**Linear Programming and its Extensions**

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**Module No. # 01**

**Lecture No. # 32**

**Min-Cost-Flow Sensitivity Analysis Shortest Path Problem Sensitivity Analysis**

Let us just recall from the last lecture. I was talking about the shortest path problem and we discussed Dijkstra's algorithm and Dijkstra's algorithm required the c i j’s to be non-negative. Then, we also discussed the Floyd and Warshall’s algorithm, which could take care of c i j’s being non-negative being negative also and actually the Floyd-Warshall algorithm gave you shortest distance between a pair of nodes; every pair of nodes.

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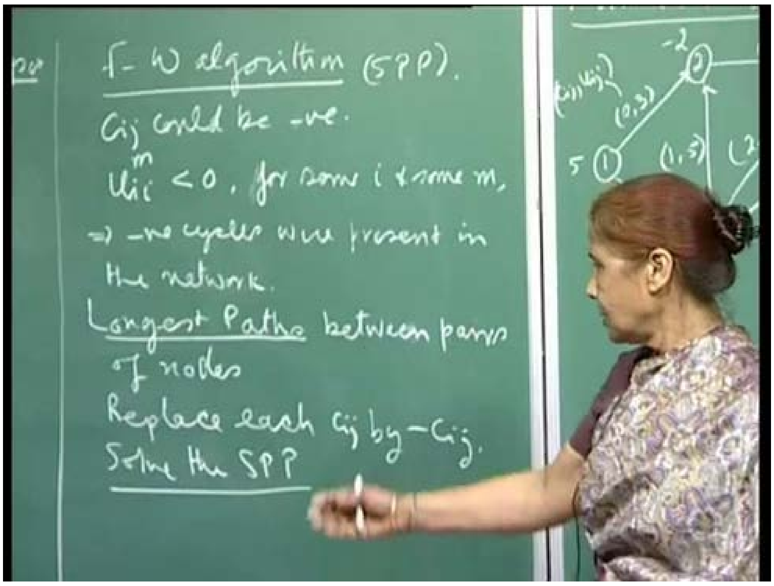


And I had showed you that, if for the Floyd-Warshall algorithm, Floyd-Warshall algorithm yeah this is when this is a shortest path problem and we put we said that c i j could be negative also; could be negative. So, in that case, I have showed you that if u ii m became less than 0 for some i and some m, then this implied that negative cycles were present in the network and therefore, shortest path lose their meaning because I can keep going around the negative cycle and reduce the distance between any pair of nodes.

So, therefore, the moment you discovered that there is a negative cycle present in the network, you stop computing the shortest path distances. And this is simply said that for some node, that means the diagonal entry because i to i, the path from i to i including notes upto from 1 to m minus 1 that became less than 0, then you said that there is a negative cycle present. So, this will happen for some i and for some n upto n plus 1 because that is the number of iterations for n, for the Flyod-Warshall algorithm. So, now, suppose you… So, then there is another problem related to shortest path problem and that is the longest path problem.

You also want to and we will discuss the problem sometime later, in which we really have to compute longest paths between between pairs of nodes. So, here is the longest path, but again, here also you see that if if the longest path problem, if the network has a positive cycle present in it, then the same argument applies that I can go on increasing the longest path distance, but traversing the positive cycle as many number of times as I wish to.

So, the longest path problem also is feasible, if we can assure that there are no positive cycles present. So, now, to look at it… So, that means, if yeah. So, what we will do is replace each c i j by minus c i j. So, if you replace each c i j by minus c i j, then solve the shortest path problem; then, that means first replace all the c i j's by minus c i j and then you look for the shortest path between the specified pair, and the shortest path computation will give you the longest path. So, again, same thing here that in case while solving the shortest path problem, we discover there is a negative cycle. It would imply that respect to the c i j's at corresponding cycle have positive length. And so, we will again say that the problem is not feasible.

So, therefore, here, depending on the others if your all c i j's are non-negative, then the minus c i j will be all negative. So, you will have to apply apply a Warshall algorithm for the corresponding shortest path problem, and then whatever results you have for the shortest path problem will apply here and then you can translate them to the the longest path problem. So, that is nothing. Then, the other aspect that we want to look at now is, since we have discussed the min-cost flow problem, we have discussed the shortest path problems. So, I want to now look at the sensitivity analysis of the post optimality analysis for this problem. So, for the shortest path problem, the only thing is to change the c i j between for a particular arc. So, here for example, now, the thing is that the moment you lose the linear simplex, the simplex algorithm structure, you you the the post optimality analysis becomes… you know you have to do more effort and it is not as smoothly done as you do it for the simplex algorithm.

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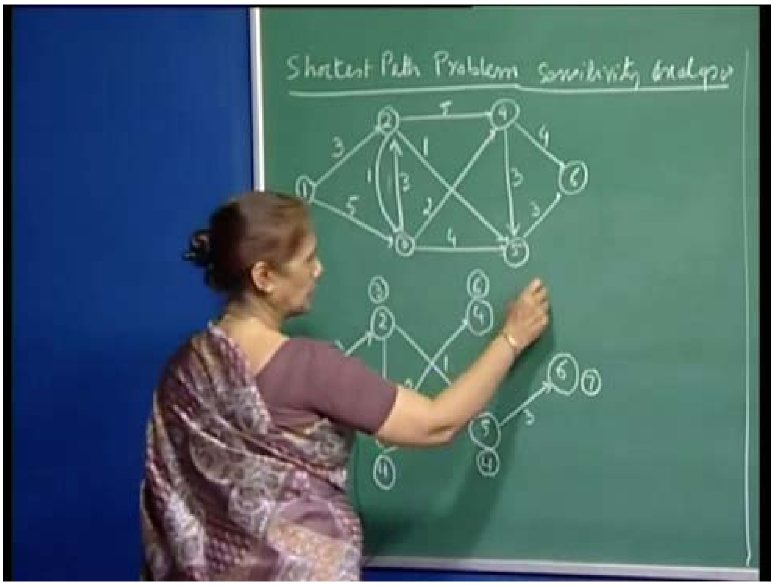


So, here, for example, yeah let me yeah. So, here, let us say look at... So, we had we had at some time ago discussed this problem with you, the shortest path problem, and this was your tree of shortest path; tree of shortest path, we had computed this from node 1. So, this gives you, like for example, 1 to t the shortest path distance is 3; from 1 to 3, the shortest path length is 4 and so on. See sensitivity analysis for the shortest path problem and I was trying to show you. The idea here is that if you change the c i j when r i j is not present in a shortest path, suppose you want to look at the how about kind of changes we can make in the cost c i j for an arc ( i, j) which is not present in the shortest path.

So, now, before I do this, I will have you recall that. See, in the 29 lecture, I had shown you that you see these w i values (Refer Slide Time: 07:01) which denote the shortest path length from node 1. So, then the corresponding dual variables because the shortest path problem we could formulate as a linear programming problem, and therefore the dual variables have a meaning here and I had shown you that the pi i’s would be minus of these w i’s; that means, pi i’s will be the negative of the shortest path length from node 1, but whichever node you fix and then you define your linear programming problem and then you get the dual variables. So, if this is this, then… and in fact, I would like you to, you know, using these pi i's then you show that these numbers satisfy that ; that means, this problem is actually an optimal solution to the shortest path problem.

By showing that for all arcs for non-basic arcs, that means arcs which do not figure in the shortest path, the optimality criterias satisfied because for the arcs in the paths - shortest paths see dual constraints will satisfy its equality.

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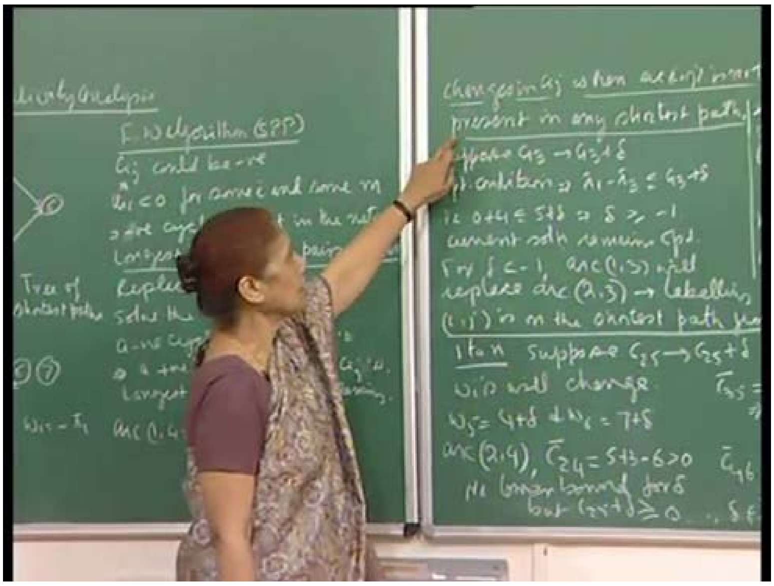

So, the first case we will consider is that of an arc which is not present in any shortest path. So, let us say, 1 3 is 1 of those arcs which is not present on any of the shortest path and let me consider the length going up by the amount delta. So, the current, the new length of the arc becomes 5 plus delta and the optimality condition is required.

So, remember, I had in the earlier lecture showed you that whatever the lengths of the shortest paths which are denoted by the circles numbers at each node, then the dual variables are negative of the length of the shortest paths. So, minus pi i is w i that means, pi i is minus w i. So, if you apply the shortest path condition when this is pi 1 minus pi 3 should be less than or equal to c 1 minus 3 delta. This is a new c i j or c 13, but. So, substituting pi 1 is 0 because w 1 is 0, pi 3 is minus 4; so, plus, it becomes plus 4 here

(Refer Slide Time: 09:00) minus 3 is w 4 and. So, this should be less than or equal to 5 plus delta which implies that delta should be greater than or equal to minus 1. So, that means the current solution will remain optimal. This will remain the 3 of shortest paths for all values of delta greater than or equal to minus 1, which is expected because as it is (1, 3) did not figure any of the shortest path. And if the length goes down even by 1 unit, even then it does not become a candidate for being one of the shortest paths. But in case delta becomes less than minus 1, then you can see for yourself arc (1, 3) will get replaced by (2, 3).

Of course, this is a small example, but essentially what will happen is that you will be you will have to start the labeling procedure again because remember we do not have the linear programming structure any more. We were because we to explode the structure, we have used the Dijkstra's algorithm, of course, this is the case when I am taking all c i j's to be non-negative. So, in that case, you will have to start your labelling again. But you can see away from here that if (1, 2) (1, 3) becomes, let us say, for example, delta is minus 2 and this would become…, this length will be 3 and the shortest path will be (1, 3) instead of (2, 3).

So, (1, 3) will become a shortest path and this is arc (2, 3) will go away and so on. So, but essentially, the moment one of the lengths changes, you will have to start your labelling procedure from where that arc can come in to the basis; can figure in the shortest path.

Now, similarly, let us now go to an arc (i, j) which is on the shortest path from 1 to n. Let me consider that case. Then, similarly, you can then see how to handle the case when (i, j) is in one of the shortest paths, but not on the path from 1 to n; different cases, but the treatment is the same. So, now, what happens is that when c to 5, for example, take the arc (2, 5) which is figuring on the main path, I mean shortest path from (1 to 6). So, (2,5) goes up from 1 to 1 plus delta. In that case, you see the shortest paths will change because if this becomes 1 plus delta, this length becomes 4 plus delta and this will become 7 plus delta; these paths lines will not be disturbed. 1 to 2, 2 to 3, and 3 to 4; so, it is only this becomes 4 plus delta and this becomes 7 plus delta (Refer Slide time: 11:12 to 11:28).

So, now, we will have to check for optimality conditions because we want to make sure that for what values of delta the current solution will remain optimal, and so, I have computed it, for example, for arc (2, 4) c bar 2, 4 will be 5 plus 3 minus 6; this does not change because your 2 and 4 have not changed. So, that remains optimal.

So, anyway, this is 8 minus 6 which is 2, which is greater than 0. So, no lower bound for delta. Why I am saying that here? yeahno nono not in this case (Refer Slide Time: 12:05) but this comes later fine yeah yeah yeah. So, this is c bar 2, 4 any way remains non-negative and then c bar 3 ,5 if you look at the arc (3, 5), then the optimality condition requires 4 plus 4 minus 4 minus delta because see this thing is the cost is 4; then minus pi 3 plus pi 3 minus pi 5. So, this becomes this (Refer Slide Time: 12:32). So, that gives you delta less than or equal to 4; similarly, if you compute c bar 4, 6 the number comes out to be this; gives you delta less than or equal to 3. Since both of them have to be satisfied, your delta has to be less than or equal to 3.yeah.

Now, from here, I am saying that since 2, 5 figures on your shortest path. Therefore, it will not hurt if the cost, if the c 2, 5 goes down and there is no limit; so, there is no lower bound, but again the algorithm that we have used here, the requirement is that also c i j's must be non-negative. So, the validity of the algorithm will be questioned in case I allow the c 2, 5 to become less than 0. So, therefore, I am saying, that logically does not require delta 2 to be a lower bound, but we will since we are using the algorithm which requires c i j's to be non-negative, we are going to say that delta should lie between minus (1 , 3).

So, in this interval, as along as delta varies in this interval your current solution will remain optimal; that means, your path length will be7 plus delta. So, anyway, from… So, let me say now this can go up to 10 or this can be 6. So, the shortest path length, this path will continue to the shortest path and the path length will vary from 6 to 10 as delta varies. And after that, if delta goes beyond 3 then this path will not any longer be the optimal path and you will have shortest shortest length path, I mean of a path lengths which means less than 10. So, this will happen.

Then similarly, then you can you can even question, you can ask a question; if arc 4 suppose the current network does not have arc (1, 4) and so, you may want to ask a question - suppose, it is considered that the new root may be added, and therefore, you would want to know when is it profitable to add the root the 1, 4. So, therefore, you will find you will want to find out currently the shortest path length; from 1 to 4 in your shortest path tree is 6, because these are the shortest path lengths from node 1; so, currently it is 6. So, immediately, you see that if you want to add this arc, if you want to construct this, then the cost the length of this root should not be… that means your c 1 4 should be less than 6 or less than or equal to 6. In that case, you will have an alternative optimal solution. But otherwise, you will have to again, if you want to add this root, then you will have to start your labeling procedure again, and then find out what are... So, you will u see 1, 4 as an unknown number and then by the labelling procedure will tell you as to or so, yes. You can you can do this way, but, immediately by looking at the shortest paths tree, you can find out what should be that cost of the length of the root 1, 4 so that it becomes profitable.

Let us say c 1 4 is equal to 5, then it will be worth considering to include the arc (1, 4) to construct the arc because your shortest path length up to 4 will go down. Of course, it does not figure this arc (1, 4) will not figure in the shortest path from 1 to 6, but you see again here I am trying to what I am trying to point out is that suppose c 1, 4 arc is added and the cost is 5, then you would have added this arc, and the path up to here is the path length is 5. Now, from here, you want to reach 6; then this cost is 4 (Refer Slide Time: 16:19). So, then 5 4 9; nine is not less than 7. So, therefore, a chain adding this root will not cut down on your shortest path, but you can compute, you can make this computation and find out if at what cost it will be profitable to add 1, 4; in the sense, that your shortest path length reduces.

Say for example, you can immediately make the computation here; if this is c 1 4, so, here, for example, direct roots. So, this will be c 1, 4. So, then if you add, this is 3 plus 3; 6. So, one root would be plus 6 and this root is plus 4 and then. So, c 1 4 2 plus 4 is the smaller one; this you want to be less that 7 and this will tell you that the c 1, 4 should be around 3 or less than 3; then it will be profitable to add. So, you can whatever question you want to ask, now you know how to answer that question. So, this is the whole idea; this kind of sensitivity analysis you can do.

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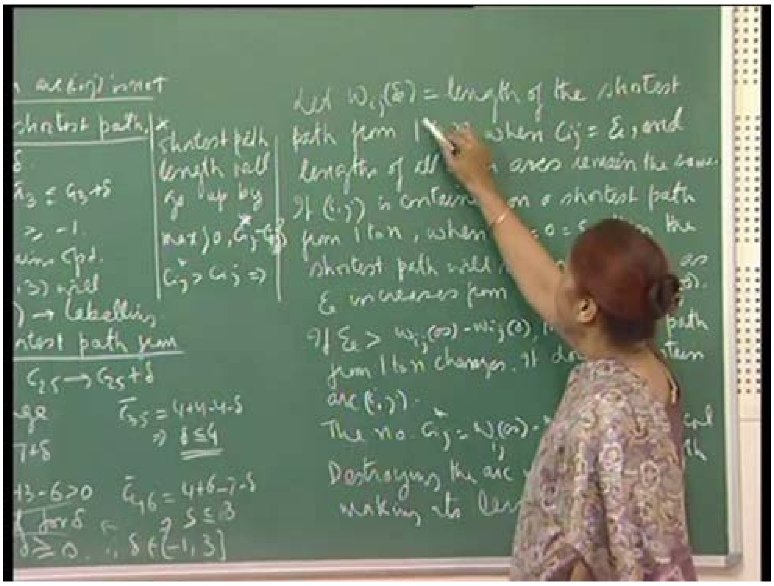

Then another interesting aspect of sensitivity analysis is that suppose now let me define w ij (psi) as the length of the shortest path from 1to n. So, remember, I am talking of the shortest path from 1 to n; I am not talking of other shortest paths because this has more meaning; otherwise, you can do the same analysis for any shortest path.

So, when c i j psi, so, suppose the current length is psi and so, I am defining w i j (psi) as the length of the shortest path and of course, all other arcs will remain the same. Now, if i j is contained on a shortest path from 1 to n when c i j is 0, then the shortest path will remain shortest as psi increases and goes from 0 to this number (Refer Slide Time: 18:04). And remember, when we say that the length of an arc has become infinity it implies that the arc is not there because obviously, you are talking in terms of…you have finite lengths. So, in shortest path length will be finite; so, any arc capacity going to infinity means that it is not part of the shortest path; either it is destroyed or it is not in the not so.

So, therefore, as long as the current path remains the shortest path from 1 to n, that means, that i j is the arc (i, j) is figuring on that shortest path. So, then the and as the capacity of the arc ( i, j) keeps going up, the length of the shortest path, just as here I showed you that from one to 6, the path length will go from 6 to 10 as delta varies from minus 1 to 3. But, after this, the current path will not remain optimal and that means, arc (2, 5) will go out of the shortest path and a new shortest path will come in. So, therefore, this path length, the psi can increase up to this number and then the moment psi becomes bigger than this number, the shortest path from 1 to n changes. So, it does not contain arc (i, j). So, this is the idea and we will define a c i j star as the critical capacity; critical length, critical length of arc (i, j) which is w ij (infinity) minus w ij (0).

So, destroying the arc is equivalent to making its length equal to infinity this is the whole idea and then. So, you want to ask a question - shortest paths length will go up by maximum of this (Refer Slide Time: 19:45 to 19:59); see if c i j is less than c i j star, then the arc( i, j) is present in the shortest path. Then, the max here will be this number; Because 0 is less than this number, this difference is positive. So, as long as c i j star is greater than c i j, your arc (i, j) can remain on the shortest path and that will be the length. So, the arc ( i, j) will remain on the shortest path. And the moment c i j becomes equal to c i j star which i said here, also moment it becomes this or c i j's bigger than c i j star, then this will become because this number will be negative and so, max will be 0. So, essentially your shortest path length will go up by this amount if c i j's less than c i j star.

So, then again, here you want to find out question that you can ask here is which is the arc which gives you the minimum increase in the shortest path when destroyed? Or the other question - when will it give you the maximum increase in the length of shortest path when it is destroyed? So, in times of strategy, you know war and so on other situations, you want to consider arcs. For example, if an arc is likely, if a root is getting very bad, leaves it pairs, then you have limited resources and then one root which needs repairs, then you would want to know which is the root which has to be repaired first; that means, if it get destroyed, if it is not used, then the length of the shortest path will go up by maximum amount. So, these are the kind of questions you would need to ask.

Say for example, if you look at the arc, non-basic arc (1, 3) then you do not expect that see. So, let us just complete, I will just do it for one this thing. So, here, critical critical length; so, you want to compute c 1 3 star; I will just give you one example and then you can compute the thing. So, c 1 3 star is equal to what? So, what is your w 0? So, when when this 0, you see, this is now is not figuring in the path 1, 3; it will not matter because… yes. So, c star 1 3 if you have c critical length for c 1 3 star, you see w infinity of course, s 1 3 infinity 7 because arc (1, 3) is not figuring in the shortest path and the current length is 7 from 1 to 6.

But when it is capacity zero, you see I have quickly computed the thing and you can verify if this arc length is zero. Then you see your shortest path would be 2 and 4. The shortest path will be 1 3 4 and 6 and the length of this path would be 6; so, that is w 0; and so, the difference is 1 unit, but, then max of this is 0,1 minus 5 because the current capacity is 5; length is 5. So, 1 minus 5 which is minus 4; therefore, this max this is 0.(Refer Slide Time: 22:52). So, essentially, what we are saying is that for arcs we should not figure in shortest path; there are change in the by…. the critical capacity would be something which is smaller than the current capacity of the…

So, fine, what I am saying is the this number would be negative and so, there will be no change in the length of the shortest path, but for an arc 2, 5 which is figuring in your shortest path, you see if this you can compute that when it is zero; when it is infinity when 2, 5 is not figuring in that, that means, its capacity is, length is very large; then the path will be I think 1 3 5 6 9, may be 11 or 12; I do not know. So, figure out 5 to 7; it will be 1 3 4 6; 5 and 2 - 7; 7 and 4 – 11; so, this is 11 because 2 5 is not figuring and then and w 0 is 6. Because this number is 1, so, it becomes 0. So, therefore, the capacity of the path will be 6 and so this number will therefore, this is this is equal to 5 and max is 4 (Refer Slide Time: 24:00).

And therefore, here, when the difference in this, the shortest path length, will the path lengths go up by 4 if the arc is destroyed? So, therefore, the question that one wants to ask here is why through this, through this analysis, you want to ask the question, find an arc which when destroyed increases the shortest path length by smallest amount.

And so, these are the kind of questions or the opposite questions also can be asked. While destroying an arc, why there is maximum increase in the shortest path and so, on. So, these kind of questions can be asked and so, the see. And what I wanted to point out here is that because we have not solved the problem, we have not by the labeling algorithm we have obtained these numbers.

So, therefore, what will happen is that when a label changes for a node, then you have to start your labeling algorithm from that point onwards because then the other labels will change. The computations before that, before this node got labeled; they will remain intact, but the other computations have to be made and this is what you will have to do for the Floyd-Warshall algorithm also because Floyd-Warshall algorithm any way cannot be formulated as a linear programming problem. Because it is a multi-objective problem you are finding shortest path between pairs of nodes.

So, there also, if a particular c i j changes, then you see for example, if this changes then you see there you are computing the numbers u i j m. So, then m is 4. So, here you are computing these paths when you are using nodes up to 3; 1 2 3 only. So, those will remain intact if this length has changed, then, obviously, the u i j 4 and 5 onwards will change.

So, you will have to recompute from that point onwards; this is the kind of thing you will have to keep track. And you see, the moment you lose the simplex algorithm frame work, then sensitivity analysis becomes… you know you have to do more work for it.

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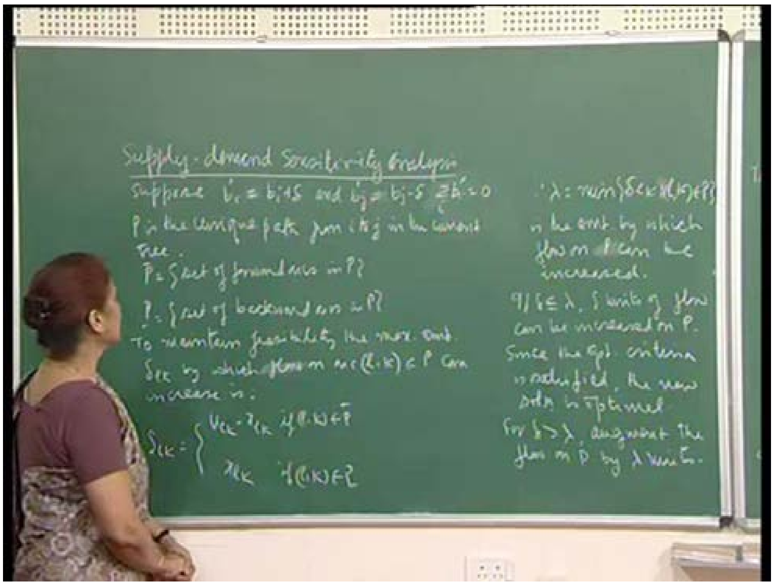


So, now, we will consider the sensitivity analysis - supply and demand; that is if the numbers here increase and decrease, when can we adjust the current optimal solution and take care of the changed demand and supply.

So, here, since we are working with the condition that your new supplies and demand should also again add up to zero, so, if I increase the suppose I am taking. So, delta has to be positive. So, if I increase the ith supply by delta amount, then some demand must also go up so; that means, the b j will be negative. So, minus delta will get added to it. So, the supply and demand will remain the same. So, let us just take some pair i j. So, also supply has gone by delta and the demand has also gone up by delta units. Now, since in the current solution, the tree that you have of the basic arcs, there will be unique path form i to j in the current tree; I should have said the current basis tree and some arcs will be forward arcs; that means, arcs which have the same orientation as from i to j and the other arcs which are denoted by p lower bar. So, p upper bar is the set of forward arcs in p and lower bar is the set of backward arcs in p.

And then, to maintain feasibility the maximum amount delta k by which flow on arc (l, k) can increase, you see now along this path you will try to because the supply has gone up by delta unit and the demand also. So, you want to see whether of this path p you can have increased flow. Now, what does that mean? It will require that. See for any path, for any arc in the path, for any l, k in the path, if it is a forward arc then its current flow must be less than the capacity because then only i can increase the flow; if the arc (l, k) is saturated, then there is no point; I cannot increase any more flow on arc (l, k).

Similarly, if it is a backward arc, it is a lower bar; it is not very clear. Let me make it p lower bar. So, if it is a backward arc, then it must have a positive flow because then I will increase the flow in the opposite direction, which means that the flow on the arc (l, k) will decrease. So, if all arcs in p satisfies this condition, then I will say that it is possible to increase the flow from i to j and by how much amount so that delta will be the minimum.

So, that means, the augmenting capacity of each arc on the path p, I will compute by this and then I will take the minimum because the minimum number of all the delta k's is the amount by which I can augment the flow or increase the flow on p. Now, if delta, so, therefore, delta lambda is the amount, has the augmenting capacity you can say of the path p, now if delta is the less than or equal to lambda, fine; because we want to increase the flow by delta and the capacity of the path allows you to do it.

So, then I can increase the flow and since the optimality criteria continues to be satisfied, I have not disturbed my basic feasible solution. Therefore, the current solution is optimal. Of course, the cost will go up because you are having increased flow on the network. Now, the problem comes when your delta is greater than lambda.

So, then of course, you will just increase the augment the flow on p by lambda and now you have to worry about the rest of the remaining flow delta minus lambda. So, the question is to see flow can be augmented by delta minus lambda units. And here we will now need to because the current solution you will say is not feasible and so, you have a situation for a dual simplex pivot because your optimality criteria is currently satisfied the basis is satisfying it, but the basis is not a feasible solution.

So, now we start say looking at how to implement the dual simplex pivot here. And again because of the simplified structure, you see that the pivot dual simplex pivot also gets simplified now, first of all because the capacity on arc (p, q) too have flow more than lambda units. So, there will be a blocking arc; in sense that blocking arc as I said it will either be saturated; that means, after augmenting the flow by lambda units on p, a forward arc; so, for p, q the flow plus lambda became u p q, and so, it got saturated. Therefore, I cannot increase the flow any more in arc (p, q). And if it was a backward arc, then the flow that you reduced on the arc was equal to the current flow on the arc, and therefore, it became zero now. So, therefore, I cannot use (p, q) as a back word arc also. So, therefore, b p q is a blocking arc.

So, we will drop arc (p, q) and we will try to see if we can change the basis or the basic tree and get another root from i to j so that I can have increased flow on that root or on that path. Now, once you drop an arc (p, q) remember, in expanding tree.

If you drop the arc (p, q) when the tree gets divided into sub trees, then let me call them t 1 t 2; disconnected- it becomes disconnected because spanning tree is a minimal connected tree. So, the moment you drop any arc from the tree, spanning tree, it becomes disconnected. So, have two portions - disconnected portions t 1t 2. s 1 are the nodes of t 1 and s 2 are the nodes of t 2. And suppose i belongs to s 1 and j belongs to s 2, then see what is happening is that s 1 s 2, the nodes sets this is a cut in the sense that and I am just later on, of course, we will have more occasion to use cuts heavily, but now here what I am saying is that a cut set means that it is the partition of the node set. So, node set. So, s 1 union s 2 equals the node set and they are just joined.

So, this is how you define a cut and the cut set. The cut set will be all arcs which are either from s 1 to s 2; that means, l, k such that l is in s 1 and k is in s 2 or its arcs going from s 2 to s 1; that means, l is in s 2 and k is in s 1. So, this is all the possible cut sets;

that means you have your partition the node set. Your i is here; your j is here; I am saying this is s 1 this is s 2 (Refer Slide Time: 32:30); then you have arcs are the going from this to this or you have arcs going this way.

So, this is all and therefore, you can see that any path from i to j will have to use one of these arcs; either going from here to here or from here to here. So, this is what we are saying; any path from i to j will contain an arc from the cut set s 1 from s 2 and this is where from we will try to now locate the arc so that i can have a path from i to j.

Few things are not being explained in detail, but later on when we do the max flow also you will hopefully get a better feeling for this. So, now, any way, the idea is to look for a a valid arc from among these should have positive capacity left on it and backward arc must have positive flow. So, these are the apart from p q; p q became a blocking arc. So, I need arcs from here to here which have positive capacity of from here to here (Refer Slide Time: 33:31). So, it is a positive capacity forward arc or it is a backward arc with positive flow on it. So, among all valid arcs choose that one which has minimum of c bar l, k; that means, a relative cost and this is your dual pivot, dual simplex pivot.

Now, please just stop here for a while and see. Why I am saying this because remember for the network problem we had shown that your y i j's are either plus 1 minus 1 or zeros. And once there is a dual simplex pivot, the element that you choose must have a corresponding y i j negative; so, we do not need the ratio; the minimum ratio will actually the minimum of the c l k bar absolute value.

So, this is the arc which will be. So, once you choose a valid arc to be satisfying this condition, then you see our optimality condition will continue to be satisfied. So, then what we will do? If you have a valid arc, then among all the valid arcs we will choose the one which has a minimum relative cost, absolute absolute term and then we will try to flow as much arc as possible on the path from i to j.

So, what you can say that if I have an arc like this, then see the flow from i to up to this point is there then since this has positive capacity i will increase the flow on this (Refer Slide Time: 34:53). So, the total m n come up to j point right. So, you can then increase the flow on this path finding out the capacity of the path the way of I have told you earlier. And again if there is still some left over, you will do the same thing; you will again find a blocking arc and try to find which is the arc which is there is a anymore valid arc or not.

So, if you ultimately cannot find any valid arcs and you still have some left over flow, then that means, this is in feasible; that means, I cannot allow for increase of delta units here since I am not able to accommodate delta units in the network and entries the flow, but if your dual simplex pivots allow you to keep in increasing the flow, you will go on doing it as long as you can and then stop if you do not have any invalid arc. So, this will this is the process we will use to find out and I will take up an example to explain the various steps that have told you.

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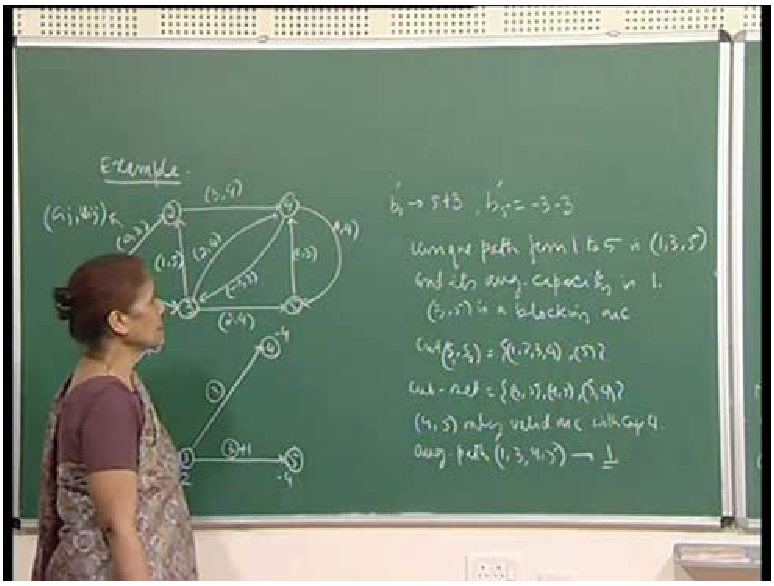


So, let me now show you for the min-cost flow problem. This is what we are considering now, the sensitivity analysis for the min-cost flow problem where the supply and the demands have been changed.

So, this is the network flow problem; the first number here is the cost and the second number is the upper bound - the flow in that particular arc. So, and of course, oh oh I have not shown the demands and the demands and the supplies could also be shown this is number 5 and this is 3; that means, 8 units are available here and this is minus 2 minus 3 and minus 3. So, the sigma b I's; sigma b i's add up to 0, i varying from 1 to 5. So, this is what you will have and this is….

Now, optimal solution you can see that the of course, this is the later step, but 5 units are leaving node 1, the supply point and since these two are demanded. So, two are reaching here; then here for example, three units are coming here and then three plus six; six units must go out. So, from here four units are going three units are going to four and three units are going to five which are that respective demands of the two nodes. So, this is and you can check for optimality; this is an optimality solution.

Now, if we want to consider in the case that b one goes up to eight units; that means, three more units are available here and at five, node five, you have three extra demands. So, six. So, again they increase in the supply is also equal to the increase in the demand.

Now, you want to consider, how I can accommodate this flow in the current solution so that I take advantage of the optimality criteria being satisfied. So, as we said, we will find out a unique path from one to five which is currently 1 3 and 5 and find out its augmenting capacity. Now, in this case, both the arcs are forward arcs. So, you can see that here, the capacity is three units because the capacity is for this arc is 6 and here the capacity is only 1. Therefore, the augmenting capacity of the path 1 3 5 is 1 unit because I cannot see. I may increase the flow by three units here, but then on the arc 3 5, the flow will not be able to send three units from 3 to 5; only one more unit.

And that is why you see. So, the augmenting capacity of the path is 1 and then (3, 5) becomes a blocking arc right.

So, anyway, I will then augment the flow here. So, that means, this becomes four and this is also four and now you see the (3, 5) is a blocking arc (1, 3) still has some positive capacity left. So, no problem; 3,5 becomes a blocking arc, and as we said we will then say that if I drop this from the tree what is the cut I will get. So, if you drop this, you see the nodes 1 2 3 4 are on one side and one belongs to that this subset and then this is your other set s 2, s 1 s 2. So, your cut s 1 s 2, as I was saying, this is the partition of the node set and node 1 belongs to s 1 and node 5 belongs to s 2 and then, the cut sets are the possible arcs, as i said, from either s 1 to s 2 or s 2 to s 1.

So, you then this network you can see because this is your cut, this is your cut and you can see that 3 5; of course, we are going to drop then the backward arc is (5, 4) and the forward arc is (4, 5). These are the three arcs that you have; this is not valid 4 5; four five has no flow; on it, there is a positive capacity. So, therefore, this is a valid arc (( )) it is a forward arc; (5, 4) is a backward arc since there is no flow on (5, 4). So, again this is not valid. So, this is what we meant by saying that (4, 5) is a valid arc; it is the only one. So, therefore, I cannot. Really if we had more than one valid arc, I would have chosen the one with the minimum relative cost, but here, I have no choice. (4, 5) is the valid arc; the capacity is 4 for this arc. So, now, you look at the augmenting path because you see 4 to 5. So, you have to have a path from 1 to 4 and then 4 to 5, and it turns out to be the path is 1 3 4 5.

I am talking about this sensitivity analysis here, but later on also, when we do max flow, hopefully, you will have a better understanding, as to how to go about finding an augmenting path or a path with positive capacity, and so on. So, we will not spend a necessary time here. Here, when we talk of max flow, the things will become clearer. So, anyway, the augmenting path is 1 3 4 5. So, small network I can show you immediately by looking at the network, but otherwise, there are regular procedures to find out augmenting paths from one node to another.

So, now, the capacity of this augmenting path is 1 3 4 5. So, (3, 4) you are have a flow of three units; therefore, increase of one unit here, of course, we saw that the flow can increase by two units. So, you have to take minimum of (2, 1) and then comma 4 because this has augmenting capacity of four units. So, the minimum is 1 and therefore, the augmenting capacity is one unit of this path and I will increase the flow by one more unit. So, therefore, this will become plus 1 plus 1 and 4 5 which I have shown you here.

So, now, the augmented flow is 5 units here. This is 4 and this is 1. And now, immediately you see that the arc (3, 4) is a blocking arc because I need still to find out if I can send one more unit of flow from 1 to 5. So, so far, I have had I have augmented the flow by two units.

Now, when I drop the arc (3,4) from the tree, then you see 1 2 3 will be your first set of the cut and 4 5 is the second cut of second part of the cut, and then the cut set the cut set you will have only two arcs (2, 4) and (3, 4) because now your cut is this. So, (3,5) is already saturated and (2,4) is the one which has positive capacity because it is not being used and (3,4) is already saturated because I am dropping it from the tree. So, therefore, (2,4) again is the only valid arc. So, hopefully, in the assignment sheet I will be able to give you problem in which you have more than one valid arc and then you will have to make the choice by the amount by the relative cost of the arc.

So, then (2, 4) is the only valid arc and augmenting capacity of this path is again 1 2 4. So, the path is 1 2 4 5 yeah. So, this is the path 1 2 4 5 and the capacity here is 1. Here, of course, it is 4 and here it is 3 now, because you already have one unit of flow on it. So, therefore, the minimum would be one and so, 1 2 becomes the flow becomes… I have i should not do this because this is see this is this thing is that yeah I had a choice of making this non-basic arc or a basic arc or more of them, but any our this thing, this has been that when you are dropping an arc, then it becomes non-basic. So, like (3, 4) was a blocking arc. So, right now, I am using the criteria that it is not in the basis. So, it is a blocking arc at its upper block; it is a non-basic arc at its upper bound. So, we will maintain 1 2, even though at its upper bound and I will maintain it as a part of the basic tree.

So, therefore, one more unit of flow you have here and now you see that eight units are being sent from 1 and six units are coming to node 5 4 plus 2. So, the cost has gone up. I have not made a calculation of the cost, but the idea here is that you are maintaining optimality for the flow and you can check.

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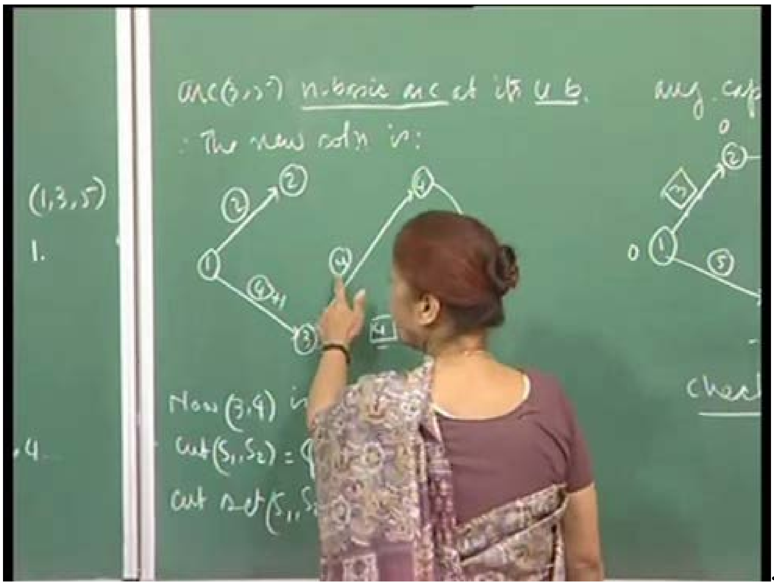

So, let us quickly check for optimality. Here, for example, what are the arcs that you need to for 2 3 or is it the arc is 3 2. So, here I have computed the dual cost; for example, for we are starting with w 1 as 0, then this because the cost for arc (1, 2) is zero. So, therefore, this will also be 0. w 2 will be 0 and then just make the calculation here for 2, 4. What is the c bar 2 4? It should be zero because this is a basic arc.

So, c bar 2 4 is 3. So, 3 minus 0 plus w 4 is 0, which implies that w 4 is minus 3, which I have written here. Then for (4,5) what is the cost for (4,5)? The cost is 1. So, 1 plus 3 minus w 5. oh why am I writing 3 minus 0 plus w 4? This is minus plus plus w 5 is zero. So, which implies that w 5 is minus 4. So, this is it.

And now let us quickly check for arc (2, 3) (3, 2). So, c 3 2 bar is 3 2 is how much? 1. So, 1 minus w 3; 1 minus w 3 becomes plus 2 and this will be 0. So, this is 3 which is greater than 0; fine and what is the other arc? The arc you have is 3 4 3 4 3. So, c bar 4 3 will be equal to c bar 4, 3. The cost is minus 1 minus 1 plus w 4; I mean minus w 4 minus w 4 is plus 3 and this will be plus w 3. w 3 is how much? w 3 is minus 2 minus 2 which is 0.

So, again, optimality criteria is satisfied for now. You have 5 4 c bar c bar 5 4 c bar 5 4 5 4 is 1. So, 1 minus w 5 which is plus 4 and then minus w 4 which is I mean plus w 4 which is minus 3 which is again positive; 5 minus 3 is 2 which is positive.

So, optimality criteria is satisfied for the upper bound, for you can check for these also the optimality criteria would be satisfied because the corresponding thing should be less than zero. So, the relative cost should be less than zero. So, this shows you that you know as I have been saying again and again, that because of the simplified structure, the things become much simpler to handle. You can program them easily and even the dual simplex pivot is easily handled and you can do this kind of analysis, the sensitivity analysis and you can find out if the increase in the capacity. The increase in the right hand side and the left hand side can be handled maintaining the same basic feasible solution and if not then you can very easily update the old solution to the one which can take care of the increased supply and demand.

And so, the next thing, the next part of the sensitivity analysis; see the thing is that working with doing this sensitivity analysis gives you a better feeling for the algorithm. So, now, here the next part that we have to consider is changes in the capacities which I hope to discuss in the next lecture, just to see if the u i j’s changes; then can sort of find out what are the kinds of changes which will permit the current solution to remain optimal. Because increased capacities would mean in some cases increased flow in the network and then you would want to, of course, make sure that the increased flow either helps you to reduce the cost; that will be the first priority and then, of course, you can see how other things can be handled; how can it be handled if the capacity goes up.