Linear Programming and its Extensions

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Lecture No. # 03

Moving from one basic feasible solution to another, optimality criteria

In the last lecture, we defined a basic feasible solution then, we also said that they can degenerate and non-degenerate basic feasible solutions. We found an upper bound of the number of basic feasible solutions that an LPP can have, and then finally we said that if there is a feasible solution to a LPP, then there always be a basic feasible solution. Actually, I showed you how we could construct a basic feasible solution from a given feasible solution.

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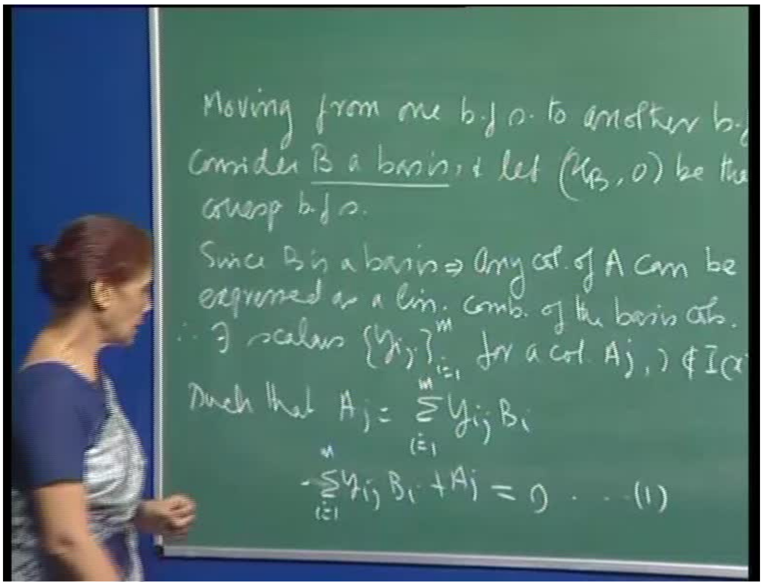

So, we further carry on the process of developing the simplex algorithm; so, the first result that I would like discuss now is that, how do you give a basic feasible solution, we want to move on to a better one in the sense that you objective function value should improve; so, first I will give you the method for moving from one basic feasible solution to another basic feasible solution, so moving from one basic feasible solution to another basic feasible solution.

So, this starts the this will give us the beginning of the simplex algorithm, the iterative procedure that we will follow for solving standard form of linear programming problem. So, suppose, let consider B a basis, and without having to repeat it I will just always be referring to the standard linear programming problem that I defined as problem 3 in the earlier lectures. So, consider B a basis, and let x B 0 be the corresponding basic feasible solution, so here again I have renumbered the variable the variables and the column, so that the first m variables correspond to the basic variables, and the last n minus m variables correspond to the non-basic variables.

So, corresponding basic feasible solution, now see our notion when we say that B is a basis, and this is the required mathematical background I will try to some up in one lecture which will come in the beginning of the scores, so that you have the definitions and the whatever mathematical background I am assuming in my lectures I will try to give that in the first lecture.

So, B is a basis which means that, since B is a basis this implies that any column of A can be expressed as a linear combination; this this concept I have used in the earlier lectures also but it does it does not harm to repeat it again and again the expressed the linear combination of the basis columns; so, this means that there exist, therefore there exist scalar’s y i j i varying from 1 to m, these are specific in the column A j (( )) there exist scalar this for a column A j, such that, j does not belong to I x or I could have said that A j is not a column in B so there exist scalars y i j for a column A j such that A j can be written as summation y i j B i i varying from 1 to m, so I expressed A j as a linear combination of the columns of this.

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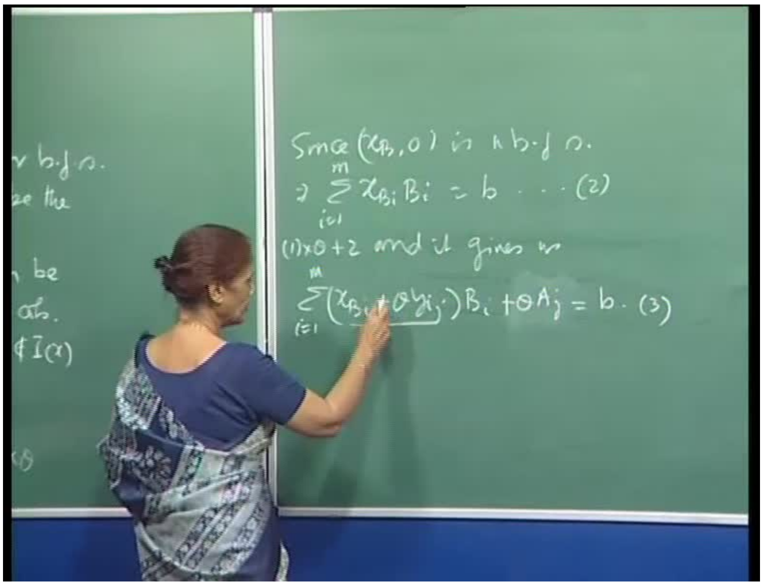

This I can rewrite this expression we can write as minus y i j B i summation i varying from 1 to m plus A j equal to 0; so, take this expression as 1, also this is the basic feasible solution, so I have that since x B 0 is a basic feasible solution, this implies that summation x B i B i i varying from 1 to m is equal to b the right hand side; and then I will do the same trigger that I did earlier I will multiplied this by theta, and then add to this equation, so this if I call this as 2, and I am looking at the expression one into theta plus two which gives and it gives me and it gives us summation x B i plus theta y i j B i, i varying from 1 to m is equal to b.

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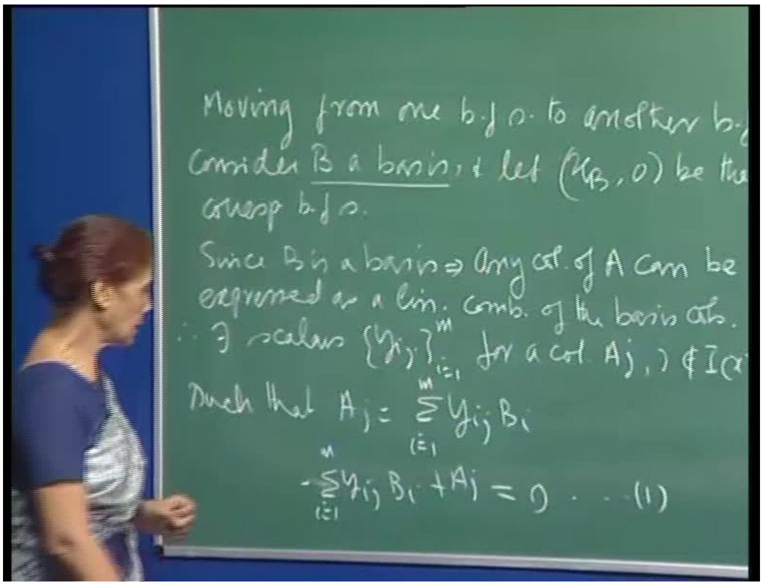

So, it is almost the same way of constructing; so, here I am using the fact that my A j is a linear combination of the columns of B, and therefore there is a set of scalars not all zeros, because if they are all zeros then A j would be 0, but certainly A j is not a 0 vector, so 0 column, therefore I have this, this scalar y i j exists, and then I rewrite this expression in this way multiplied by theta added to this equation and I get this.

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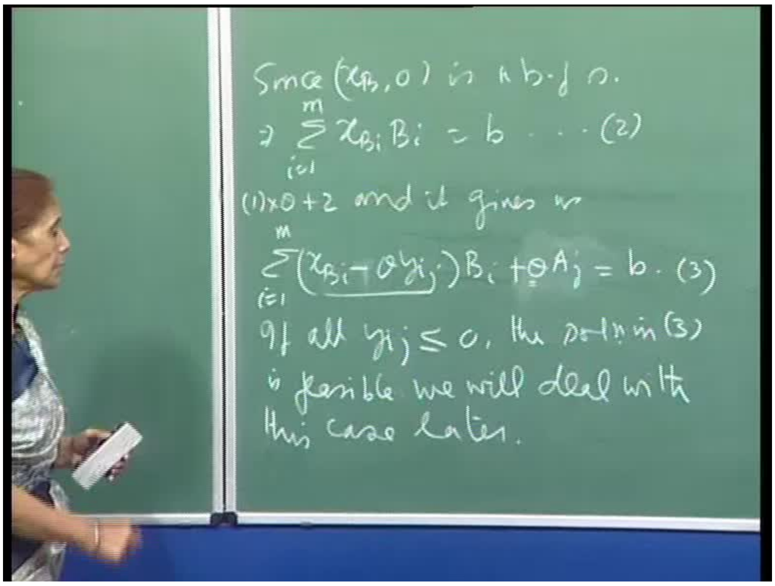

Sorry there is something missing, so I should have…, because there is a plus, so there will be p plus theta times A j then it is equal to b; let me call this as 3; now, you see that if we look at I have these scalars which are used for combination of the columns of B, and then I have theta here, so this could be called a solution, I am not calling it a feasible solution, because I do not know whether all these numbers are non-negative.

So, because that depends on theta and it depends on y i j(s), as so here, of course, let us say I specify that theta is positive or non-negative, because non-negative does not make any sense, because I am trying to construct a new solution; so, let us take theta to be positive in that case y i j(s) could be large of numbers, so that this expression need not be non-negative, x B i(s) are non-negative, theta is positive, but signs of y i j and the values of y i j they matter therefore; and since we trying to construct a new feasible solution, so I required that this these scalars must be non-negative, so here if all y i j(s) are…, now again please there was a minus sign here I could see therefore this has to be minus.

So, in any case y i j(s) can be both positive and negative; so, let us see if all y i j(s) are less than or equal to 0, then of course the signs of these numbers will be non-negative, and so this will be a feasible solution; if all y i j(s) are less than or equal to 0, the solution in 3 is feasible, it cannot be basic feasible solution, why, because I have m columns here, and I have another column here, so total number of columns is n plus 1, since the rank of the matrix a is m, it follows that m plus 1 columns cannot be linearly independent.

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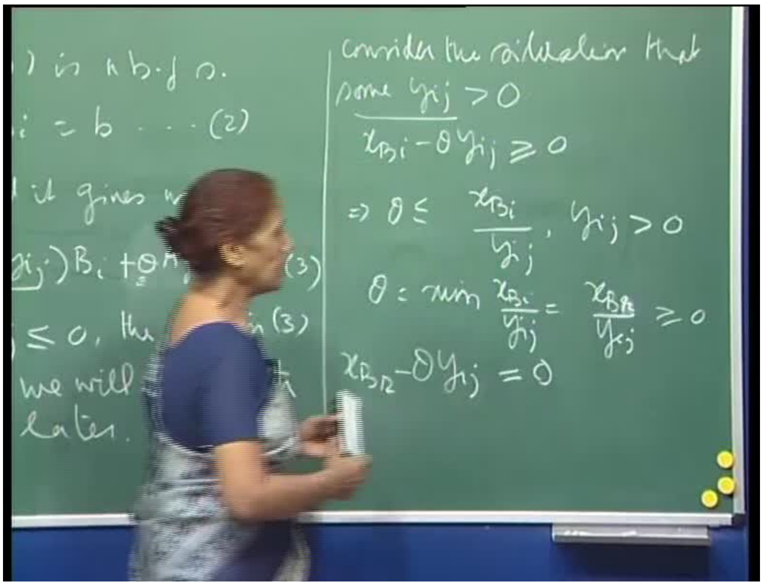

Therefore, this is at most if all y i j(s) are less than or equal to 0, the solution in 3 is feasible; we will deal with this case later.

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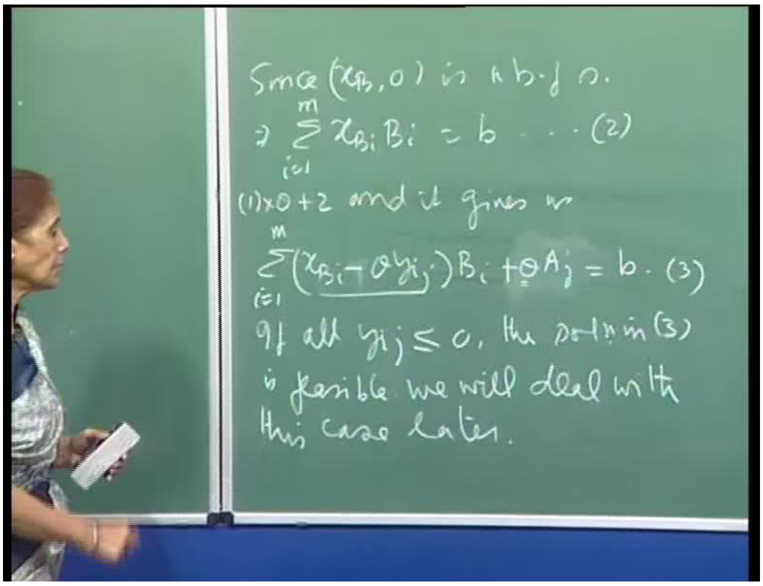

Now, let me take the situation or consider the situation, so consider the situation that some y i j(s) are greater, than 0 some y i j(s) (( )), at least one of them is positive right; so, in that case I required that x B i minus theta y i j should be non-negative, which implies the theta less than or equal to x B i by y i j, when such that, y i j is positive; for negative ones I do not have to worry about the sign, it is only about whenever so y i j is positive, I have to worry about the sign of this term, and so I put a condition on theta that theta must be..; so, in that case we say that, we choose theta to be minimum of x B i upon y i j, and let us say that this is equal to the x B r upon y r j.

So, that means, now, you see that the relationship, so theta, if I choose theta to be this value, then what happens to the value of this term x B r minus theta y r j; see, if I substitute theta equal to this, when you see this become 0, so what will happened is that your rth basic the new value of the rth basic variable, if I want to has become 0 right, and this theta value…, this is certainly I am choose this is…, this is non-negative, because I do not know about the value of x B r, it could have been a degenerate basic feasible solution, and therefore x B r could have been 0, but it does not matter whatever it is it is non-negative number, I am dividing by a positive numbers, so again the ratio is non-negative.

Theta is positive here; now, what I have manage could do by making this choice of theta, I have manage to make one of the basic the new value of the new basic variable equal to 0, so it becomes non-basic, and non-basic variable originally A j, because j was not A j was not a basic column, but now I have given it a value theta.

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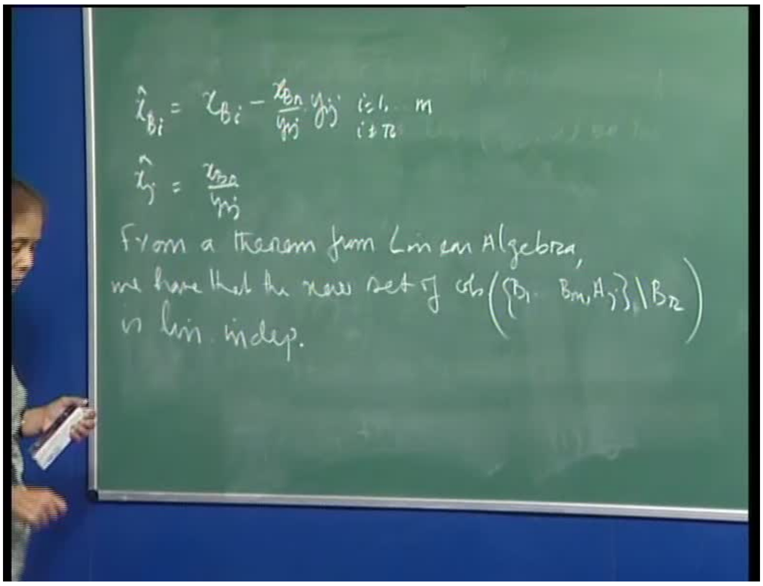

This is how we are now use to reading the solution the coefficients, which are the coefficients of the columns, which you use in expressing the system of equation, they become your solution; so, the new solution if you look at it, this is let me call it x hat, so x hat i or x hat B i, because again this is same index, this is equal to x B i minus theta time or let if you want me to write the value here of theta, this is x B r upon y i j into y i j, this the new value, and you see that this is valid for i varying from 1 to m i not equal to r.

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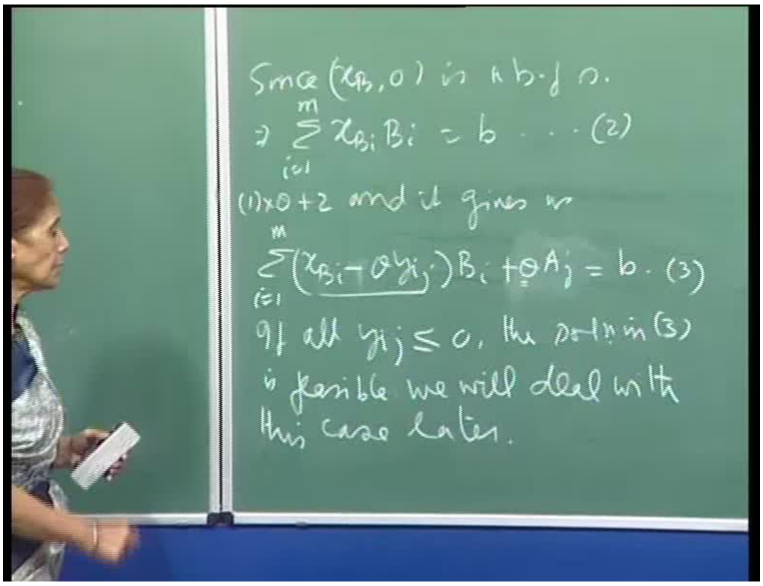

Because for i equal to i, it be it is 0; and you have that x j hat is your x B r upon y r j; so, you have m minus 1 positive or non-negative variables values here, and you have another non-negative value here, so you have m of them. Now, of course, since our idea is to move from one basic feasible solution to another, so obviously we would want that the corresponding columns here should be linearly independent; and this again is the theorem from linear algebra, we have that the new set of columns that the new set of columns and this new set is actually B 1 to B m, and A j and then minus B r, so this is the new set the columns B 1 to B m, but B r is missing, and then A j set of columns is also is linearly independent.

The theorem in linear algebra says that, if you have a set of linearly independent columns which forms a basis, then I express non-basic column in terms of the basic columns, and then I can replace one of the basic columns by the new column provided that the coefficient used in expressing A j as a linear combination of B 1 B 2 into B m is non-zero.

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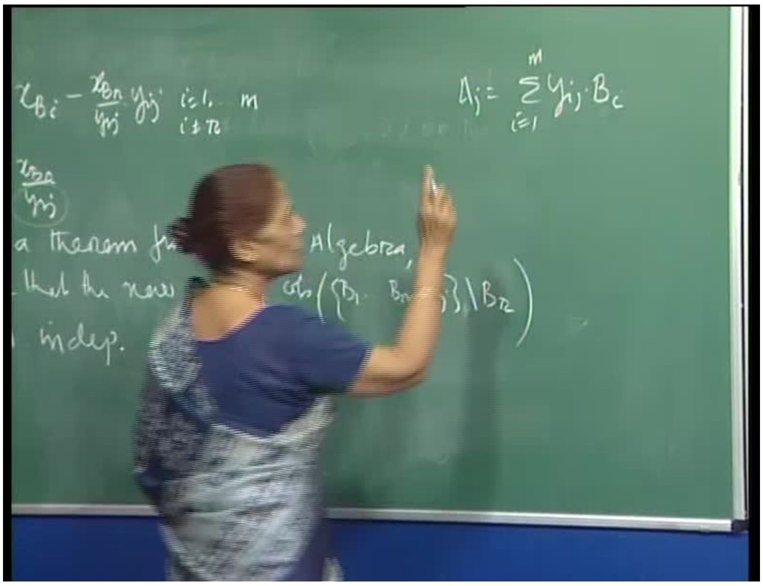

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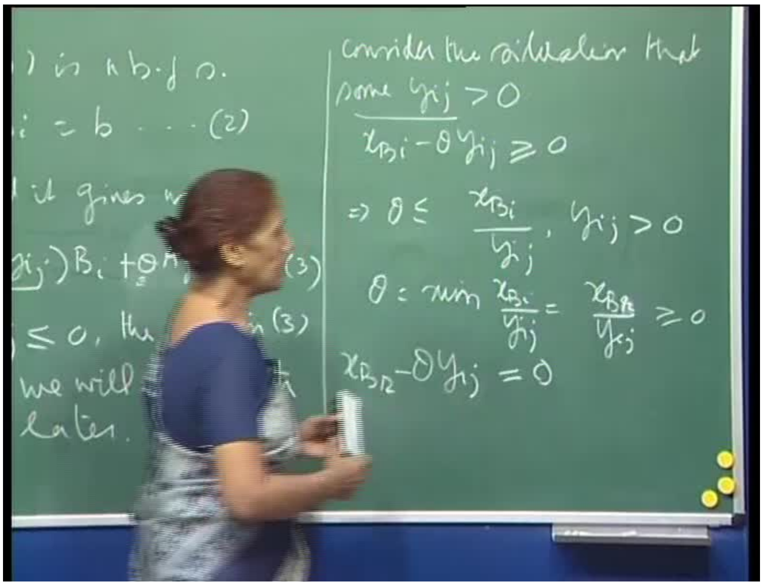

And remember what was our y r j, our y r j was the…, because I said that when you express A j as a linear combination of the basic columns, then this is…, this I be refer to 1 to m, so y i j is the coefficient of B i, so y r j is the coefficient of B r, and we assume that y r j is positive, so we did this ratio for all y i j(s) which are positive therefore, and took the minimum, so y r j is positive number; therefore, linear algebra theorem says that if you replace the corresponding basic column here by A j, such that, the coefficient used here is positive, the a is non-zero, linear algebra does not have do anything with the sign of y r j, it is here that we are concerned with the sign of y r j.

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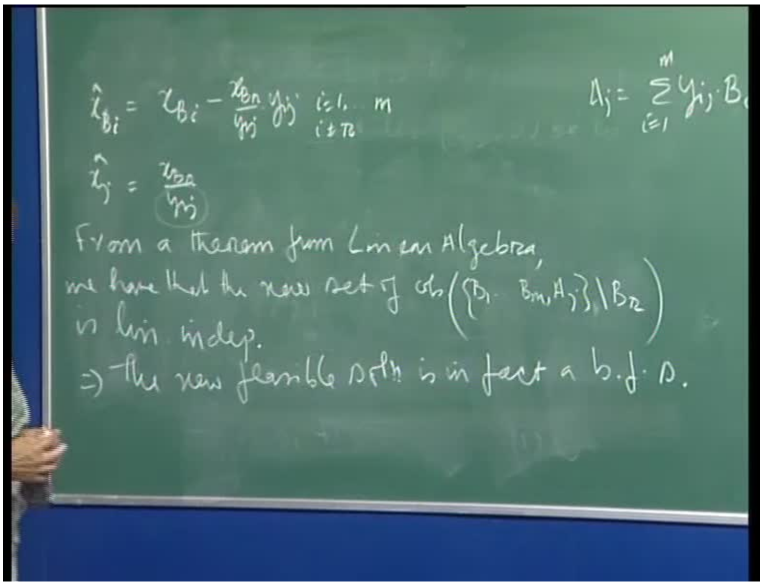

Because if y r j is negative, this number will become negative right or non-positive, because x B r is non-negative, so we want to go from one basic feasible solution to another, therefore we need the sign of y r j to be positive; so, then this is a linearly independent set, which implies that the new feasible solution is in fact a basic feasible solution; so, we have a achieved what we set out to do, that is to derive to construct the basic feasible solution from a given basic feasible solution, and the at that what we do is, we replace one of the basic columns by one of the non-basic columns.

Now, this is not enough, because we want to improve the value of the objective function, so we want say that we should have…, so we need two things we need two say whether given basic feasible solution will give us the best value or not, if it does not then I should be able to move from that basic feasible solution to another such that the value of the objective function is improved.

And for the third question that some of you it may have come into your mind is that, why am I concentrating on basic feasible solutions, and that question will get answered may be after two more lectures or third that 4th of this lecture whatever it may be; so, right now I am just constructing I am developing the theory to basic feasible solutions short while it will be clear why we are doing that. Now, the next move is to move from one basic feasible solution to another, and in the process improve the value of the objective function.

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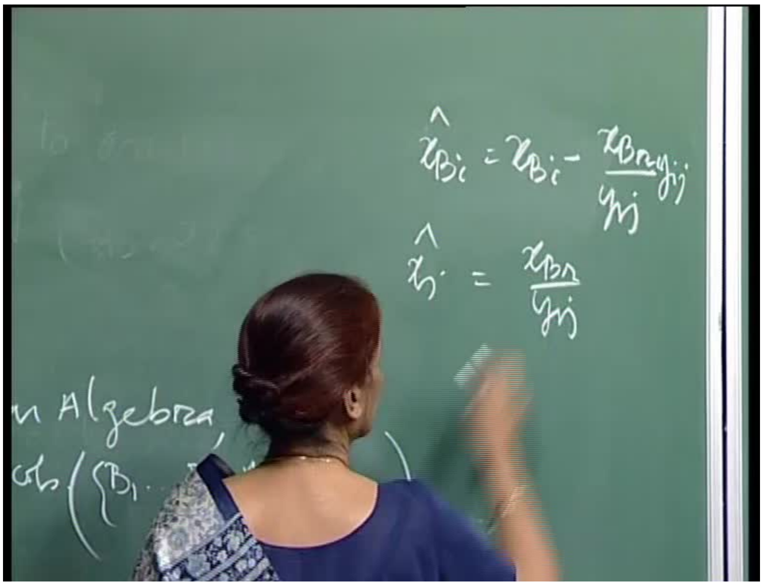

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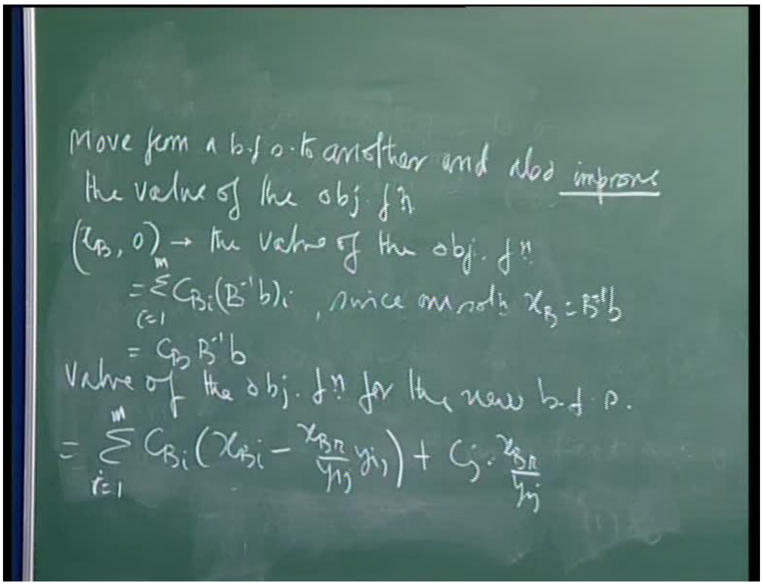

Let me just keep it up on this things now our new basic feasible solution that I obtain, so this x B hat high is x B i minus x B r upon y i j into y i j, and x j hat is x B r upon y i j move from basic a feasible solution to another and also improve the value of the objective function, so there key word here is now improved; and we can quickly recall that, I had a basis, I replace one of the basic columns by a non-basic column, then the and added this by making the value of the new value of the basic variable corresponding to the column, I am replacing 0, so that it gets the status of a non-basic variable, and the non-basic variable I made the value is positive, so that it becomes a basic variable, so this is what we are doing.

So, this is your new feasible solution; for the original one, for the basic feasible solution x B 0 the value of the objective function objective function is given by is equal to C B i and then B inverse B i summation i varying from 1 to m remember, because our since our solution x B is given by B inverse b, remember, because b is a non-singular matrix and we put all other variables equal to 0, so this is our basic feasible solution the non-negative ones, the others of zeros, so and these are the corresponding coefficients of the basic variables in the objective function, so this gives me the value of the objective function which in short I can write as B inverse b.

Now, it is understood here that C B is a row vector, I can write the transpose, then it becomes tedious writing it so many times. Now, consider the…, so let us compute the value of that objective function for the new basic feasible solution; so, here the coefficients remain the same and for the except for the this thing, and you see that since x B r hat is 0, I can include that in the summation, what I mean to say is that will be equal to summation I will write as C B i x B i minus x B r upon y r j into y i j i varying from 1 to m, so I do not have to exclude the r h term, because anyway this number is 0, so the contribution of the r h term to this sum will be 0, therefore I can keep it, and that could be convenient, because I want to read and plus it will be C j into x B r upon y r j right, this is my new value; so, let us rewrite it to get the condition as to which basic feasible solution I should move to in order to improve my value do not get a function.

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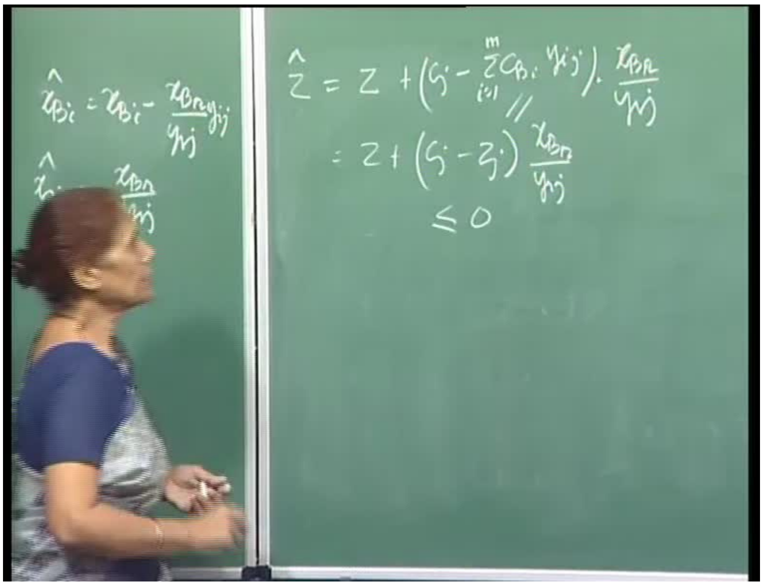

So, if you now look at this summation C B i x B i, that is my old value of the objective function, so that will be simply; so, if I write Z hat as my new this things when this is equal to Z, because C B i x B i is my old objective function, then this is plus, you see C j minus; and now look at this here, x B r y r j is something which is common, then you multiply C Bi with y i j, so your summing up over i so if i quickly rewrite this this is summation C Bi y i j i varying from 1 to m, this into your x B r upon y r j, so the expression; so, you see this quantity is specific to your…, because the y i j(s) were the scalars used for expressing the jth non-basic column in terms of the basic columns.

So, these scalars are specific to your jth column, so then C Bi are your basic coefficient basic variable coefficients, so this becomes something specific to your jth non-basic column; so, this I can write as Z plus C j minus Z j into x B r upon y i j right, so Z j is something which is equal to this; so, you see the change in the very nice way of expressing the change in the objective function value when you go from one basic feasible solution to another, and obviously if we want the value since we are considering a minimization problem, and once I give you the condition for minimization problem you can immediately write down the condition for a maximization problem.

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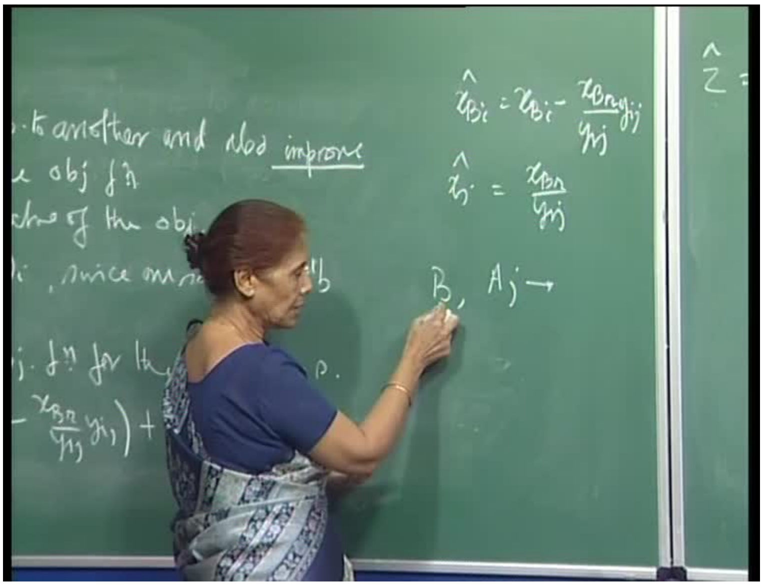

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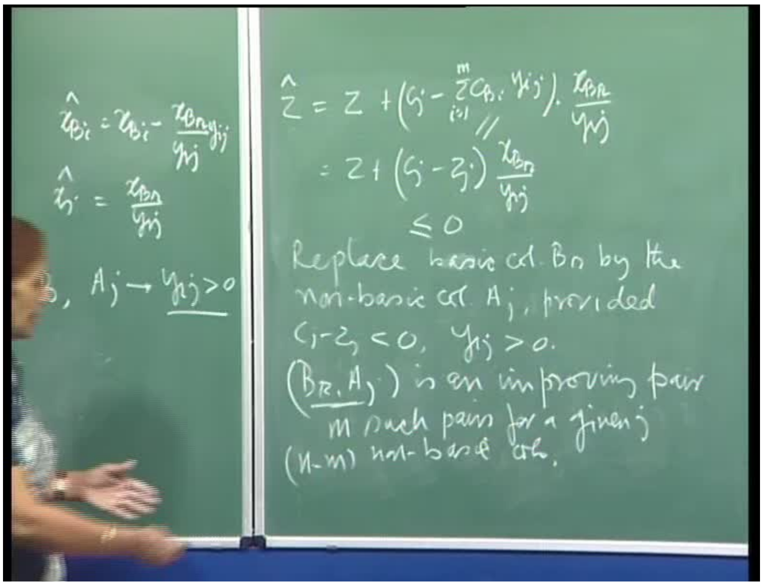

So, there is no problem there; so, here if my the LPP is of the mineralization form than I need this to be less than or equal to 0; in fact, if I wanted to into improve then it should be less than 0, but we can less than or equal to 0, this is something which is non-negative; so, if I want Z hat to be less than Z then C j minus Z j should be less than or equal to 0, therefore what is being said is that, here I arbitrarily choose given a basis B, I arbitrarily choose A j, the only condition was that I will replace A j, I will bring in A j into the basis and remove one of the basic columns here, you are replacing a basic column B r by the non-basic column A j provided y i j(s) positive, and then the r s come from the fact that thus ratio x B i upon y i j was minimum, therefore the r k from that right, and so you got that you replace B r by A j such that you get a new basic solution which is feasible, first that is the condition, and then C j minus Z j is also less than 0, then in that case when you replace B r by A j your value of the objective function will go down.

That is Z hat is equal to Z plus C j minus Z j into x B r upon y i j; so, if this is negative and this can be non-negative, so it can be 0 or positive; so, many case your Z hat would be less than or equal to Z; if you replace B r by A j provided you get a new basic feasible solution, and this condition is satisfied; so, then we say that B r comma A j is an improving pair, if it satisfies are the three conditions, that means, y r j is positive, and then you get a new basic feasible solution by replacing B r by A j which use that you are only replacing that particular basic column for which this ratio is minimum, so that you end up with the new basic feasible solution, and then C j minus Z j is also less than 0, in that case this is an improving pair.

And so the next step would be an…, as you can see that this may not be there may be more than 1 j for which this is less than 0, and in that case it is not…, I mean A j I may replace a particular a particular basic column may get replace by more than one non-basic column and still improve the value of that objective function; so, therefore, this not a unique pair, this can be more than…, for a given basic column to be replaced by a non-basic column the a choice may not be unique all the time, m such m such pairs for given j for a given non-basic column, m such pairs at most upper bound, so they can be m such pairs.

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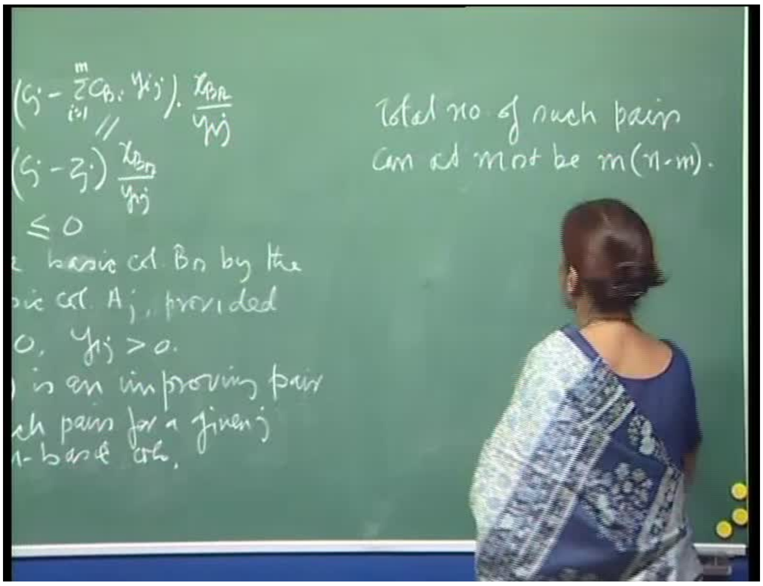

And then we have n minus m non-basic columns, n minus m non-basic column, so you might say that, we have how many m into n minus; so, the total number of such pairs that you can consider, so total number of such pairs can at most be m into n minus n.

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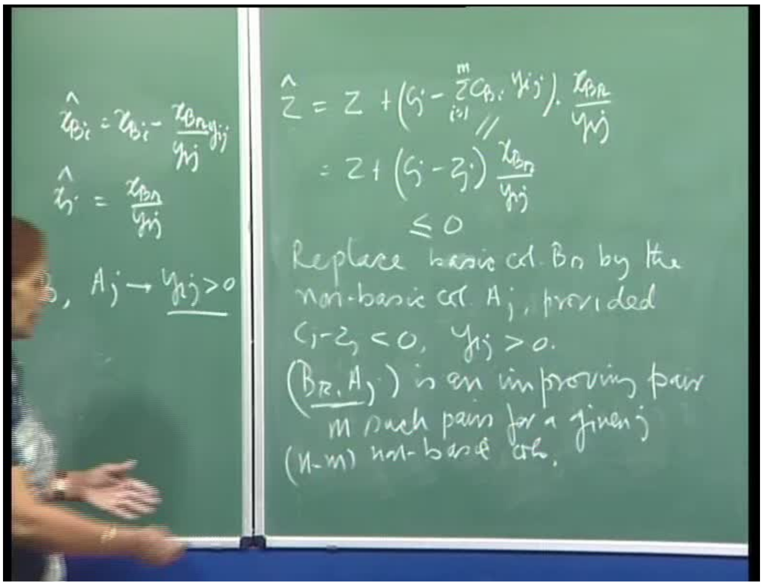

Now, let us come to the second question; since we are wanting to minimize, obviously you would want to make the maximum improvement of the objective function value, and for that what would you need to do; see here the improvement is product of C j minus Z j into x B r upon y r j, so you can say in other words that, this is the.., this is be level at which your jth basic variable is entering the basis right; and this is the rate of improvement in a sense, because the total improvement is C j minus Z j into the value, the level at which the variable is coming into the basis, so this the total thing.

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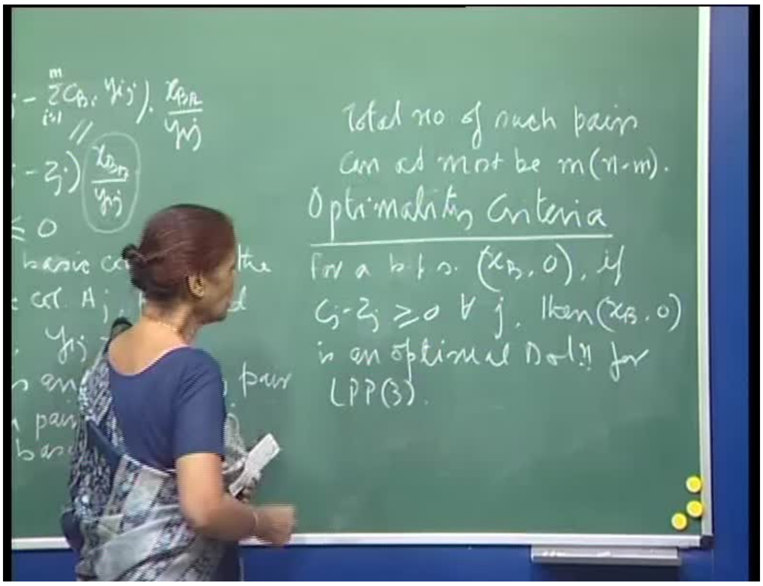

So, now, if we want to find out the what is the best possible improvement; if the value of the objective function you will have do this exercise, that means, you will have to compute this and this for all the m into n minus m pairs right to get the best improvement, but then that gets to be very in efficient, because your n see ah reasonably sized linear programming problem can have a few hundred variables and a few hundred constraints; for example, this may run up to 1500 variables, and let say 400 to 500 constraints, let this number is very large and if you have to compute this quantity for each of this such pairs, then it will the algorithm will become very inefficient.

So, at least that is not done; the idea is that you should be able to move from one basic feasible solution to another such that there is some improvement; if you stick to that it has been and of course later on we were discuss all these aspects and show you that these things work, you do not really have to go for the best improvement, and because one twice to strike a balance between the improvement in the value of the objective function and the amount of work that has to be done, so the two things are very…, the balancing is very important, and that is what makes the algorithm efficient.

So, this is what we have covered so far. Now, immediately you can see that from this equation you can derive the optimality criteria, so what we are saying is…, and again it will not be…; so, the next thing I am going to talk about is optimality criteria, this is what is needed to determine whether a given solution is an optimal solution or not.

So, we definitely need this kind of thing, and because if we have this optimality criteria, then it will help us to stop the algorithm, because it will tell us that there is no more improvement possible, and therefore you stop right; now, what we will going to…, again I am trying to motivate how the optimality criteria comes about, and you see that here what is happening.

If you cannot find j for which C j and Z j is less than 0; then what we are saying here is that, from this basic feasible solution B or from this basic feasible solution that we have, if at this point I cannot find any non-basic column for which a C j minus Z j is less than 0, then I will stop here, and I will say that my x B comma 0 is a basic feasible solution; this is a just a claim, I need to substantiated, because all right now I have said is that given a basic feasible solution, if I cannot find a non-basic column such that C j minus Z j is less than 0, then all I am saying here is that from the given basic feasible solution, I cannot make a move to another basic feasible solution say that the value of the objective function improves.

This is the only claim I am making, but we will be able to show you may be in a after another lecture that this is enough, is this condition is sufficient; that means, if I have…, if I have a basic feasible solution, and such that I cannot find a non-basic column for which the corresponding C j minus Z j is less than 0, then I should stop and my current basic feasible solution is an optimal solution.

So, this is what we want to now start proving; so, the optimality criteria is that is, now we see that if for a basic feasible solution, so I am just continue to write it us this saying that we have renumbered the variables and so on; for a basic 0 if C j minus Z j are less than or greater than or equal to 0 or all j then x B 0 is an optimal solution an optimal solution for LPP 3, and I will said about trying to proof part this condition, so for a basic feasible solution if all C j minus Z j(s) are non-negative, then this is the…, then x p is optimal solution.

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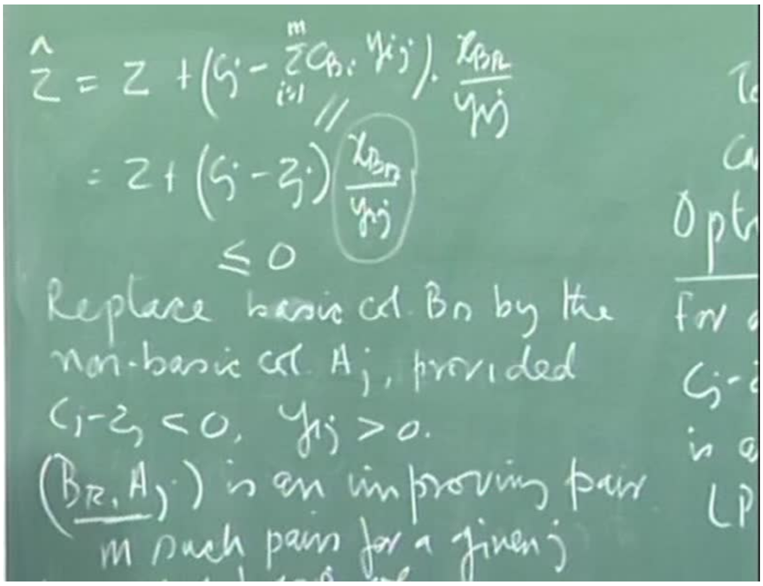

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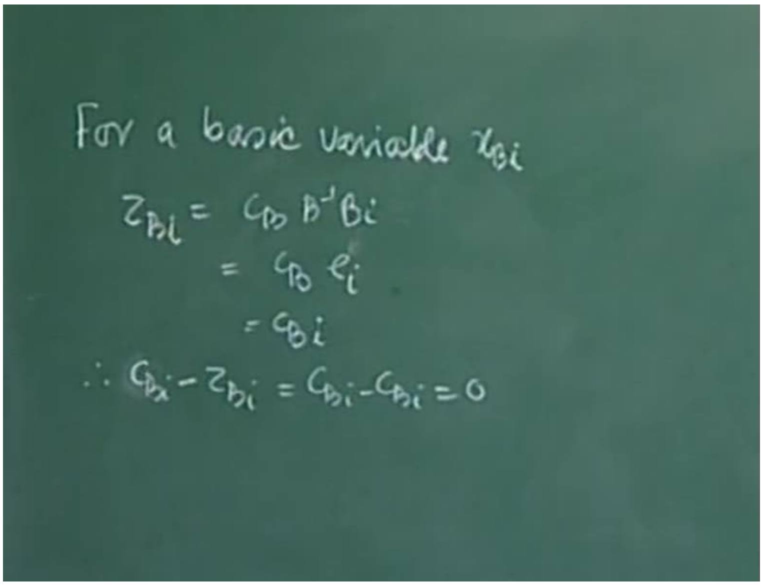

Now, from the computation you see that, here, yes, let us look at the computation for a basic variable, I wanted to show you that computation; so, you see, if you want to compute the Z B I, this will be C B B inverse A j, now A j for the ith basic variable is B i, then B inverse B i is E i by our definition right, B is the basis matrix, so this is the ith column, and this is the inverse of the basis matrix, so B inverse B i will be E i; and therefore, when you multiply this with this, it will be the ith component only, so C B actually I remember I write this even though this is a column vector I am writing this is the row vector, so I am not writing the transpose here anyway, so C B; this the m dimensional column, this is the m dimensional, so they product could be simply the ith component of the vector C B the basic variable; so, therefore, for a basic variable the z B i reduces to C Bi and hence when you want to compute C Bi minus z B i, this will C Bi minus C Bi which is 0.

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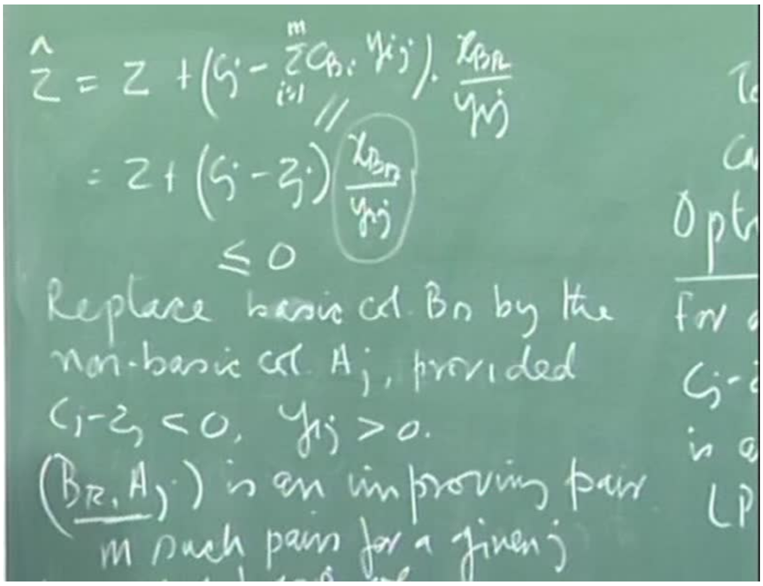

So, for a basic variable the relative cost is 0, then of course we will spend some time some more time in interpreting these things in detail; but right now as I said that if you want to say that this is the rate at which the value of the objective function will improve corresponding to the jth non-basic variable; then what this says is that, for the variables which are basic the rate of improvement is 0, because whatever improvement was possible was done, so no more…; therefore, for the basic variables, the coefficient the numbers or the quantities C j minus Z j(s) are all zeros; therefore, when I say this right, this is an optimality criteria, and it actually refers to the non-basic variables, and it say that for all non-basic variables your C j minus Z j should be a non-negative.

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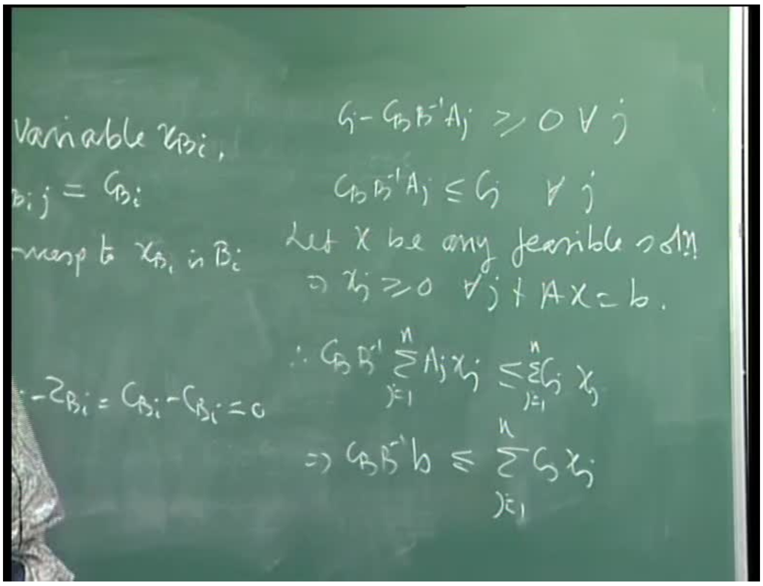

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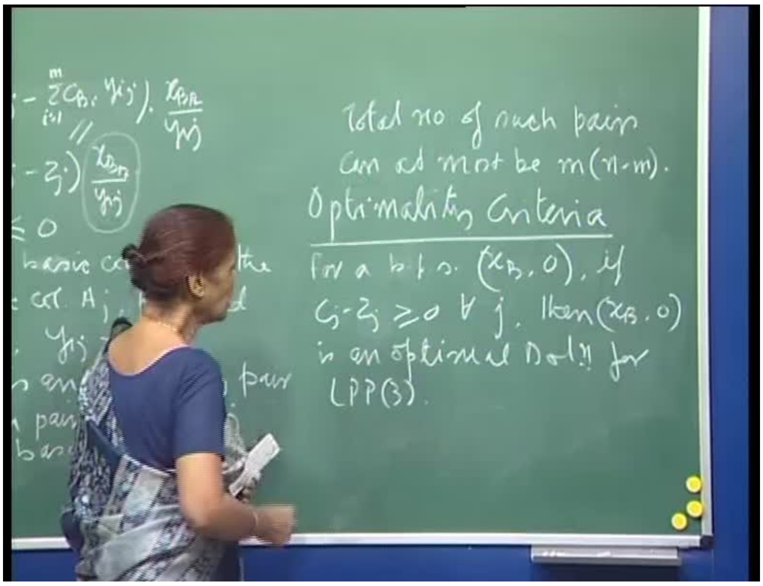

For the basic variables the C j minus Z j(s) are non-negative; now, suppose, and therefore I can state this as a condition for all j(s) right; so, let us look at this, here the says that C j minus C B B inverse A j are greater than or equal to 0 for all j, which means that C B B inverse A j is less than or equal to C j for all j; let x b any feasible solution, any let x b any feasible solution, so that means all its components are non-negative, which implies that x j is greater than or equal to 0 for all j, and a x is equal to b right; so, if i multiply each of these equations by the corresponding x j and add, see this quantity is common, therefore, I can say that therefore C B B inverse summation A j x j j varying from 1 to n is less than or equal to C j x j summation j varying from 1 to n, the inequality does not change, because I am multiplying by a non-negative number on neither side, so this is what you have; now, from your since x is a feasible solution, this summation adds up to vector b.

So, therefore, which implies that C B B inverse b is less than or equal to summation C j x j j varying from 1 to n, and that gives you the result, because this is the value of the objective function; remember, your basic feasible solution is B inverse b, this is the cost coefficient corresponding to the basic variables, so this is your objective function for the basic feasible solution x B 0. And this is the cost for the x to be any feasible solution. So, what we saying here, we saying that if x is any basic feasible, if x is any feasible solution, then the value of the objective function is always higher than the value of the objective function for the basic feasible solution x B 0 for which this condition holds.

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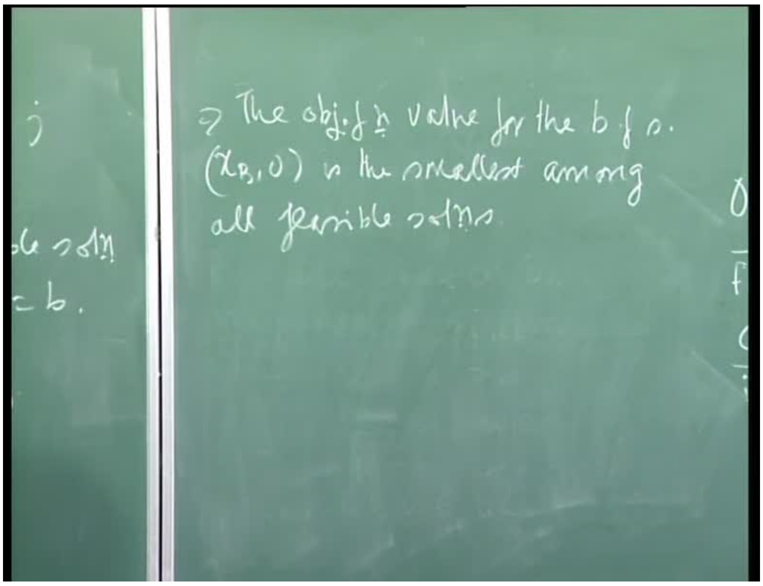

See, you see the simplification is evident here, therefore this implies that the objective function value for the basic feasible solution x is a smallest among all feasible solution, and therefore this is the valid optimality criteria at this if a basic feasible solution satisfies this condition for all j, then we can stop the algorithm there, and we will say that the current basic feasible solution is an optimal value.

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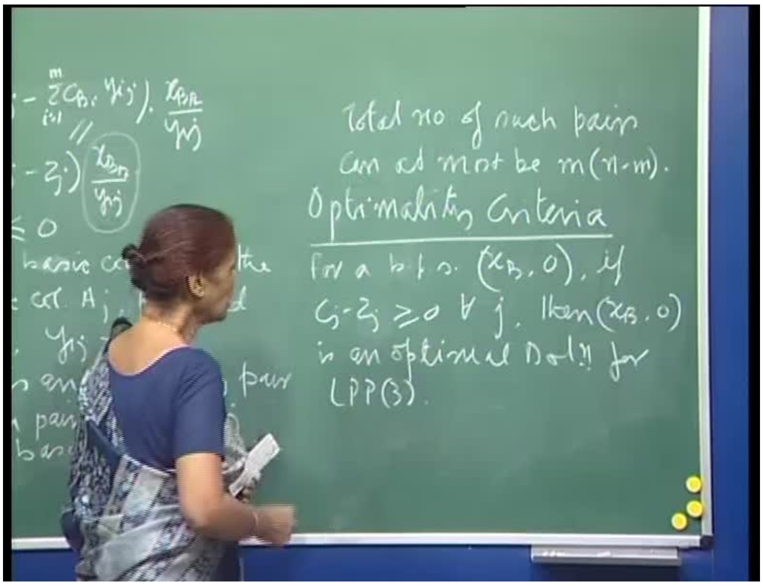

Now, lot of things can be derived from here, for example, so what had be shown, we have shown that a basic feasible solution is an optimal solution, that is all, but I have not claim…, and of course the questions can be asked that you may have a basic feasible solution, but then this condition may not hold; if this condition does not hold, then we said that we will move to an optimal, because if there is some C j minus Z j which is less than 0; then I can move to a new basic feasible solution, and improve the value of the objective function, so this goes on, but when a question asked for the that is…; right now we are not established in a comprehensive way that a linear programming problem will always have a basic feasible solution as an optimal solution.

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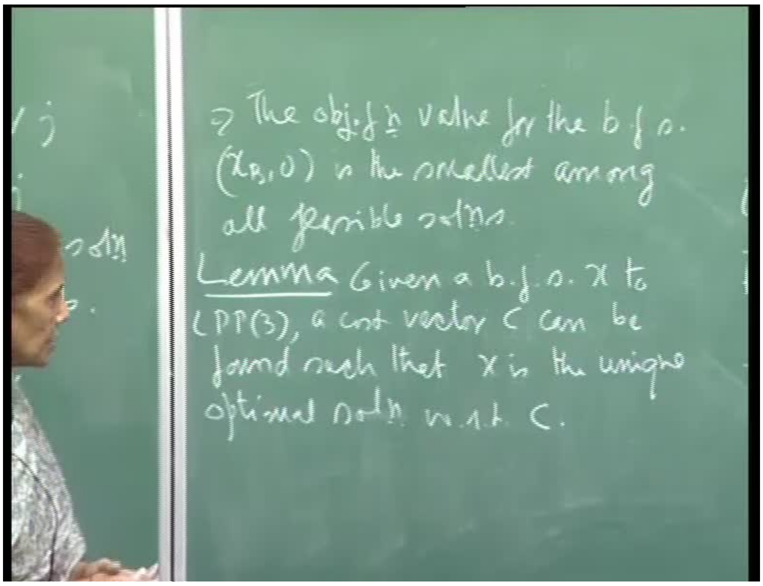

So, I still have to do some more work; but right now what we have concluded here is that, if you have a basic feasible solution, and it has this property, it satisfies this condition, then your basic feasible solution is an optimal value, so we have proven this; and now, just to make the things more interesting, let me just also recount a lemma here, which simple and that again gives you some insight into the…; so, the lemma says that given a basic feasible solution x to LPP 3 given a basic feasible solution a cost vector C can be found such that x is the unique optimal solution with respect to C.

One needs to understand, it may be the implications were also become clearer as we go along with the course; so, what we are saying is that, if you have a basic feasible solution when you can always kind a cost vector C such that this basic feasible solution is a unique optimal solution respective to the C; and the constructions are interesting, and therefore, it will give you an insight into the theory of LPP, and the and how we that look at…, how we construct this into x algorithm.

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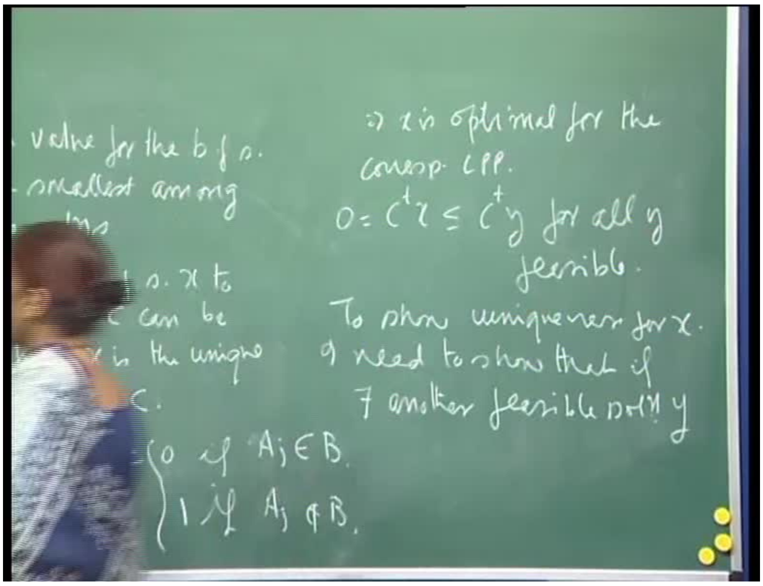

So, let say defined, so proof, so let define C j as 0 if 0 if j belongs to I x, so again I am using the same notation, that means, could be coefficient of cost coefficient C j equal to 0 corresponding to the basic variable which is positive right; I x is the collection of induces is corresponding to x for which the basic variables are positive; so, if a basic variable is positive then put the corresponding cost coefficient as 0 and 1 if or to be no I will redefine because this is not be good enough; and you see, so there were you can also learned by these kinds of small mistakes in the sense that if j if I would be…, if A j belongs to b, so I want to do it for the whole basis it will not be enough to…, because my basic feasible solution may be degenerate.

So, then I need to define, so I have to say that C j is 0, if A j belongs to b and is 1, if A j does not belongs to b for this is my definition of a cost vector right; and now, so obviously with this the cost function the…, so if you have this cost coefficients for corresponding to the basic feasible solution x, then we are saying that…, so this implies that x is optimal for the corresponding LPP.

Let us spend some a few minute from this, why is it optimal, see I have chosen the cost coefficients such that the cost coefficient are 0; if the variable is basic variable and the cost coefficients are 1; if now, since your LPP requires that variables are non-negative, and C j(s) are all non-negative, you can immediately say that see here since what are you says that is optimal because c transpose x which is 0 is less than or equal to C transpose y for all y feasible, yes, just think on it for a few minutes; what we are saying is that C j(s) are non-negative, your variables are non-negative, so this number cannot be has to be non-negative or positive C transpose y, for any feasible y; and for C transpose x I know the value is 0, because I have chosen the C is to be 0 whenever x is the basic variable; and when x is non-basic then the variable has to be 0, and the corresponding coefficient is 1, but does not matters; so, the total thing is 0; so, this is always less than or equal to C transpose y for the all y feasible.

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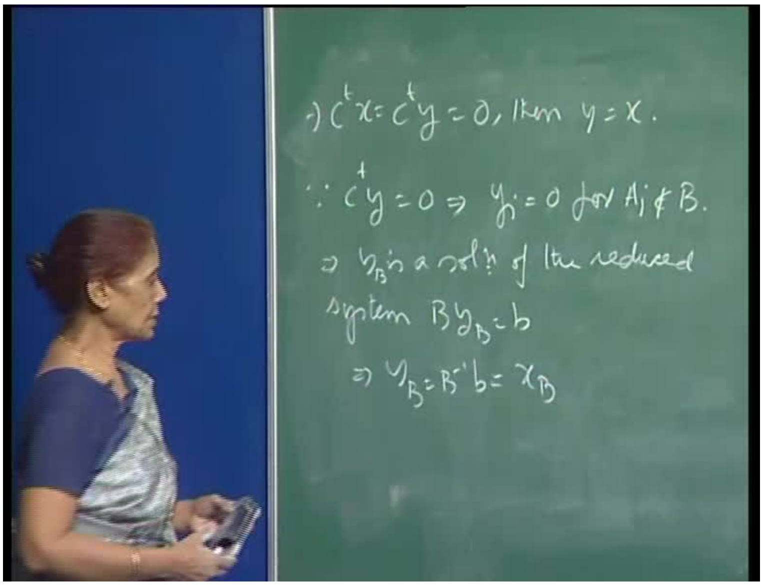

So, therefore, this is optimal; now, you want to show uniqueness; to show uniqueness for x what do I need to say that, suppose, so to show uniqueness for x, I need to show that if there exist another feasible solution y right, such that the two values are equal, then y must be equal to x, that is what we mean by uniqueness; so, if there is another feasible solution y such that c transpose x equal to c transpose y is 0.

So, to show uniqueness for x, I need to show that if there this another feasible solution y such that C transpose x this; then y is equal to x, this is what I have to show; now, since C transpose y is 0, this implies that y j is 0, the number for A j not in B; see, my definition for C j(s) says that, this is 1, if A j does not belong to B right; so, since C j(s) are positive, and this objective function value is 0, this implies that y j must be 0 for A j not in b, because the corresponding c is positive, so this is what happening right; and then therefore, this implies that y is a solution of the reduce system or in fact I should be able to say y B, because I have put all y j(s) as 0 whenever the corresponding column is not in B, that means, for all the non-basic variables the y j is 0, so if the reduce system which implies that y B equal to B inverse b is equal to x B.

And the non-basic variables have already the same by definition, because I have said that y j must be 0 whenever the corresponding column is non-basic; and this is non-basic will spec to the solution x that you had, therefore y must come as the reduce from the reduce system B y B equal to b, and so let say that y B is B inverse b which is x b, so this solution is unique fine. So, this after all this theory let me just now take up an example and as try to illustrate all these things, that means, moving from one basic feasible solution to another, improving the value of objective function, and the demonstrating the optimality criteria.

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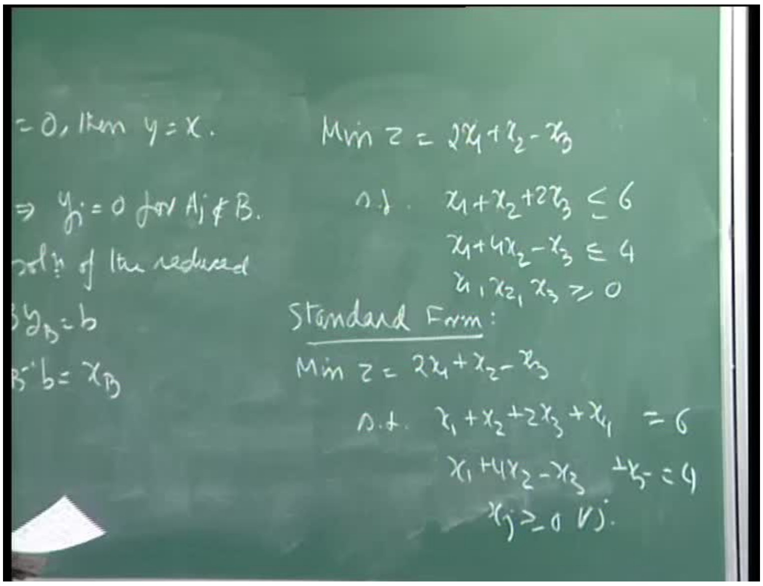

So, I write the example here; so, this is a LPP minimize Z equal to 2x 1 plus x 2 minus x 3 subject to x 1 plus x 2 plus 2 x 3 less than or equal to 6, x 1 plus 4 x 2 minus x 3 less than or equal to 4, x 1 x 2 x 3 greater than or equal to 0; I will convert this to a standard form, so the standard form is standard form would be minimize 2x 1 plus x 2 minus x 3 subject to x 1 plus x 2 plus 2x 3 plus x 4 is equal to 6, x 1 plus 4 x 2 minus x 3 plus x 5 is equal to 4, and x j(s) are greater than or equal to for all j, this is my standard form.

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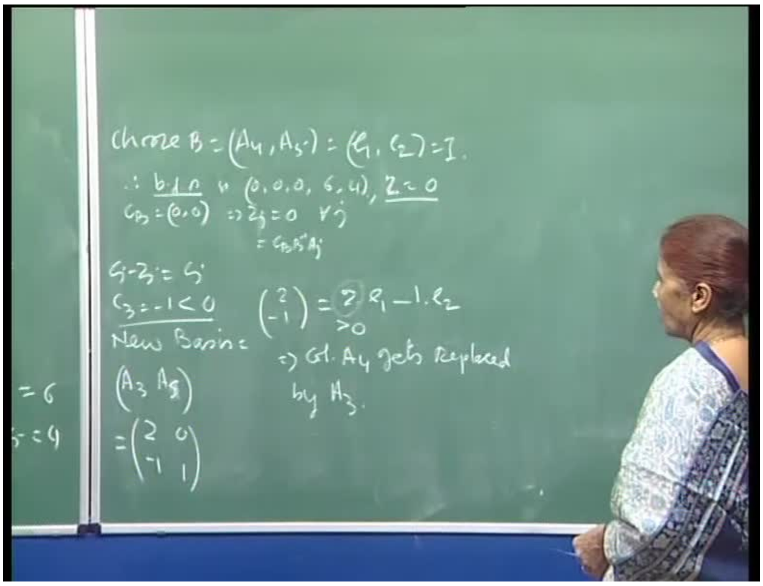

And starting immediately without checking anything I can immediately say that these two columns are linearly independent, because this is e 1, this is e 2, they form a basis for the column space; so, I will use my starting basic feasible solution given by the columns A 4 and A 5, which means that i put x 1, x 2, x 3 equal to 0, and I immediately read my basic feasible solution as x 4 is 6, x 5 is 4, so choose b as A 4 A 5 which is nothing but e 1, e 2, and so this is an identity matrix.

And so, here and therefore, your basic feasible solution basic in the complete form; if you want to write basic feasible solution is 0 0 0 6 and 4; and now, I need to compute the I want to check the optimality criteria, which means that I have to compute the C j and Z j(s) to see whether I can improve the value of the objective function are not; so, what we will do is here we will compute all the C j minus Z j(s) for each of the non-basic columns 1 A 1, A 2, and A 3; now, here things are very simple, because your C B is 0 0, the stack variables do not appear here, therefore C B is 0; if C B is 0 which implies that Z j all Z j(s) are 0 for all j, because Z j was C B B inverse Z j were C B B inverse A j.

So, if your C B has all 0 components, everything is 0 here; therefore, your C j minus Z j(s) are simply C j(s), and this we have to compute for the non-basic variables, so the which is already known to us, this is C 1, this is C 2, and this is C 3, we see that C 3 equal to minus 1 is less than 0; so, your optimality criteria is not a satisfied, and in fact if we induct, if we induct the third column into the basis, then we can improve the value of the objective function, and by the way what is the current value of the objective function.

See, remember, x 1, x 2, x 3, are all 0, so the current value…, this is this, and your Z is 0, so the starting value here Z is 0, he can improve on it because I have this condition satisfied that C 3, your C 3 minus Z 3 is minus 1 which is less than 0. So, if I induct this into the basis what is my basis become…; so, the new basis is…, and so how do I how do I know which….

So, yes, so, the next question that has to be answered is which of these basic columns a 4 and a 5 which of them will be replaced by this, and remember for that what we are doing, so here before I write the new basis's I am inducting A 3, so that means, I will write my A 3 column is 3 and minus 1, and there is no problem, because my basis is identity matrix, so this is actually 2 times e 1 minus 1 times e 2, and remember we use the positive y i j(s), we use the positive y i j there is only one positive y i j the other one is negative, so there is no need to take the minimum the ratio and compute the minimum one among all y i j(s) which are positive, so this simply tells me that, even will get replaced by…, because this coefficient is positive.

This is greater than 0, so this implies that the column A 4 gets replaced by A 3, this is and therefore your new basis is A 3 A 4, which you can write in terms of the this A 3 is 2 minus 1, and sorry A 4 got replaced, so this is a 5 0 1; now, I need to compute…, is it right.

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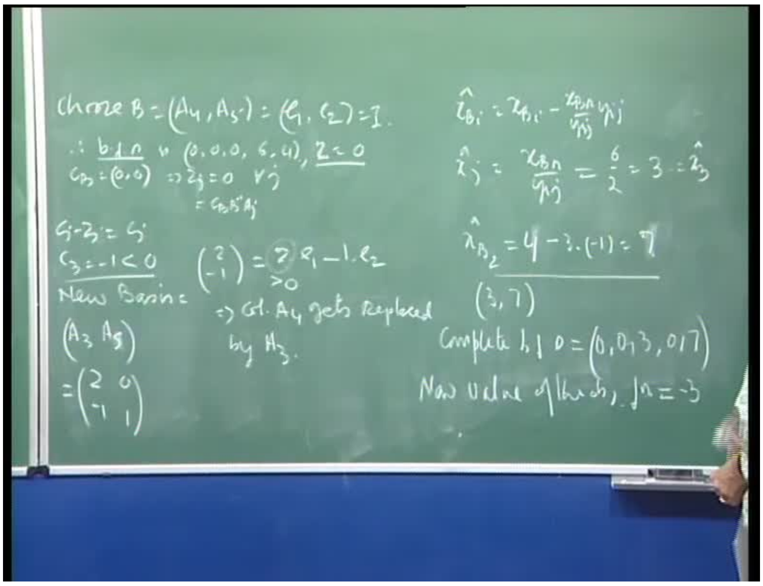

Now, you need to compute your new basic feasible solution; and if you if you can recall the formula; so, what was the formula x B i hat we said is x B i minus x B r upon y r j into y i j, so we have all the numbers of, for example, here your new basic variables, and your A j x hat j was x B r upon y r j which you already know, because this is 2, the y r j is y rj is 2, and x B r what is the value of x B r, x B r is 6, because the value for the outgoing basic variable was 6, so you are this thing is 6 upon 2 which is 3.

So, the new value…, and this this is actually your x 3 hat; so, the new value of the non-basic which was a non-basic variable x 3 is now become 3, and what is the value of your this thing x hat B 2; this is the second basic variable, when this was the first basic column, this was second basic column, so x hat, this is equal to what was the original value of this thing was 6 minus x B r upon y r j(s) 3, and what is your y i j, the corresponding this thing minus 1 here, remember these are the…, y i j(s) are the scalars, I need to express the incoming column in terms of the old basis right, so this is minus 1.

So, this into minus 1, this is becomes 9; so, actually this value goes up, and these things also we will later on keep focusing on the here, because in this equation you see these two are 0s, x 3 became positive, so the value of x 5 actually goes up, this is 4, so why is it showing 9, sorry why do somebody correct me, this is 4, this is 4, so 4 minus 3, so this is 7, sorry this is the new value itself; so, your new basic feasible solution is 3 and 7, yes, I am just writing the basic variable, but in the complete basic feasible solution is 0, 0, 3, 0, and 7, this is your new basic feasible solution, and of course you can compute that value of the objective function directly, new value of the objective function is equal to…, yes, from the objective function is x 3 is positive, x 1 x 2 are 0, its minus 3, which is definitely better than what you had what you started with z equal to 0; now, it is minus 3, quickly, I will just complete it now here so minus 3 is the value has the improved, and of course you could computed the by the formula that I gave you. So, now, we need to just quickly verify whether this is optimal or not; and if it is not optimal then we will move on to a new basic feasible solution, but…; so whatever we discuss so far I try to demonstrate that through this example.

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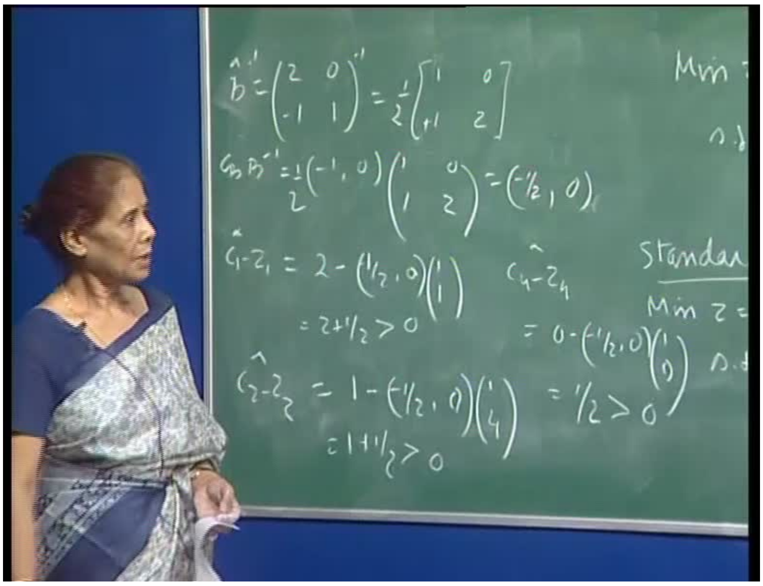

And now this is your basis, and this is this, what do I need to compute my C j new C j minus Z j(s), and for those I need the B inverse; so, let us see if your new basis you want to call it b hat, b hat is 2 minus 1 0 1, so b hat inverse now you should be familiar with this that you can immediately for a 2 by 2 matrix it is not a problem at all, this will be 1, this will be 2, and this will be plus 1 and 0, is it okay, quickly verify 2 0, this and this will be half, because the determinant is 2, so I divide by 2; so, this is 1, then this is 0, this is minus 1 plus 1 0, and this is so this is the inverse.

This is the inverse, and therefore you need C B B inverse, now what is your C B, C B now in this case is minus 1 m 0, because x 5 has a 0 coefficient x 3 has minus 1, and this is there is a half here 1 0 1 2; so, let us quickly compute this, this is minus 1, so minus half and this is 0 0, so that is 0.

So, this is your C B B inverse; and now, only two non-basic columns, well, x 4 is there, A 4 is there, but I will quickly do it for A 1 and A 2; so, A 1…, so if you want to compute C 1 minus Z 1, hat you can say, so then C 1 is 2 minus Z 1, Z 1 will be minus half 0 into 1 1, A 1 is 1 1, this is your C B B inverse, so if you compute this thing this is minus half plus half, so this becomes 2 plus half which is positive.

Then compute C 2 minus Z 2 hat, so this is C 2 is 1 minus minus half 0, and this is 1 and 4, so this again is 1 plus half, this is positive; and you need to do it for C 4 minus Z 4 hat which is equal to C 4, this is 0, and minus minus half 0, and this is 1 0 which is again is half (( )). So, you have arrived at optimal solution, that means, the current basic feasible solution satisfies the optimality criteria, and the minimum value of the objective function is minus 3.

So, right now I have tried to show you that the algorithm that we will finally develop for the for solving a linear programming problem is in iterative method, and I have tried to show you how we will go from one basic feasible solution to another and improve the value of the objective function, and there is a stopping criteria also that the algorithm will be able to stop, now able to stop I have to explain further, and we will talk about in the next lecture.