plus y is also in w for any real number c.

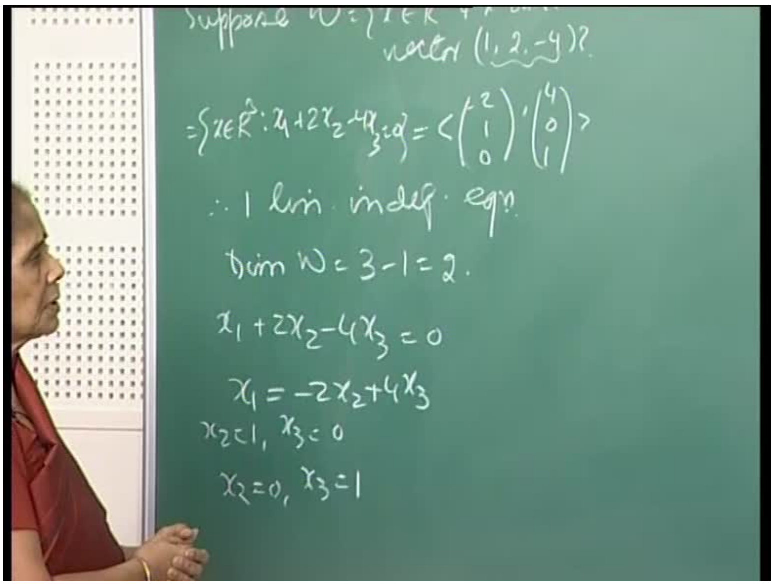
So, therefore, the solution space here is a subspace, now, the of course, the whole idea is that, how do you go about finding the matrix corresponding matrix a and of course, they can be more than one such system of homogenous equations, for which the solutions basis w, so all those things are there, but we will not going to that, but I just want to give you an idea and so, of course, the way you go, then I want to relate the, because when you write down these equations, then you see the usual way of solving this system of equations, is that, you row reduce the a matrix, you find the echelm form row, reduce echelm form of a and because elementary row operations on the, on the equations will not change the solution space; this also, you know, from your elementary linear algebra and so, here, if I, you will be able to reduce the matrix a, in this form, where this is a r by r unit vector, unit matrix sorry and these m minus r rows, these are m minus r rows which are all zeros and you have some non-zero entries here; so, immediately you can say that, the rank of the matrix is r, that means, the number of linearly independent equations is r and then we know that, because you have these, this identity matrix here and you will have x 1, x 2, x n.

So, immediately you will be able to, you know fix the, because the remaining see here you will take the first variables x 1, x 2, x r, then x r plus 1 to x n and you will be able to fix the r variables, x 1 to x r in terms of the n minus r variables. And then, so, the number of the and therefore, the number of linearly independent equations are r and so the number, the I should say that, the number of linearly independent solutions, here would be, would be n minus r why, because you are expressing these r variables in terms of then minus r variables, which are again, if you use a language, we say that the n minus r variables are free to take any value. And so, the r variables get fixed in their values, corresponding to the n minus r variables which are free and so then, I can find n minus r linearly independent equations solutions here and so the dimension of w and this is the relationship, I will try to demonstrate it through some examples.

So, dimension w is then n minus r, because I can find n minus r linearly independent solutions and they will form the basis for w, because then any solution in w, any vector in w is a solution and can be expressed as the linear combination of the n minus r linearly independent solution, that you will find by giving a some values to the free variables the n minus r free variables.

So, this is the idea. Now, suppose I describe just take a subspace w which is described as follows, that this is collection of all vectors in R 3 which are orthogonal to the vector 1, 2 minus 4, I am writing it is the row, but it is actually a column vector in R 3 but does not matter, because of you know, sometimes you do not know want to take so much space, so you write them as I mean this is common.

(Refer Slide Time: 47:20)

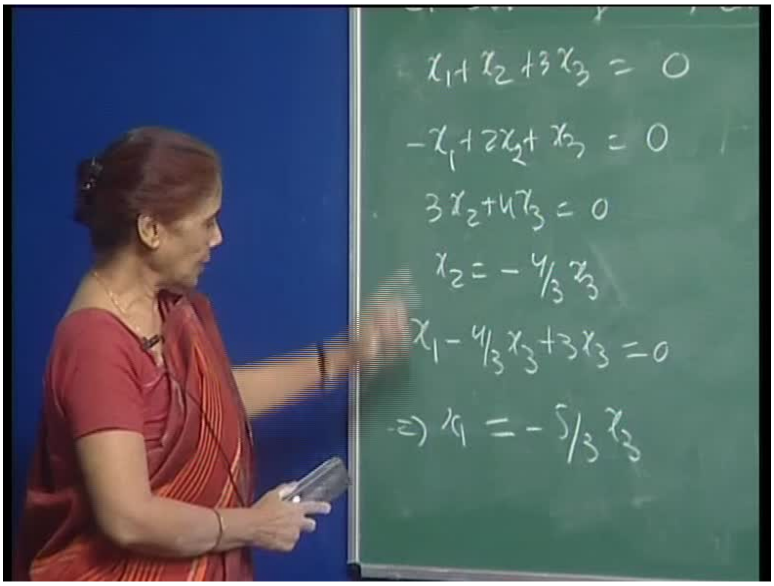


So, this is my subspace, my this description of the subspaces of collection of all vectors in R 3 which are orthogonal to this vector, which means that, I am looking for, so the description of w, now you see is immediately through once linear homogenous equation; this is x 1 plus 2 x 2 minus 4 x 3 0, because any vector x 1, x 2, x 3 which is orthogonal to this, must have its dot product with the vector equal to 0; so, x 1 plus 2 x 2 minus 4 x 3 is 0.

So, all x which satisfy this equation are in w; so, this is what I am trying to say that, you had one description of the subspace here, but then you can also give a description of w as a solution space over system of homogeneous linear equations and then what we see is that would see from here; this is what I want to explain that, if you have x 1 plus 2 x 2 minus 4 x 3 is equal to 0, this implies that suppose I want to fix x 1, this is minus 2 x 2 plus 4 x 3, so what I am doing is, I am putting x 2 equal to 1 and x 3 equal to 0, so that give me one solution, so x 1 becomes minus 2, so I immediately get the solution minus 2 1 and 0 because x 2 is 1, x 3 is 0. Similarly, if I put x 2 equal to 0 and x 3 equal to 1, then I get the second solution because it will be 4 and x 2 is 0 and x 3 is 1.

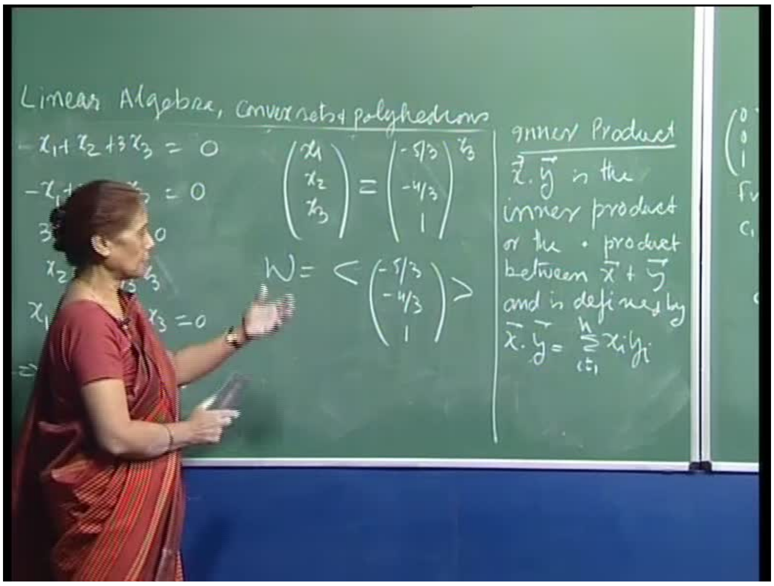
So, you see that, the whole solution space of this subspace, w is the linear collection of all linear combinations of these two vectors, it is a subspace and since you had one linearly independent equation here, describing w; therefore, dimension of w is 3 minus 1 equal to 2, and so, may be some more examples I will try to give you feeling, that how you, of course, this is a little going a little away from the topic, but essentially, I just want to make you feel comfortable with the concept of dimensions and so on and therefore, I am spending a little time here, trying to show you that how you can look at sub space in terms of as a solution space of system of linear equations and then derive the dimension of the sub space from there. Let me just spend in time on this, what we have saying is, see you have learnt this, that say x 1 plus x 2 plus 3 x 3 is equal to 0 and minus x 1 plus 2 x 2 plus x 3 is equal to 0.

(Refer Slide Time: 50:19)



How do you solve this system 2 x 2? You have learnt this method of elimination, that is the right word method of elimination, so say for example, here I add up, add up the two equations what do you get, 3 x 2 plus 4 x 3 is equal to 0, which gives you that x 2 can be written as minus 4 by 3 x 3; so, x 2 can be written as 4 by 3 x 3, now I substitute here the value of x 2, then I get this, this gives me x 1 minus 4 by 3 x 3 plus 3 x 3 is 0, which implies, that x 1 is equal to, so here 3 3 is 9 which is 5 by 3, so this is minus 5 by 3 x 3 . So, therefore, I have been able to in this system, I have two equations, three unknowns x 1, x 2, x 3 am considering the system of equation in R 3 and you see that, here the best I can do is, I can determine x 2 in terms of x 3 and one in terms of x 3, because these two equations are given.

(Refer Slide Time: 51:55)



So, this another way of saying that, you have the number of degrees of freedom for this system of equations is one, because only one variable is free to take any value and correspondingly, I will, the values of x 1 and x 2 get fixed and so I have a solution; so, let me give you the general solution here, because these concepts of dimensions and so on, have to be understood clearly. So, here for example, what is a general solution x 1, x 2, x 3 a solution for this system is can be written as minus 5 by 3 x 2 is minus 4 by 3 and this is 1 and I multiplied by x 3 see.

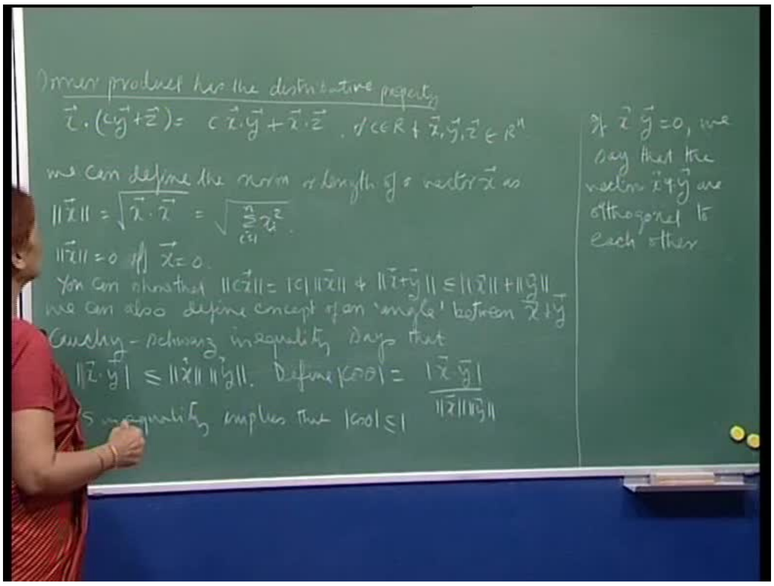
So, this solution space to the system of equations can be written as a scalar multiple of this vector, you see that and so the combination, that means, my, if I write w as the solution space, then this is the linear combination of the vector 5 by 3 4 by 3 and 1 so any solution and of course, you know that, this you already know, that if you have a system of equations had been using the word it is a solution space, that means, I have been calling it a vector space and we know that, if they, if x 1, x 2, x 3 is a solution and y

1, y 2, y 3 is a solution then the sum will also be a solution any scalar multiple of this solution vector will also be a solution; therefore, this is a sub space and here I have shown you that, if the system of equations, if the dimension or if the rank or whatever you want to call it a number of or you can say that the number of equations is linearly independent.

So, if the number is m and you are talking in R n and the solution dimension of the solution space is n minus m; so, in this case, it is 1, why is it 1, because the spanning set contains only one vector, it is non-zero vector; so, it has to be linearly independent, this is another thing, see a single turn set containing a single turn non-zero vector will also been linearly independent. So, the whole solution space can be represented as a linear combination of scalar multiple of this vector right.

So, I think, this, the subspace is there, yes, now I think, I can just equip something more to R n to make it a more useful space and that is inner product, which again you have been inner product up to R 3, you have been already using this concept of inner product. So, we say that two vectors x dot y, so it mean two vectors x and y, I define this, this is called is the inner product or the dot product or the dot product between x and y and is defined by equal to summation x i y i, where i vary from 1 to n.

(Refer Slide Time: 55:15)



So, component y is product and then add up, so the inner product is defined as this, the inner product has the distributive property also, by which I mean, that if you take the dot product between the vector x and c y plus z, where c is a real number, any real number, then you can, when you can open up the brackets and you will have c x dot c x dot y bar plus x dot plus z z bar and this is true for all x y z in r n and all seen in r.

So, the distributive property is there, when, with the help of the inner product, we can define the norm or the length of a vector and it is done by taking the inner product of the vector by itself and then the under root, so the norm or the length of vector x is given by this number and you can show immediately, because you have real numbers and sum of squares of real numbers, if it is 0, then each number must be 0; so, that means, norm x 0 if and only if, this stands for if and only if, short form i f f, if and only if x bar is 0, if the vector is 0, the norm is 0 and if the norm is 0, the vector must be 0.

Now, you can easily show using this definition, that if you take the norm of the vector c x, c is a real number, then this can, this come out as absolute value of c, that means, the positive part, because remember I should have said here also, that this is greater than or equal to 0, because it is, you talking about of real numbers and then taking the under root, so that must be a non negative number.

So, here this has to be absolute value of c times the norm of x t and then this triangle in equality which also you can sit down and prove by yourself, that norm of x plus y will be less than or equal to norm x plus norm y, that means, if you take a triangle, if you take two directions with the vectors x and y and length norm of the x and norm of a y, then you can show that the third side must be less than or equal to the some of the two sides the length of the third side.

Then, we can also have a concept of an angle here and that is the Cauchy-Schwarz inequality helps us to define concept of angle and that says that, if you take the absolute value of the dot product between two vectors and that must be less than or equal to the norm of product of the lengths of the two vectors and then with the help of this, we can define the angle cosine theta, so the absolute value of cosine theta would be this number divided by inner product absolute value divided by the two norms. You see, therefore, in other words, if you just want to define theta, it will be equal to x dot y divided by the norm x norm y and so, this is less than or equal to y, because of the, a Cauchy-Schwarz inequality, I have written that, here so Cauchy-Schwarz inequality implies that, the angle, the absolute value of the angle will less than or equal to 1; so, it make sense to have a concept of an angle here also in n-dimensional, you built in space and then we see immediately say that, if this dot product is 0, then x and y are orthogonal to each other.

So, through this, these definitions we have try to show you that, the structure of R 3 that you are very familiar with goes through to R n, most of the concept. So, and therefore, in, you can, R n you can feel very familiar with the R n space and the whole idea here and of course, in the next lecture, I will continue with some more definitions and you know introducing concepts, which you will need like convexity and the polyhedral and so on and you will see that, with the, with this back ground, it, you will have a better in site into the simplex algorithmic and so, this is the whole idea that, because essentially the algorithm develop through geometrical concepts, I always feel that, it is important that you must be able to understand, while certain things were done and why the algorithm took particular shape; so, it will help you, if we go through a bit of the mathematical concepts that are needed for understanding the algorithm.

So, I set out to give you concepts of linear algebra, then would also want to, so I think in this lecture, we have almost covered everything, that I think will be needed from linear algebra and of course, hope fully, if there are something left out, you can supplemented through books or I can come back to them later on. So, the next lectures, the next lecture I will try to now introduce the concept of convex sets and polyhedrals which are very essential for giving us a geometrical picture of the simplex algorithm that we want to develop and analyze for solving linear programming problems.