Linear Programming and its Extensions

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Module No.# 01
Lecture No.# 18
Shortest Path Problem Primal-Dual Method Example

Let us continue with the shortest path problem, so remember we say that it is a directed graph. So, E is the set of edges, V is the set of nodes, and then we had weights cij; cij for each i, j belonging to E, and we said that cij's are greater than or equal to 0 for all ij - non-negative weights and later on we will consider an algorithm where the cij's are not necessarily all non-negative.

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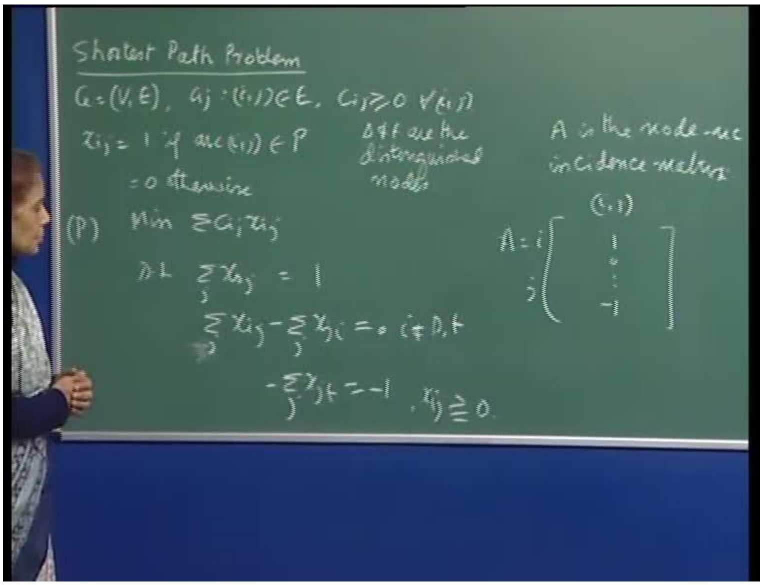

So, then we formulated and I said that xij will be 1 if arc ij belongs to the path P and 0 otherwise then, we said that the primal problem would be to minimize summation cij xij, so this will be subject to, remember. So we are finding a shortest path from 2 distinguish nodes.

So, let me mention it here s and t are the distinguished nodes and we want to find the shortest path from s to t, so the shortest path would mean that you add up the weights of the edges or the arcs which make up the path and then we want to find a path from respective which has the shortest weight.

So, then the constraints would be, because we want outcome the solution here a feasible solution to this formulation to be path from s to t which will require that, see, summation xsj summation over j has to be equal to 1, because one arc must leave s to begin the path shortest path, because you want to find the path from s to t. We had gone through this formulation earlier, and then this was xij summation over j depending on whether ij belongs to the edge set or not minus summation xji summation over j this is 0, when i is not equal to s, t. And then we said that the final, this would be summation xjt summation over j is equal to minus 1.

So, because see here, for the source node we call it the node from which the path has to be selected, we call it the source node and t is the destination node. This is also terminology, so anyway we want to leave it we should have at least exactly 1 arc leaving the node s, and exactly 1 arc reaching node t and I told you that you can write down the node-arc incidence matrix, so we had gone through all that, A is the node-arc incidence matrix. And we said that for example, so you have A here, and if you have an arc ij here and you have a node i here and node j, then because i, j is leaving node i, so it will have a plus 1, and it is entering node j, it will have a minus 1.

So, this is the term, all the others are 0s, this is the concept of the node arc incidence matrix and so the constraints can be written like this. So, any intermediate node if the arc is entering the node and it must also leave it, and that is why these 2 sums must always be equal.

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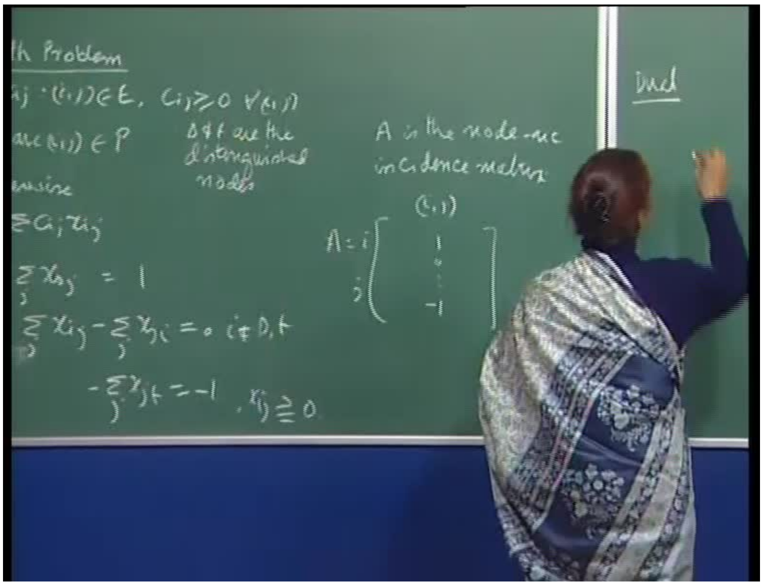

And then, now the linear programming formulation, here of course, we require the xij is to be integers, but we will show later on discuss, why we so right now because it is a linear programming formulation I will just put the constraints that xijs are all non-negative. Then we write the dual, so dual is from straight forward because any column here has only 2 entries plus 1 and minus 1.

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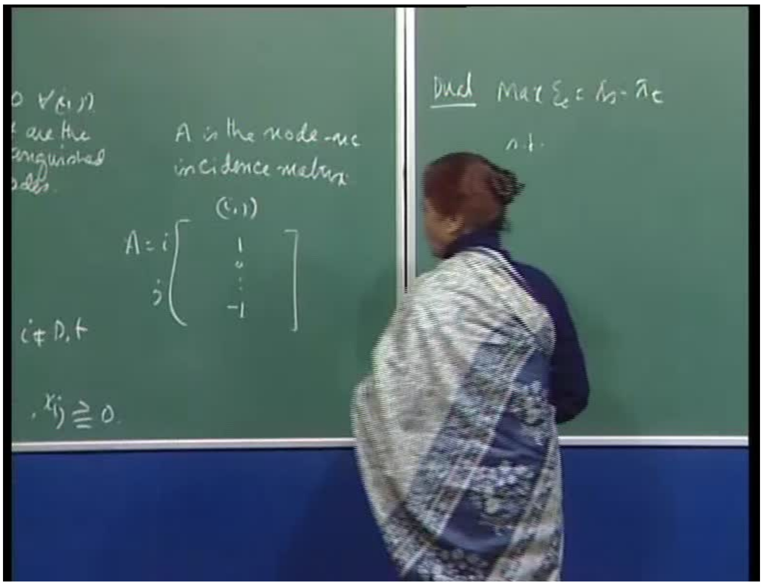

So, this will pi i and this will be pi j and therefore, when you write the dual constraints this becomes a transpose of to the node-arc incidence matrix. So, this will be, yes, so it will be pi s and pi t. So, you will be maximizing psi equal to pi s minus pi t subject to and then you have here.

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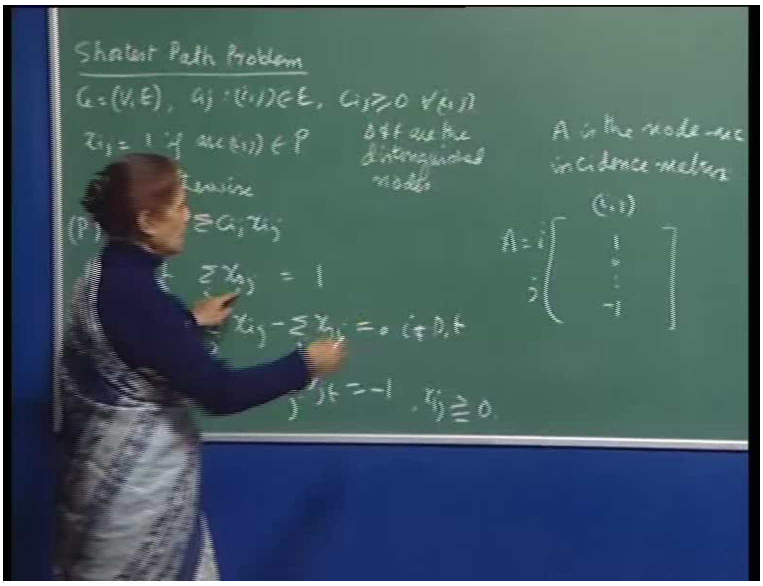

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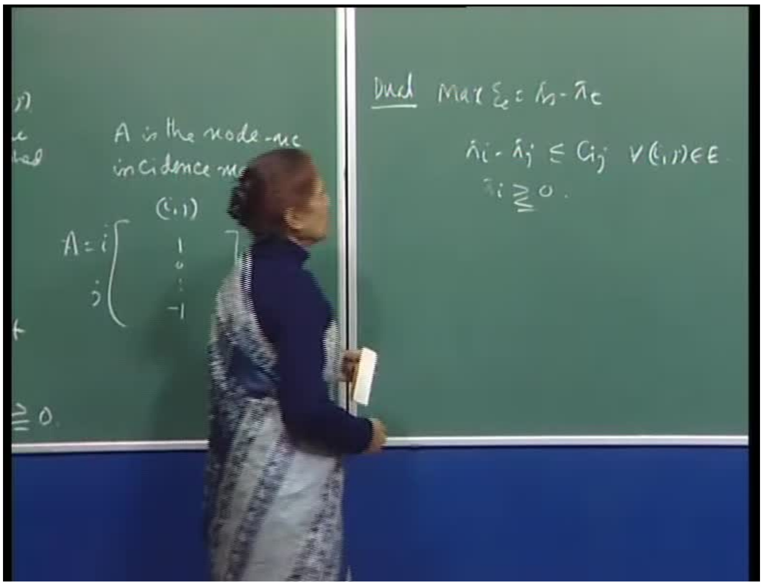

So, for example, when have the node s, if you have the node s here, all this will be plus 1, for any arc of the kind s j it will be plus 1 here and minus 1 here, so summation xsj is summation over j is 1, sorry, so this is, say for example here or don't have to specifically mention, this thing corresponding to this, your dual constraints will look like pi i minus pi j less than or equal to cij for all i, j are belonging to E.

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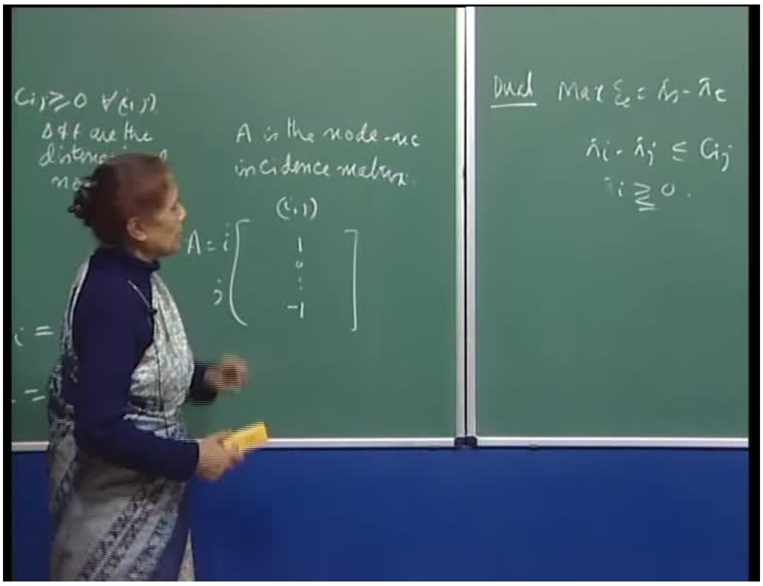

And since, your constraints are equality constraints, therefore no sign restriction on the dual variables. So, this is your simple formulation of the dual problem, because any column has only just 2 entries with pi i minus pi j.

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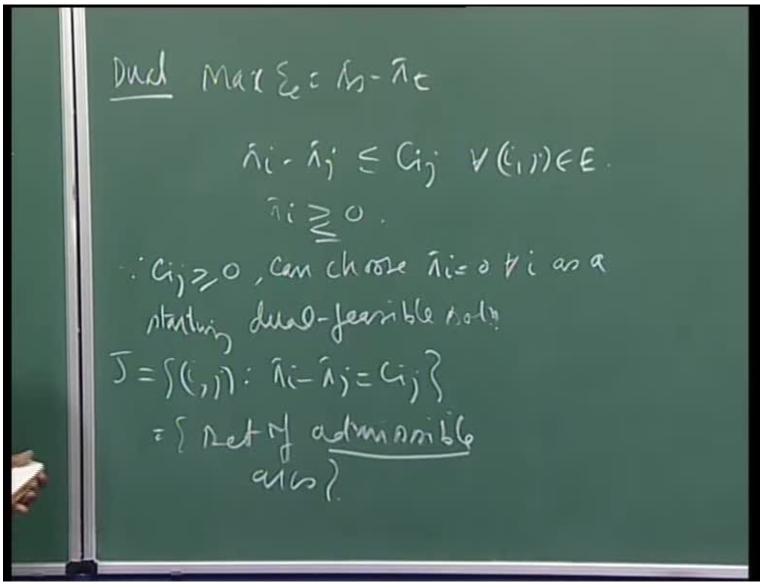

Now, since we are choosing cij’s are all non-negative, so since cij are greater than or equal to 0, we can choose pi i equal to 0 for all i as a starting dual feasible solution. You see, it is a left hand side will be all 0’s and these are all non-negative numbers therefore, the constraints are also always satisfied.

Now, corresponding to this, choose j as all i, j such that pi i minus pi j is equal to cij, so that will happen for the starting case it will happen only when the cij is 0, because pi i and pi j for the starting dual solution are 0’s therefore, equality will be satisfied when the right hand side the cij number is also 0. So, this is your set of this thing, and remember the idea is, that I am trying to show you a good application of the primal-dual algorithm. And so what we will do is will try to now, so I will call this set this is the set of admissible arcs.

So, the idea is that, if I can find a path from s to t, just consisting of the admissible arcs then I have a feasible solution for the primal, I have a feasible solution for the dual, such that they together satisfy the complimentary slackness conditions, and therefore, the both are optimal, so the corresponding for the respective problem. So, this is a set of admissible arcs, this explains that only the arcs for which the corresponding dual constraint is satisfied is equality would be admitted for forming a path from s to t.

So, this gives you this thing and therefore, your restricted primal what would be your restricted primal problem, you would be wanting to minimize, because, I do not know whether I can find a path or not so that is the idea. The idea finding a path from s to t consisting of admissible arcs, so we will use the artificial variable technique.

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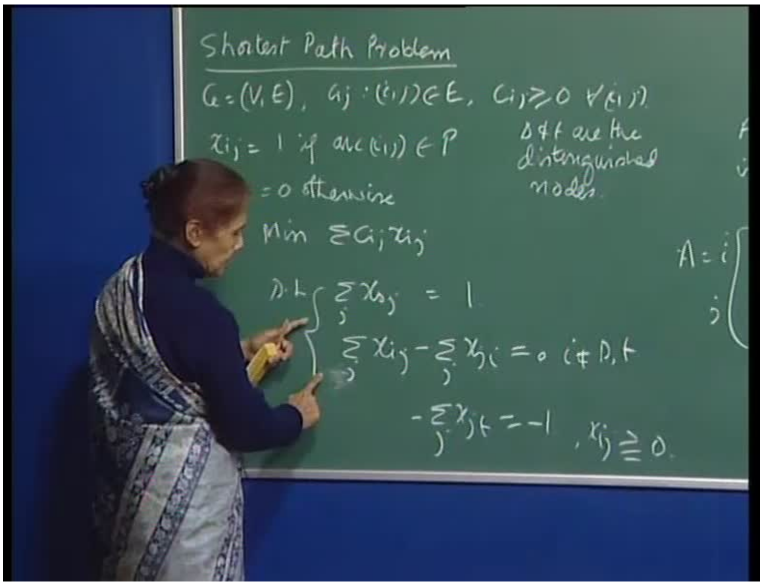

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Now, there is a simple modification here, you see that, if you add up these constraints and last time I mentioned it, but I did not show you exactly. So, if you add up these constrains you see what is happening? An arc sj is a plus sign here, but for node j it will be appearing with the minus sj, because the arc xj is entering node j, so therefore, they will be minus sign here.

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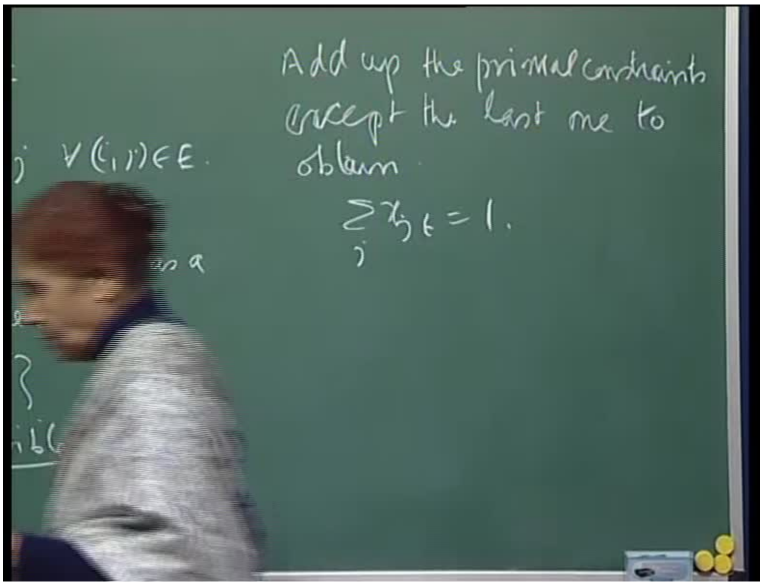

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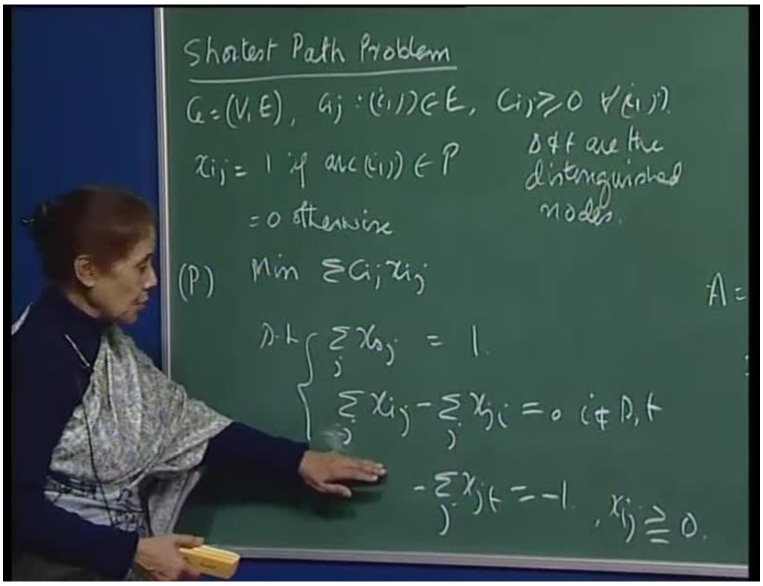

So, let me say, that if you add up the primal constraints except the last one to obtain summation xjt summation over j equal to 1, this what I wanted to last time, I did not show you exactly, see because any arc which is appearing as a positive sign in one constraint will be obtaining with the minus sign in the other, because it is leaving node s here, then it is entering and similarly, for an arc ij, it is leaving node i but it is entering node j. So, where I write the constraint for node j, it will be whether minus sign the arc xij and therefore, when you add up use the simply be left with arcs xjt, because they do not appear anywhere else, so that will be risk.

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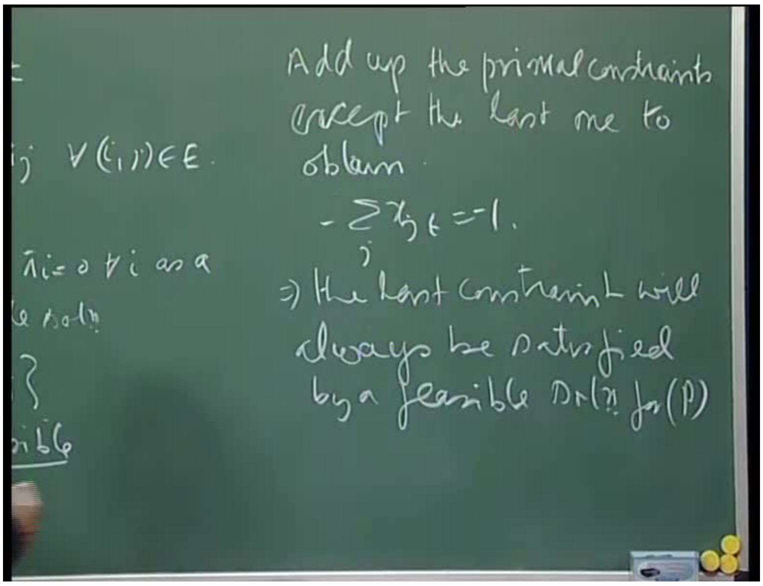

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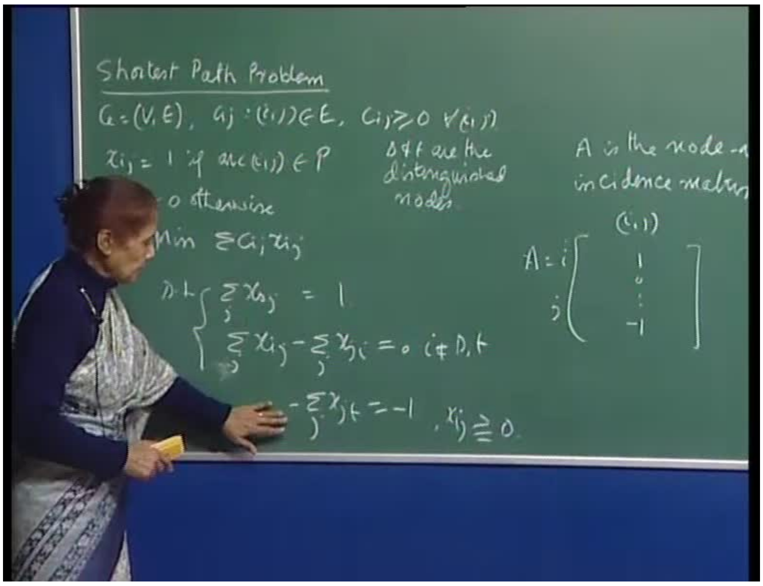

And therefore, the constraint, so if I multiply this by minus sign this will be minus 1. So, this implies that the last constraint will always be satisfied by a feasible solution for P. So, the idea is it I will simply, so therefore, we will ignore this constraint, because as long as I do not have a feasible solution this will not be satisfied, obtain a feasible solution, once I obtain a path from s to t that will be a feasible solution here.

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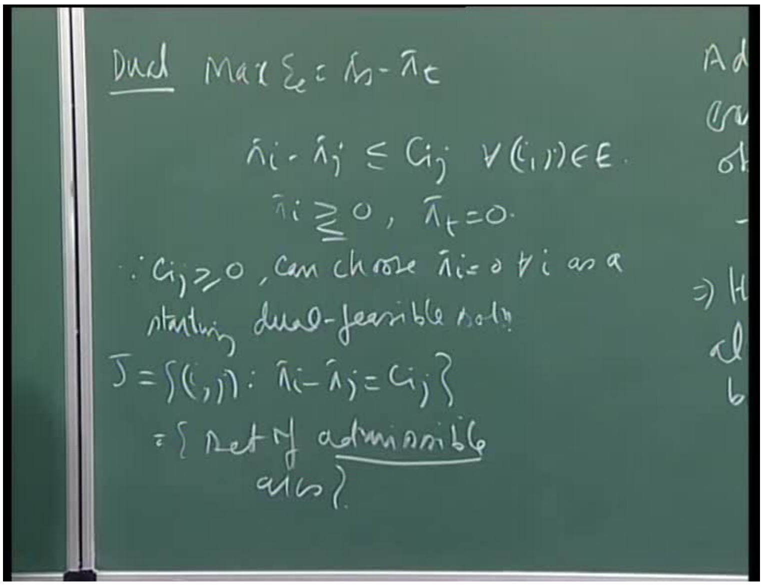

So, when I am in the process of constructing a shortest path from s to t, then this constraint will not be satisfied, it will only happen at the end. So, then I will simply ignore this, but I have to maintain complimentary slackness conditions. Therefore, will say that pi t - just keep pi t - is 0 this theory. Therefore, will ignore the last constraint and by keeping the corresponding dual variable always 0.

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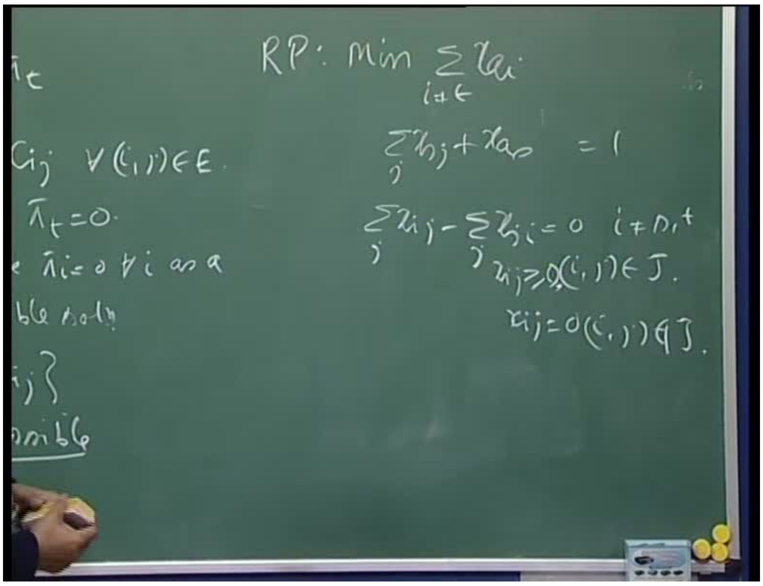

So, now let us write the restricted primal, so the restricted primal of the minimize summation xai; i not equal to t, otherwise you are summing overall. And then you will be saying that summation xsj summation of a j plus xas is equal to 1 and summation xij summation j minus xji summation of j 0; where i is not equal to s and t and of course, in the restricted primal, this is for ij belonging to J. What I am saying is that, xij greater than or equal to 0 for this and xij is equal to 0 for i, j not belonging to J. So, this will be our restricted primal problem. Only those xij's I permit to be greater than 0 for ij belonging to J.

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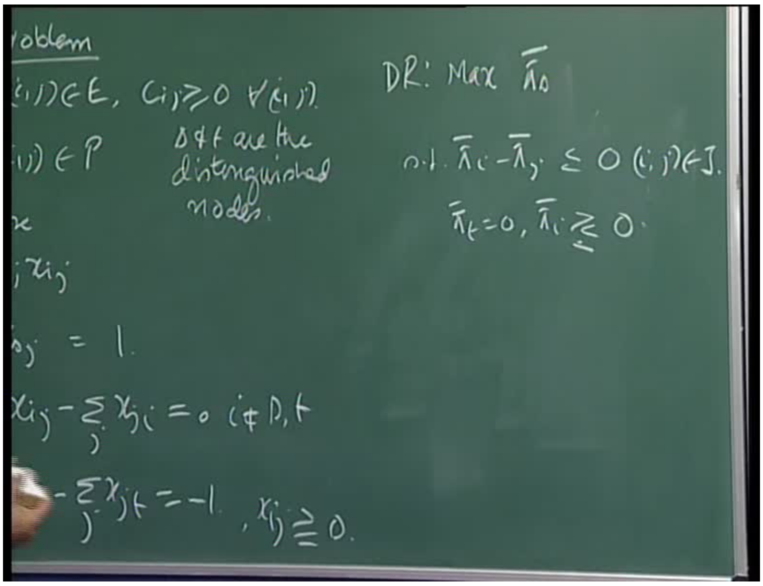

Now, then what would be your restricted dual, so the restricted dual would, therefore, be maximize. Now, let me write here as pi s bar, so the restricted dual variables have with the bar pi s bar because your 1 there, and subject to pi i bar minus pi j bar less than or equal to 0 and this is for i, j belonging to J.

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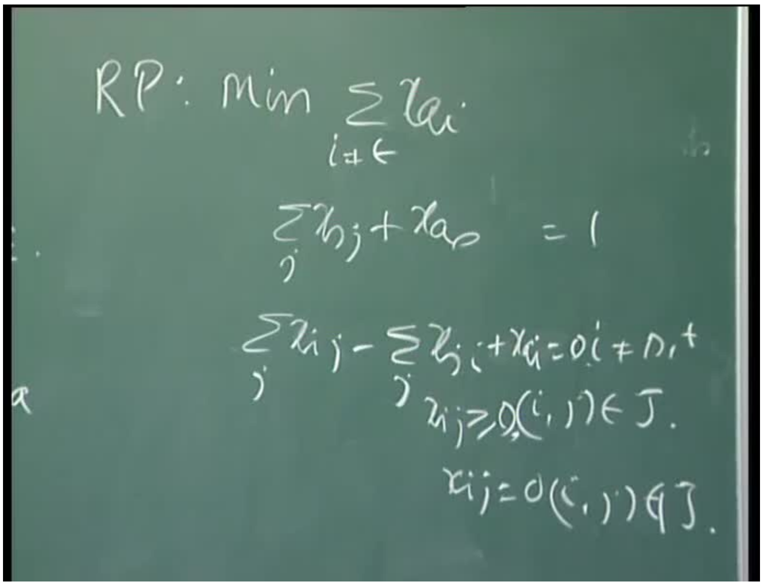

Because, I only permit those columns here, for which the corresponding arc is admissible. So, the restricted dual this become this and of course, we will also maintain that pi bar t is 0, so that will be maintain, and the pi i bar is greater less than or equal to 0. Now, there will be something else also there, sorry, you have to add xai equal to 0 dual variable.

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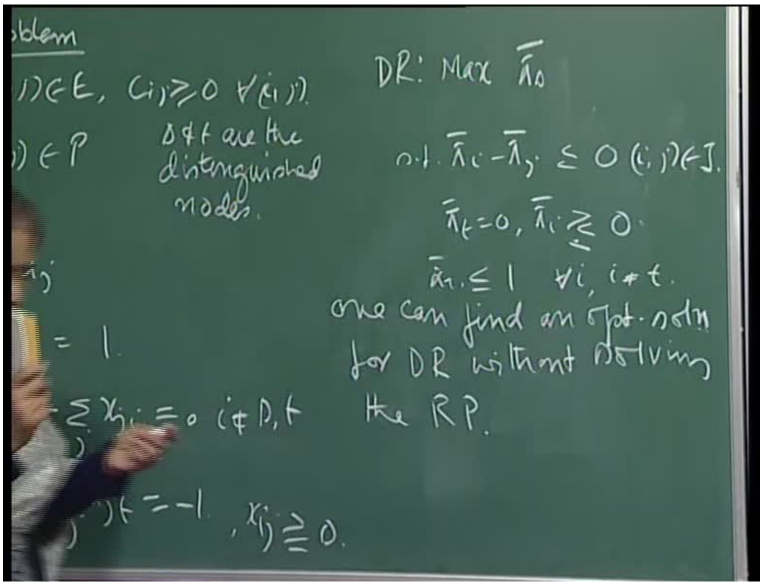

So, now corresponding to these dual variables, it will be pi r bar less than or equal to 1 for all i not equal to t, because pi bar t is 0, because all xai's are there in the objective function and the columns corresponding will have only 1 entry for the artificial variables. So, this is pi s bar less than 1 pi bar i less than or equal to 1, so this is your restricted dual problem. Now, the very simple way to find one can find an optimal solution for DR without solving the restricted prime; this is the beauty of it.

And so we will throughout keep on solving the restricted dual problem to construct feasible path or the shortest path for the primal problem. And this is where you see how beautifully primal-dual algorithm does it for you. So, let us see one can find an optimal solution for DR without solving the arc. So, the idea is, you see all restricted dual variables have to be less than or equal to 1, so the max value can be only 1.

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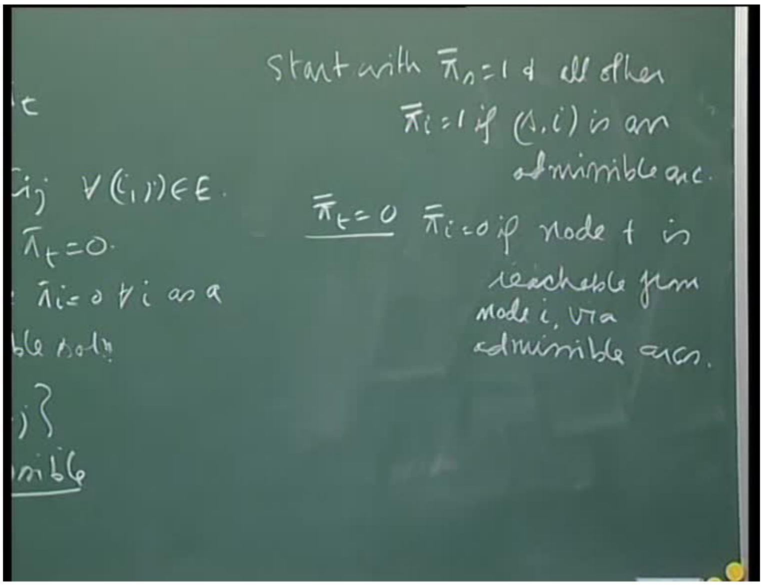

So, therefore, let us see, try the solution, so will start with pi bar s equal to 1 and all other pi bar i equal to 1 if s, i is an admissible arc. So, actually I should write down the rule for constructing the optimal solution for the restricted dual, start with pi s bar is 1, so this is an admissible arc and we have already chosen pi bar t as 0.

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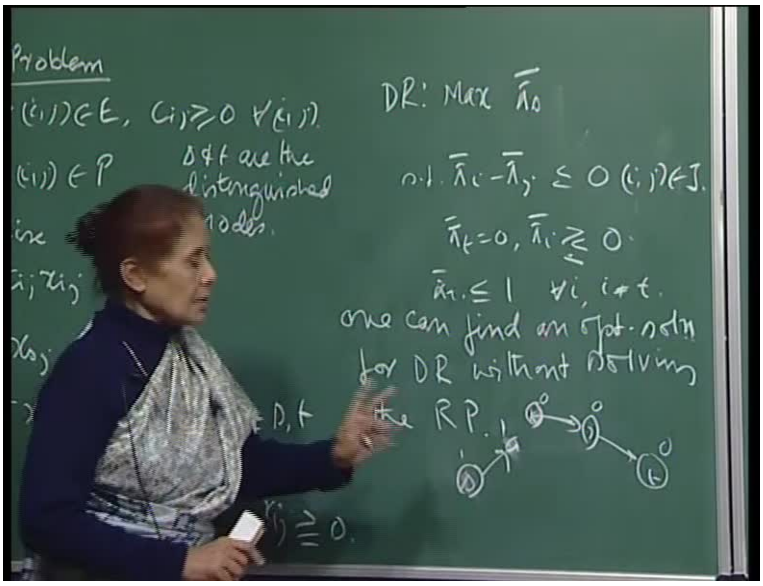

So, therefore, the rule would be pi bar t is 0 and we will say that pi bar i is 0, if node t is that is a better way to write, it is reachable from node i via admissible arcs. See, you have a graph here, so we have at node s here, we have a node t here. If there is an admissible arc, I am starting with this as 1 - I mean - if this arc is there is an admissible arc, then this also takes a value 1 and so on.

And I start from here, so this is 0, if there is an arc which is admissible and which is reaching t. The idea is this, and similarly here, if you have this arc here k, so if these are admissible arcs then all this will become 0 this is the idea. So, obviously, the thing is that, so we will start building it from here, if there is a path from s to t, then through this process I will ultimately end up putting the value of pi bar s as 0.

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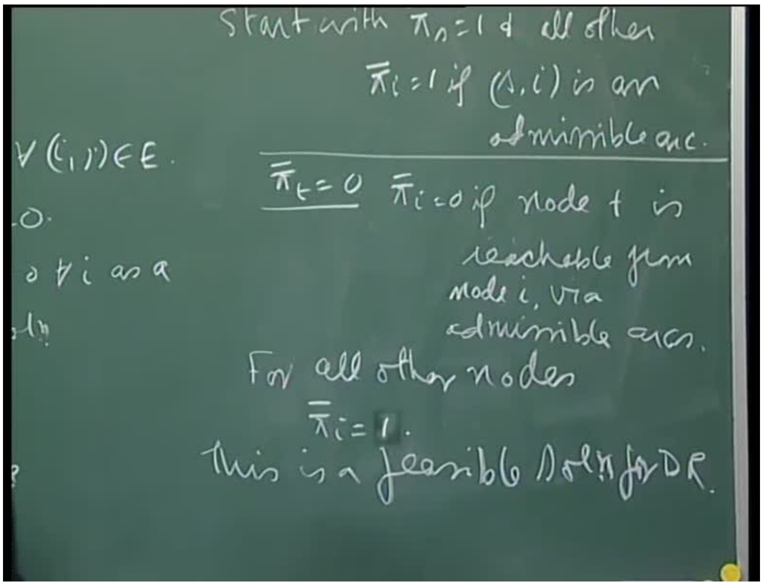

So, therefore, you might say that we start from here, pi bar t is 0 and pi bar i is 0 if node t is reachable from node i, via admissible arc. And this we have said, if they are nodes, which can be reached from s, then we will give them all values 1, so these are the 2 things. And for all other nodes; that means, nodes which are not reachable from s and from which t is not reachable, both the things, so for all other nodes pi bar i is equal to 1, this is then i, see you can see that this is a feasible solution for DR.

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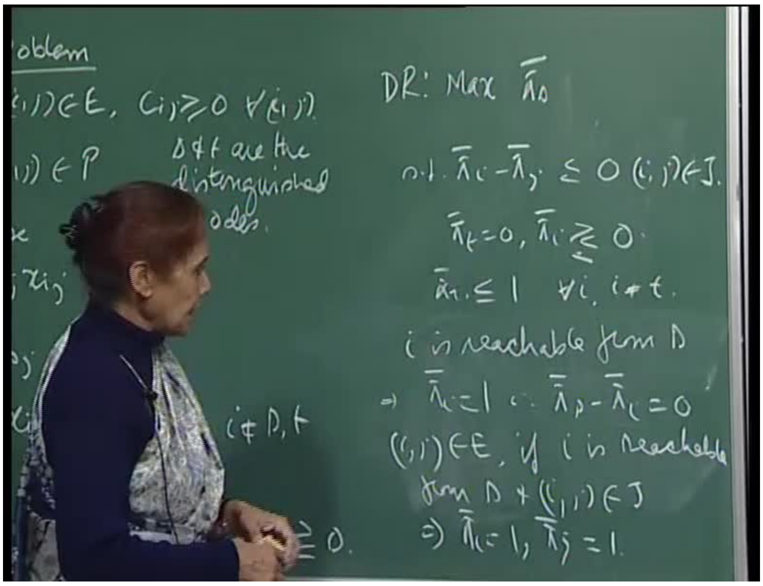

Why it is feasible quickly? Let us check it, so here I can show you the feasibility of this. So, what is the possibility? This is a feasible solution for DR which means I want to make sure that these constraints are satisfied. So, what do we do it? Now, if i is reachable from s, this implies that pi i bar is also 1, therefore, constraints of the kind pi s bar minus pi i bar is 0. So, this is satisfied, and this will continue for any i j that means or how do you want to do it, so this is this.

Now, you take any arc this you take any arc ij belonging to E, if i is reachable from s, and i, j belongs to J, this would imply that pi i bar equal to 1 and so the pi i bar is 1, and there and pi bar j is also 1, because if i is reachable from s and ij is an admissible arc, then obviously, by our definition by our construction pi j bar also takes the value 1, and so this constraint is satisfied.

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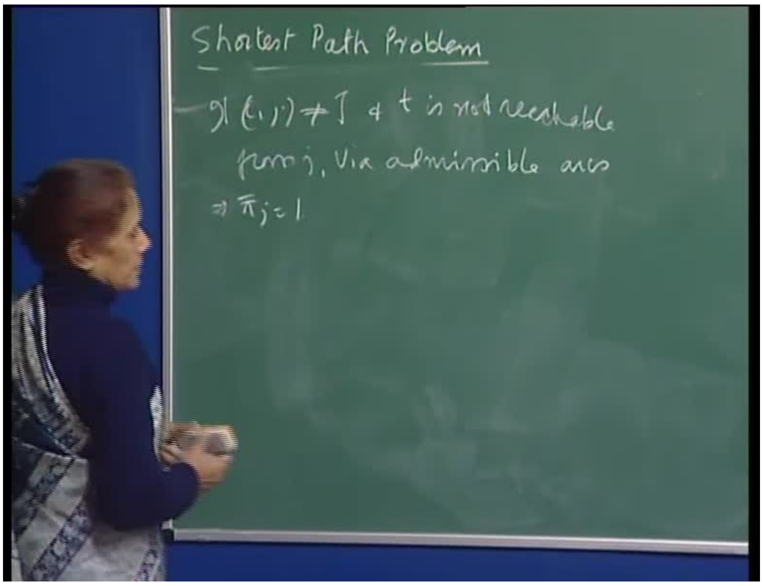

And if ij does not belong to J and t is not reachable from j via admissible arcs. Actually, this should be understood where, reachable means that I am using only admissible arcs, - via admissible arcs - and t is not reachable this would then imply that pi bar j is 1.

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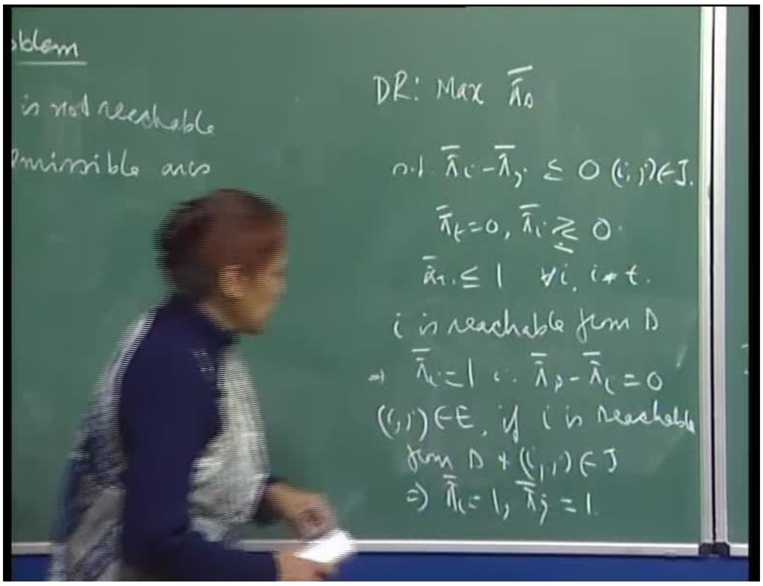

So, you have that i is reachable from s, so pi bar i is 1, but ij is not an admissible arc and also j you cannot reach t from j via admissible arcs, so by our construction again pi bar j is 1.

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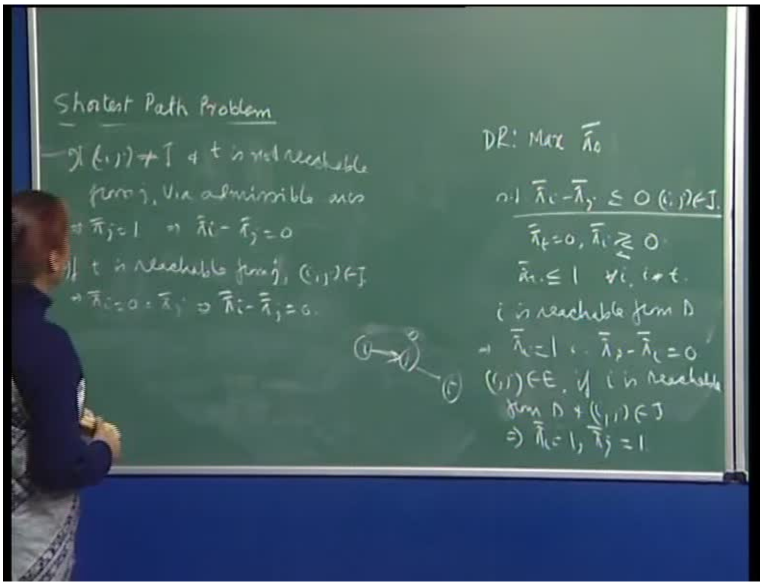

So, this again implies that, pi r bar minus pi bar j is 0, now the final thing is that if for the arc ij if t is reachable from i, and they are 2 cases; i, j belongs to J then we are done. Then, this implies that pi bar i is 0 is equal to pi j, and therefore, this again implies that pi bar i minus pi bar j is 0. And if ij does not belong to J, and what will happen? and t is reachable from j, so and if ij is the admissible arc then t will also be reachable from i, because up to j, from j to t I can reach via admissible arcs, ij is also admissible, so i can reach from i to t.

And therefore, pi bar i will also be 1 and so this will be satisfied. If ij does not belong to J, this implies that pi bar i is 1, and pi bar j is 0. So, again the constraint is – hold on- what is the problem? I want pi bar i minus pi bar j to be less than or equal to 0, and we are saying that t so you have an arc I you have at this and you have reachable from t, so this is not admissible. We are saying that if this is admissible, then of course, this is also 0 both are 0. But if this is not admissible, then this is this will be 0. And if it is not an admissible arc, then I should have pi bar i minus pi bar j less than or equal to 0 will not be satisfied.

This is for only arcs, which are admissible, so I should not have added from j, via admissible arc, it is only the constraints are only for the admissible arcs. So, let me just for so this part is not there otherwise you not have feasible solution. So, it is only so that means, if this constraints are only for admissible arcs.

So, if i is reachable from s, then this takes the value 1, and since ij's admissible so this also takes a value 1, so this will be satisfied. If t is reachable from j, and ij is an admissible arc that pi i bar is also 0. So, then the difference is 0 again, so feasibility is there, so this is the feasible solution, so this will not be admissible.

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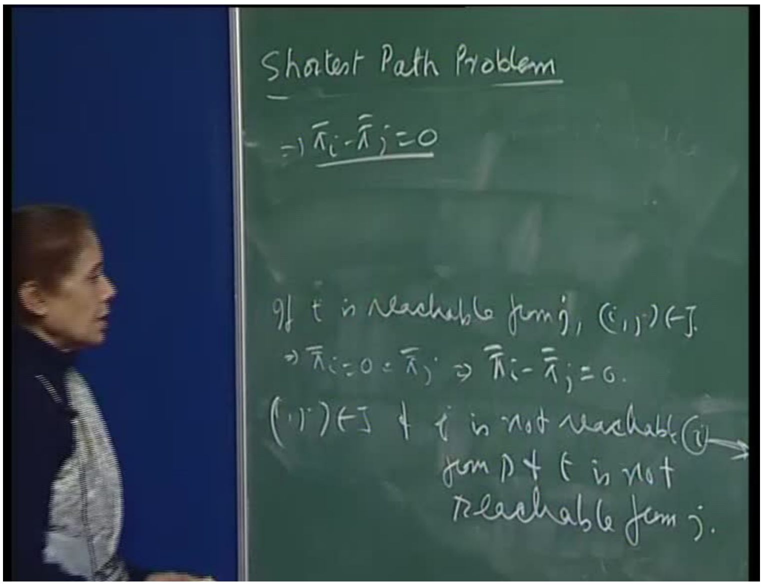

So, if now, so I simply want to say and this I have already said it here, that is, i is reachable from s then pi bar i is 1 and for any ij in E, I am saying that if i is reachable from s, and ij belongs to J then both have value 1and so the constraint is satisfied, and therefore, this implies that pi bar i minus pi bar j is 0. This takes place and if pi bar j and if t is the reachable from j, and ij is the admissible arc, then both have value 0 and so this happens. So, the third case is, that ij belongs to J and i is not reachable from s, and t is not reachable from j.

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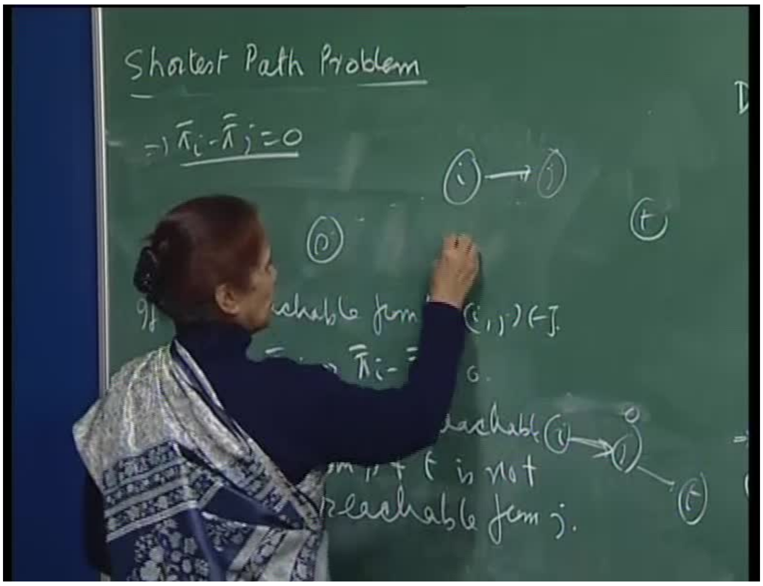

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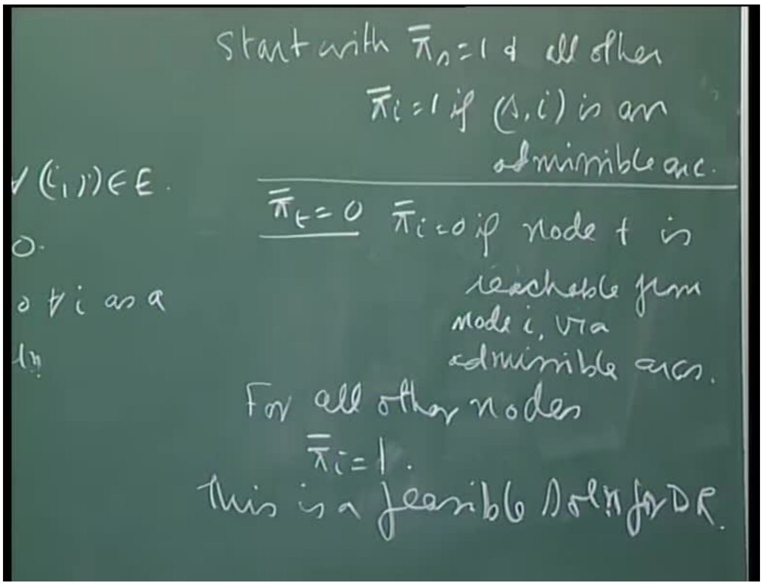

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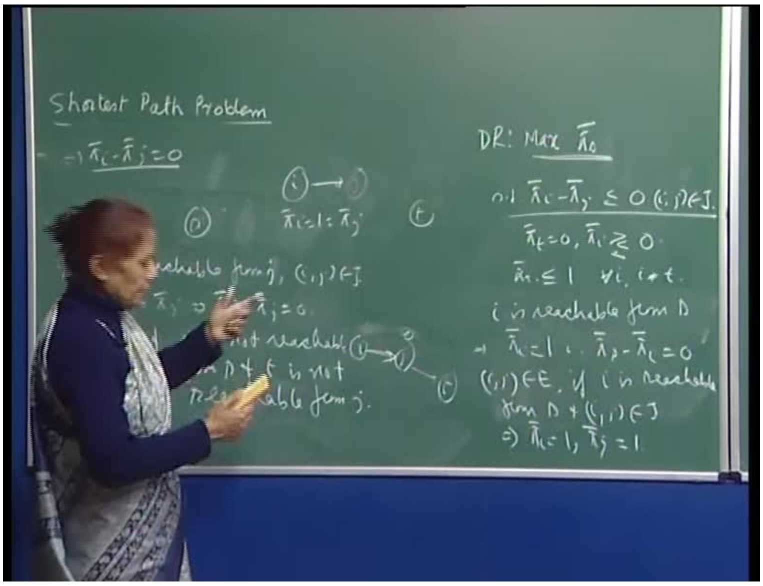

So, that means, it is simply an arc ij and there is no path connecting it to s or there is no path connecting from j to t. So, then in that case, by our construction we said that here, so for all other nodes it should be 1, therefore, again the 2 values will be the same in this case, this is admissible, so therefore, this will imply that, pi bar i is 1 is equal to pi bar j. The construction has to be sort of very clear in the mind, so this is pi bar this is 0 and therefore, again the constraint will be satisfied for the admissible arc.

So, this is a feasible solution and since the maximum value for this cannot be more than 1 and I have already, and so this feasible solution has that maximum value, therefore it is optimal. So, I did not have to use the optimality criteria or anything, it is just by observation we are saying, we have found a feasible solution to the restricted dual, and the maximum value is attained, therefore it must be the optimal solution, so this is how our things become simple why because, the structure of the problem, you see the constraint set as a very simple structure here, the dual problem, the dual constraints are very straight forward.

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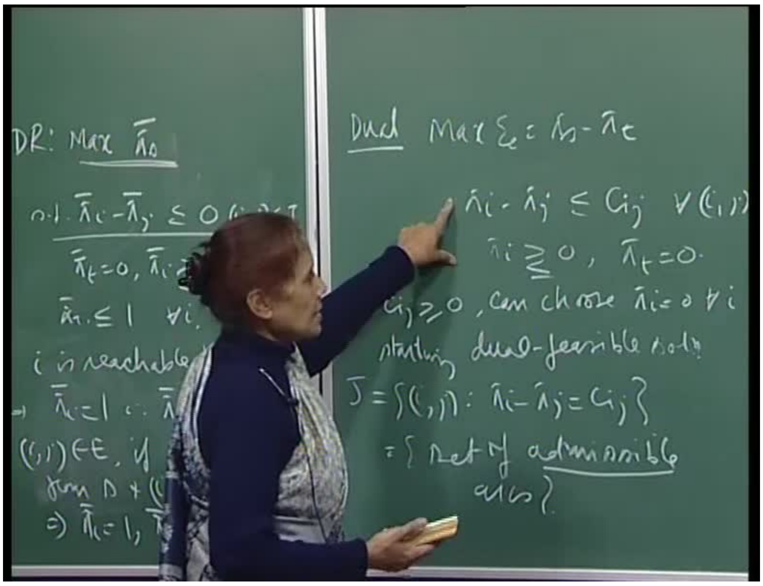

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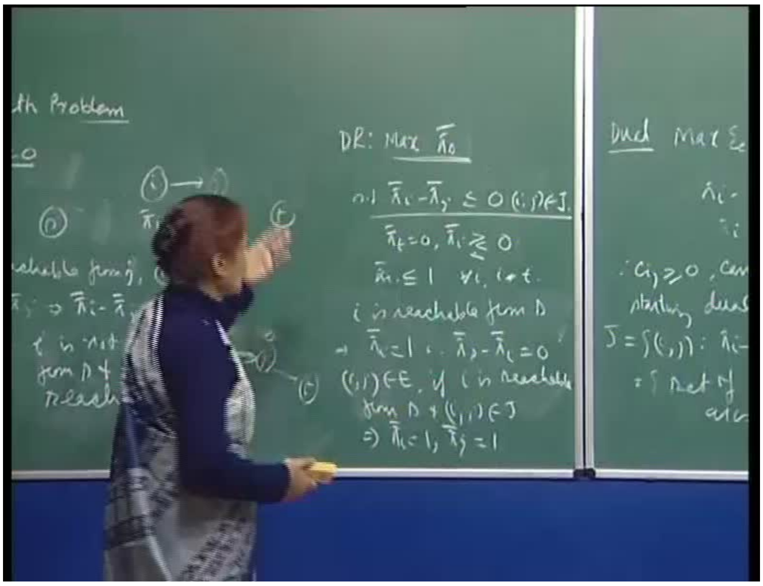

So, we can by observation look at obtain solution for the restricted dual, so as long as, max pi bar s is positive which is equal to 1; that means, your restricted dual or restricted primal it does not have the 0 value, remember, all the artificial variables have to leave the restricted primal in other feasible solution.

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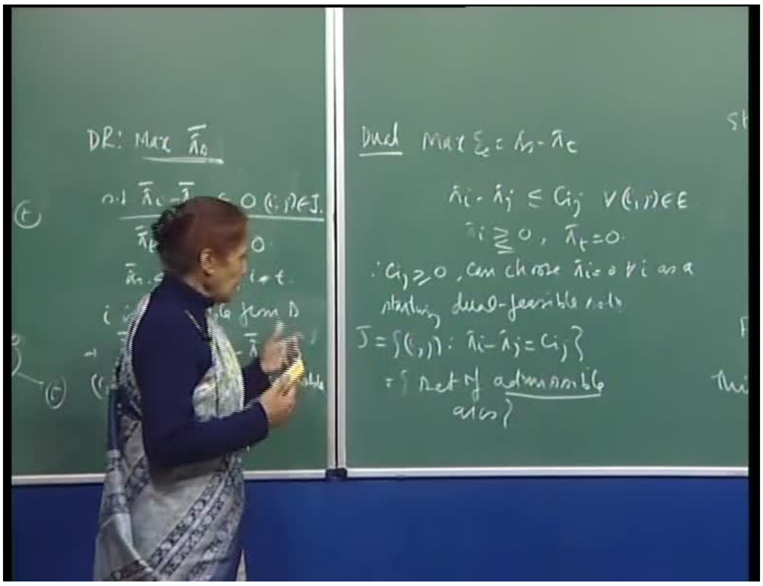

So, what we were saying is, that we are exploiting the structure of the problem here and since, the value here is not 0, therefore, I do not have a path from s to t, which is the idea behind this whole exercise. So, now I need to therefore, modify the dual solution, because the current dual solution is not yielding primal feasible solution; that means, the admissible arcs are not enough to give me a path from s to t. So, we will try to get some more arcs, which can become admissible that means I have to modify the dual solution and remember, our this thing was…

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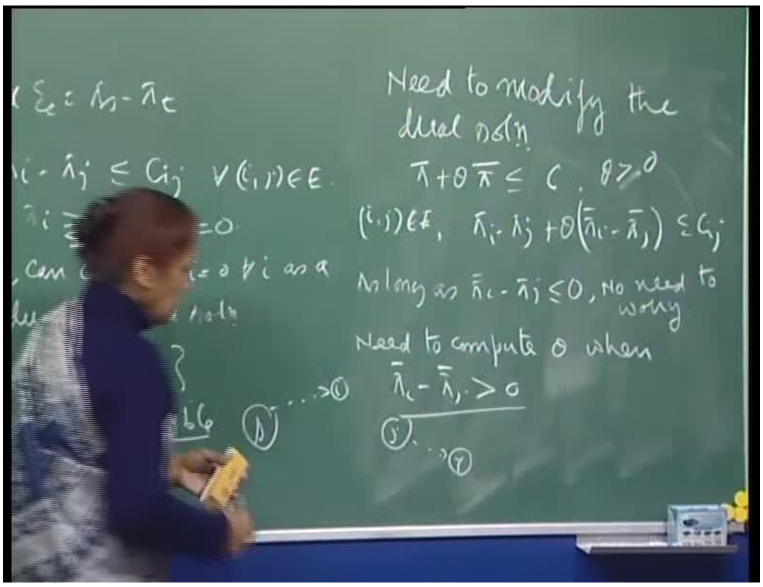

So, we need to modify the dual solution, and here again you will have to recall, so under our method was to do it pi plus theta pi bar and we will choose theta in such a way, that this solution is dual feasible. And by pi bar I already showed you the construction for the restricted dual, so this is this, where of course, we have to first compute theta. So, you see here again, in pi bar what is happening? So, now we will be adding an arc which is not in J, remember, I need to expand my set j.

So, you can see from here, that an admissible arc, you want to this to be less than or equal to cij, I am just trying to thing what is the best way to lead you to obtain the value of theta, so anyway, let us see this is this. Now, if pi bar, sorry, this is less then of equal to C, so a typical constraint that means, for ij belonging to E, the typical constraint will be pi i minus pi j plus theta pi i bar minus pi j bar less than or equal to cij.

This is what, if I have modify my dual solution and this is my new dual solution, then this is what I have. Now, as long as, pi i bar minus pi j bar is less than or equal to 0, I do not have to worry, because and of course, here this is another step, that theta is greater than 0 in fact, because, you want to modify so theta has to be chosen positive. So, if theta is positive, and this thing is less than or equal to 0, then I do not have to worry about it. And this will happen when, see, corresponding to my restricted dual solution, see some pi bar i's have value 1, some pi bar i‘s have value 0. So, when will this be negative or less than or equal to 0 either of course, the 2 values are the same, in that case, the arc has to be admissible remember.

The way I constructed any admissible arc will allow you to fix the same values for the restricted dual variables and therefore this is will be satisfied as equality. So, when this is 0 and this is 1; that means, you are so this will be, this case less than or equal to 0, we do not want that, because that is not a problem for us, the constraint will be satisfied as long as less than 0, no need to worry.

I will just writing our, no need to worry; any value of theta will satisfy the dual constraint. So, need to compute theta when pi i bar minus pi j bar is greater than 0, and when will that happen? When pi bar i is 1 and this is 0; that means, you have a path from s to i, either this has a value 1, because there is a path from s to i, and then, you have pi j, sorry, j here and there is a path from here to t. This is one situation, so this will have a value 1 and this will have a value 0, so 1 of them has to be because both one again will not give me 0, so I need this to be 1 and this.

So, either node i is not reachable from s, then also we said that, the value would be 1 or it is reachable, in that case the value is 1 here and this is 0, because, so you see now, I will when once I say that you do this, and then, we will need to compute that means, I need to compute the value of theta for such arcs.

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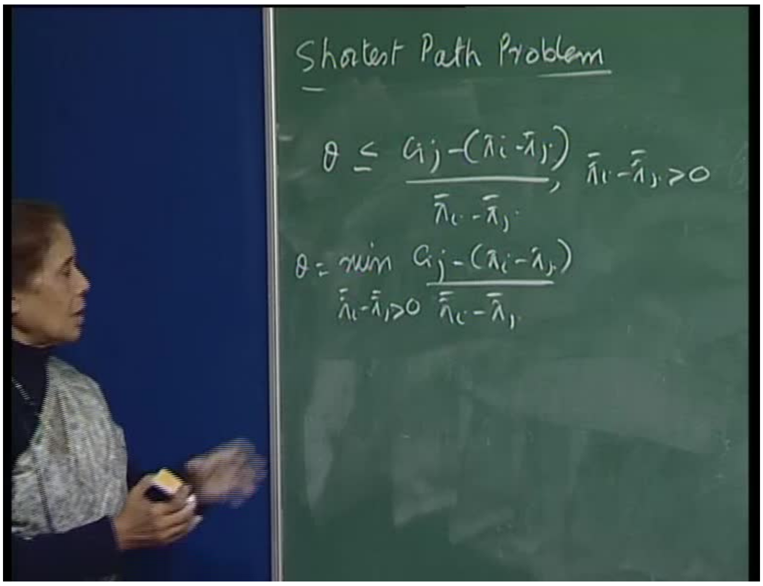

Because, once this is positive, then depending our t value of theta, this may violate this constraint, therefore, I need to fix the value of theta such that, it does not violate. So, then what do you get? You will get the theta should be less than or equal to cij minus pi i minus pi j divided by pi i bar minus pi j bar. And so we choose theta to be minimum, where of course, pi i bar minus pi j bar is greater than 0. So, it will be minimum of cij minus pi i minus pi j divided by pi i bar minus pi j bar and pi i bar minus pi j bar is greater than 0.

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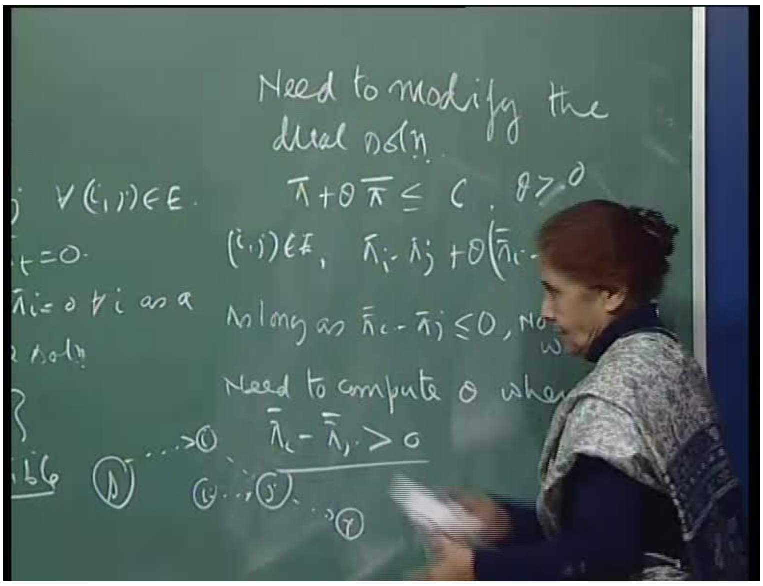

So, among all such things; that means, you will certainly we adding an arc of this kind, from j, there is the path to t and now, you by adding this arc and there will be many others here. So, choosing the minimum of all these for this ratio, so you will have at least 1 arc which was not there in J and why? Because your pi i bar minus pi j bar is positive.

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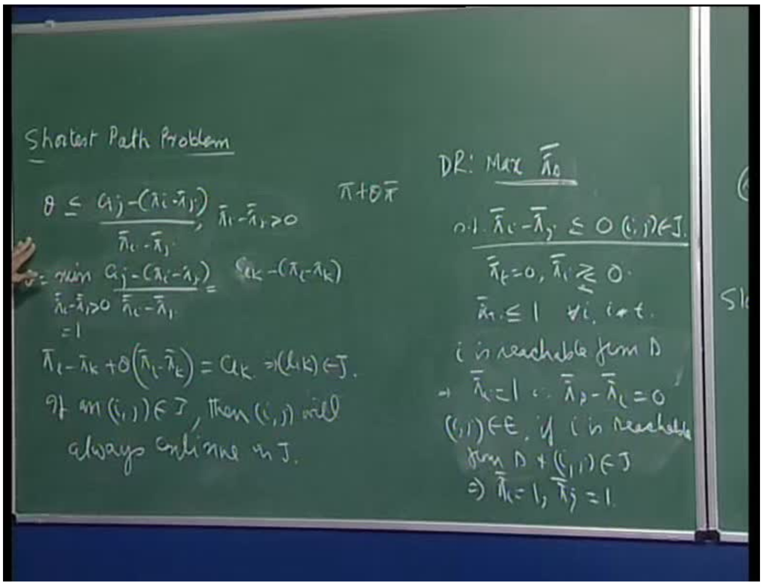

See, for admissible arcs, this would always be equal as we saw by our construction. So, this is what do, we so you choose your theta to be such, then if this minimum occurs for let us say, what do you want to say here something like ok, take some arc l k or something, so this was our theta will be computed this way, now for whichever arc let us say this is attained for clk minus pi l minus pi k.

And actually this the value, because this is equal to 1, whenever this number is positive, it has to 1, because we are giving the value is to pi bar i and pi bar j as 0 or 1, so whenever this is positive this has to 1. So, actually, your theta is nothing but this number, and for this number you see, your when you do this pi l minus pi k plus theta times pi l bar minus pi k bar, this will be equal to clk, that because of the definition. This is theta say you multiply this by this is 1, so when you have updated your solution this way, this becomes equal to, see, from the definition, because this is 1, so just see from here, this when you bring it to this side this thing this become clk so this is equal.

So, therefore, you have 1 more and so this implies that l k has been added to J, and we will also, this can be an exercise or we will discuss it in the next lecture, that if an ij belongs to J, then ij will always continue in j; that means, once an admissible arc it will always remain an admissible arc. And I will continue with this later, but let me now show you how the algorithm progresses. So, the idea would be that I update my dual solution with pi plus - the updated dual solution is, pi plus - theta pi bar and then again I have a new set J and extended set J, I will again try to find a path from s to t.

And this searching for the path is very straight forward, because you will be at the dual restricted, you will able to get a solution, and if you can label in the sense if you can give put pi as bar also as 0 then you are done. So, we will continue with this algorithm, till we or I write a path from s to t. So, the aspects that how much effort will it be required and that we will always find an optimal solution to the shortest path problem, all those things I will discuss in the next lecture, but here let me just show you the iterations of the algorithm, so that you get a feel for it and then we can look we can understand the proof better, once you know how the algorithm is progressing.

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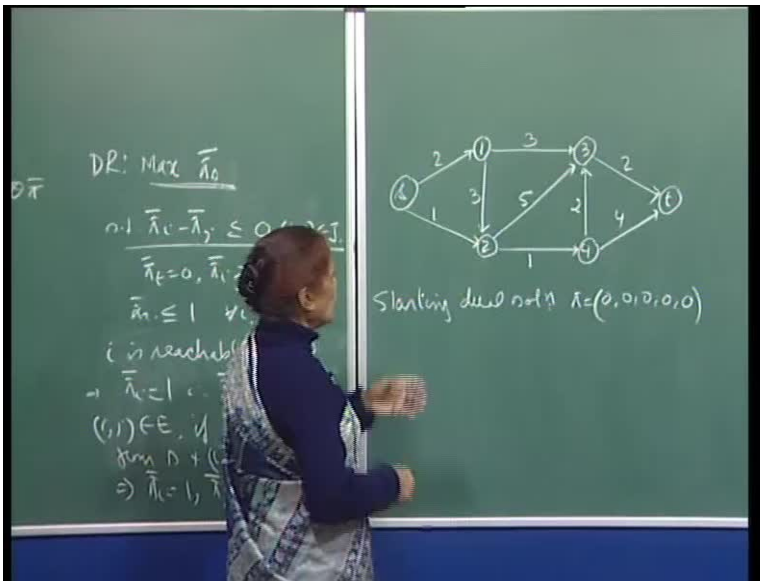

So, here, you see the numbers on the edges on the arcs give you the weight the cij’s, there are all non- negative. So, your dual constraints if you want to write them some where here, so if you write I can straight away, write the dual solution, I do not need to write that primal.

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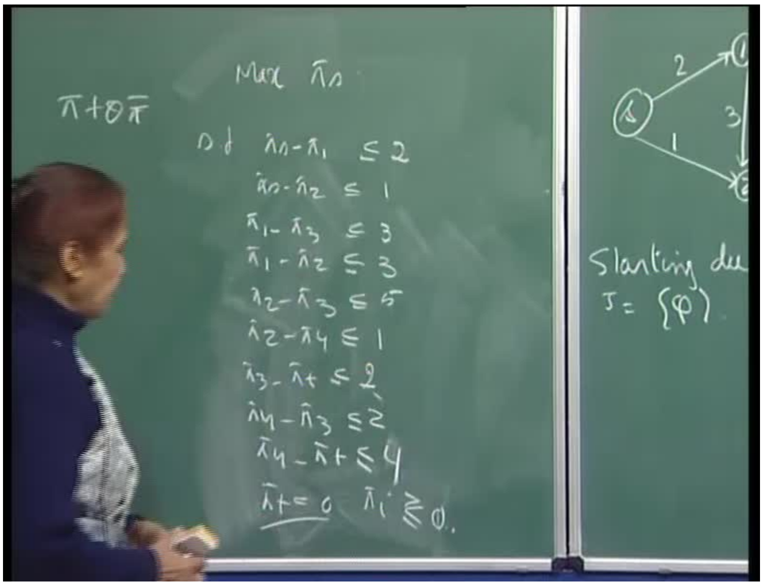

Because, for each arc this will be maximize, and so this is simply pi s subject to pi s minus pi 1 less than or equal to 2, and these are the kind of equations you will write pi s minus pi 2 is less than or equal to 1, pi 1 minus pi 3 less than or equal to 3, then pi 1 minus pi 2 is less than or equal to 3, for the arc 1,3.

Then, you have 2 to 3, so pi 2 minus pi 3 is less than or equal to 5, pi 2 minus pi 4 less than or equal to 1. Then, you have 3 to t, so pi 3 minus pi t less than or equal to 2, and from 3, you have nothing, so even then from 4 you have pi 4 minus pi 3 less than or equal to 2 and pi 4 minus pi t less than or equal to 4.

And we are saying that, so you could say that why I am I writing pi t it we does not matter pi t is 0 and all other pi i's are can take anywhere, so this is your dual solution, so we can choose the starting dual solution as all 0's I am not writing pi t 0 because, that is understood no point carrying it all the time, because all these things are non-negative, so when I put all pi 0, I have a feasible solution.

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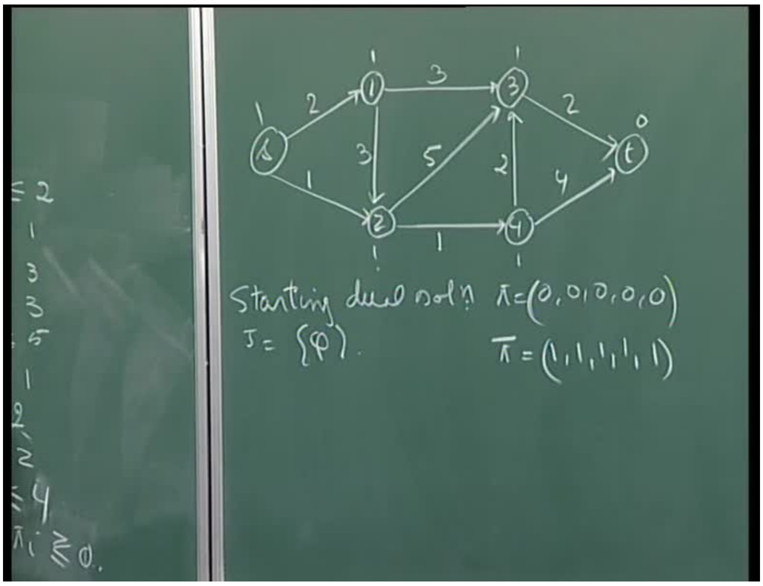

And so the starting here J is empty, then no dual constraints satisfy it as equality, because all these numbers are positive. So, J is empty, and therefore, you simply now need to and so what would be this thing, so remember since J is empty there are no admissible arcs. So, I will write the values of the restricted dual here, so this is only 0 and all others are 1, 1, 1 and 1, is it ok, or if you want you to use starting those solutions and then, you so you will have to write pi bar as 1, 1, 1, 1.

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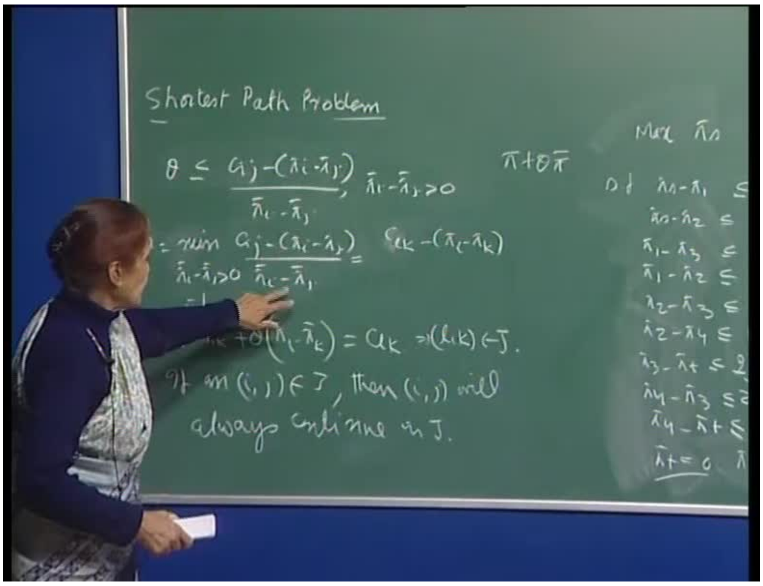

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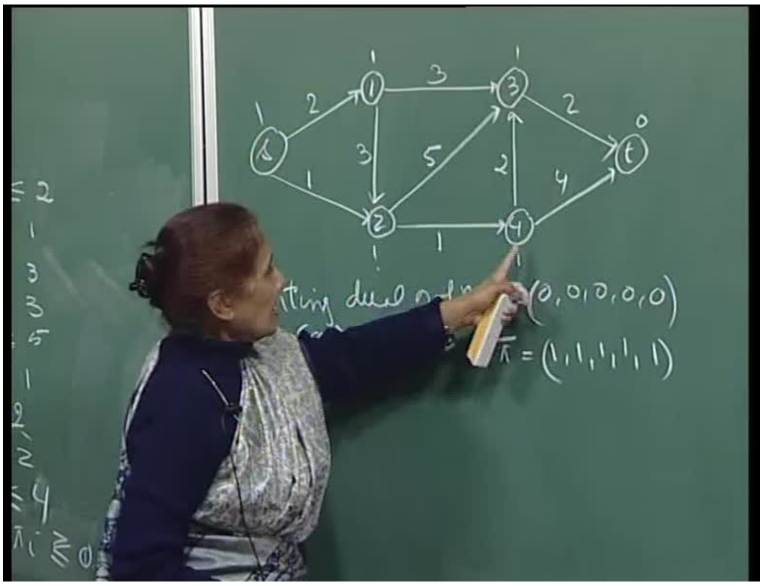

So, you need to compute theta, and remember, to compute theta, we have to go for arcs for which this is equal to 1. And since this is 0, you see for this arc and for this arc, the value is 1.

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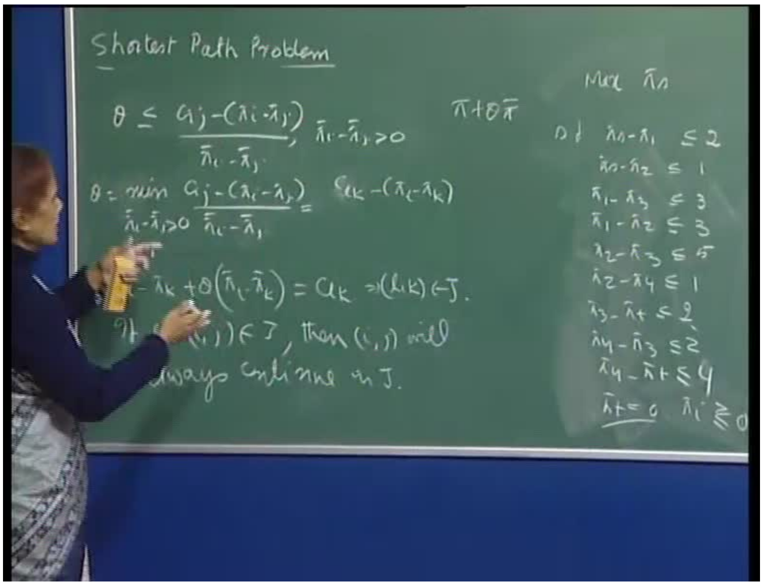

Pi 3 minus pi 3 bar minus pi t bar is 1, and pi 4 bar minus pi t bar is 1. So, yes, and what are your cij? The thing one has to stream line here is you understand where you get the quantities.

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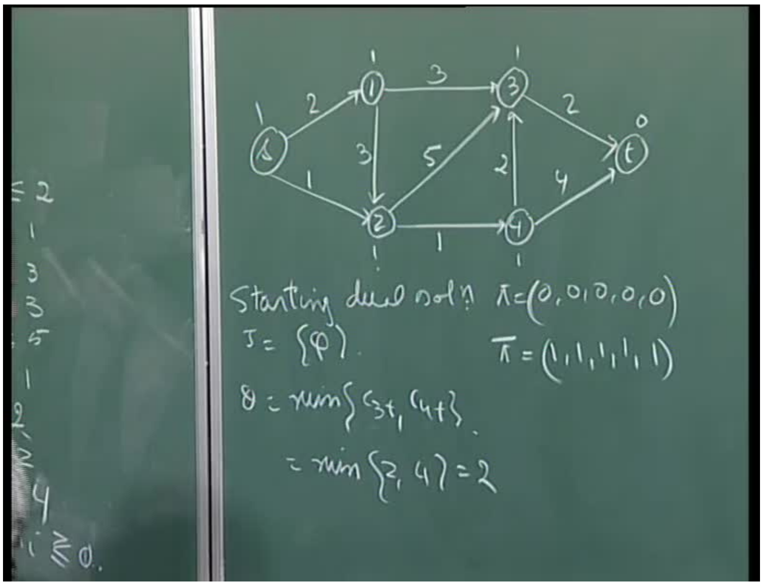

So, this is right now since, all pi i's are 0s I simply have the cij's, so I have to that means, take theta will be simply minimum of your c3t and c4t - and I will try to show you how to get the values as you proceed - so this is, c3t, c4t which is minimum; 3t is, 2 and this 4 so this is 2, so your theta is 2 and therefore, your new dual solution, which is pi plus theta pi bar, so twice this you have to add here, so this becomes 2, 2, 2, 2, 2 this is your new dual solution.

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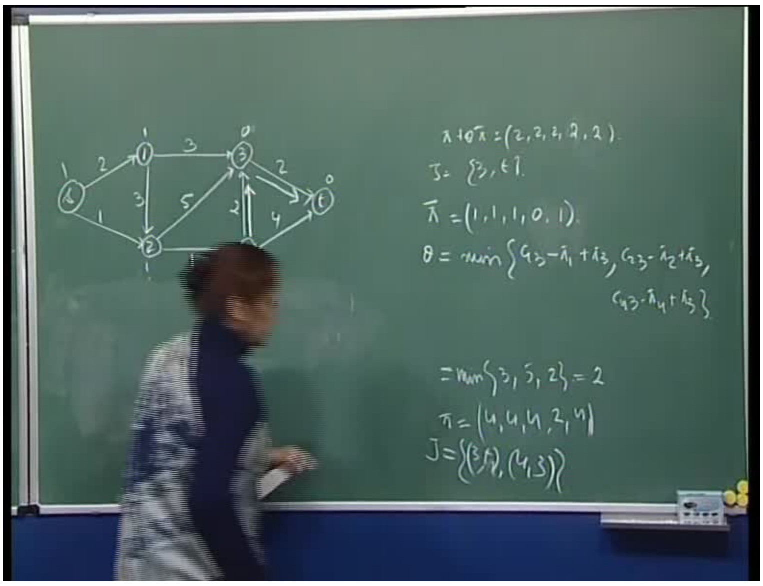

And since, the minima occurred for arc 3t you see, your J now contains are 3t, this J contains, so this becomes admissible, so therefore, the restricted dual solution is immediately available, because this is 0 and this only arc which is admissible arc, so this become 0.

So, this is your new dual solution and your pi bar now becomes 1, 1, 1, 0 and 1, well lets see it will be this way because node 3 comes first, this is your new restricted dual solution, so this remains as 1 pi bar s, the maximum value, because I have do not have any other admissible arcs, so you can see actually see that this is how you constructing the path.

You have chosen r 3t as admissible arc and you are able to connect 3 to t. Now, you might say that, so here either you keep writing the dual solution here, well in the next lecture, I just write now I want show you the iterations, and then we will come back to stream lining the presentation so that you can get the quantities as you want.

So, therefore, need to compute theta, so compute theta again and for theta again, you see because this number is 0. Now, you have what are the possible candidates? This is 1 pi bar, remember, pi bar 1 minus pi bar 3 is 1 then pi bar 2 minus pi bar 3 is 1, because this is 1 that is 0 and pi bar 4 minus, so the theta would be minimum of, so let us write it down quickly, this will be c13 minus pi 1 plus pi 3, this is 1.

Then, it will be minimum of c23 minus pi 2 plus pi 3 and finally, this will be c43 minus pi 4 plus pi 3, this how we compute, so this will we have to choose minimum of, now c13 the number is given to you here, so 3 minus and we have the dual solution here, for 1,3. So, remember 1, 2, 3, 4, well, s, 1, 2, 3, 4, so what is this value here? So, this is again its coming out be minimum of them, because these numbers are the same pi 1 and pi 3.

So, therefore, the simply we c13 is, 3 c23 is 5, and c43 is 2, is it fine, because all these numbers are the same, the dual solution is the same, therefore they do not contribute to your computing the minimize it is only the cij's which are required, so this is 2, so your theta is 2.

And therefore, your new dual solution pi would be twice this, you add here, so that will become 4, 4 s, 1, 2 and the 3, 1 this is 0, so that remains 2, and that becomes 4. Twice this you have to add here, so these 3 become 4, this remains 2 and this 4, so this is your new dual solution, and you have chosen the arc 4, 3, therefore, you see this is your step by step, you are constructing your shortest path.

And so once you have this, then I need to, so J becomes that, and you still have the value of pi bar s as 1 for the restricted dual. So, I need to update my dual solution further, and that means, that I need to, and so what would be your restricted dual solution now? So, let me update it here only, or you mant me to write it here, I will write it here the restricted dual in that case, once you know your J, so what is my J? Remember, you have to keep updating your J, so your J now consists of 3, t and 4, 3 this consists of this.

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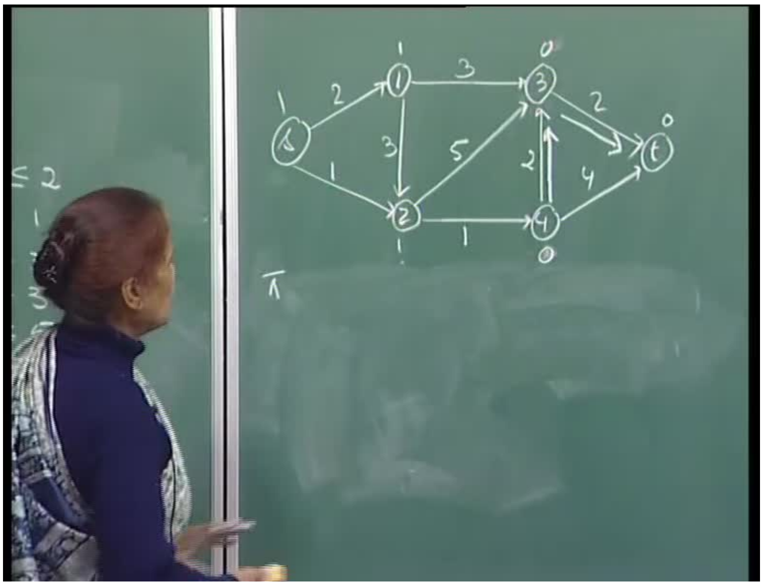

Now, what is my restricted dual solution? pi bar, so this will also become 0, and all others remain1, because nothing is no admissible arcs here, so this all will remain 1.

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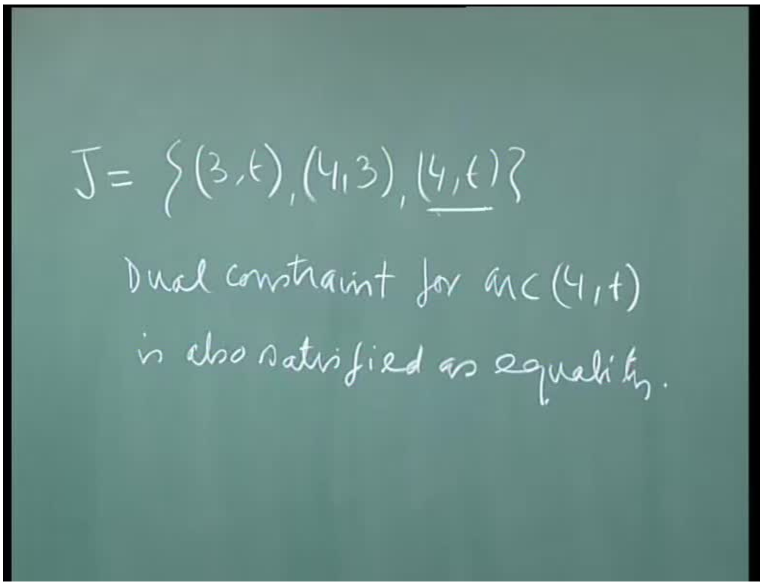

See; want to give you the correct form of set J which is the collection of all the arcs of which the dual constrains are satisfied as equality. So, arc 4 comma t was left out, but if you look at the dual variables, when you can immediately check that the dual constraint for arc 4 comma t is also satisfied as equality, therefore, arc for 4 comma t should figure in the set J.

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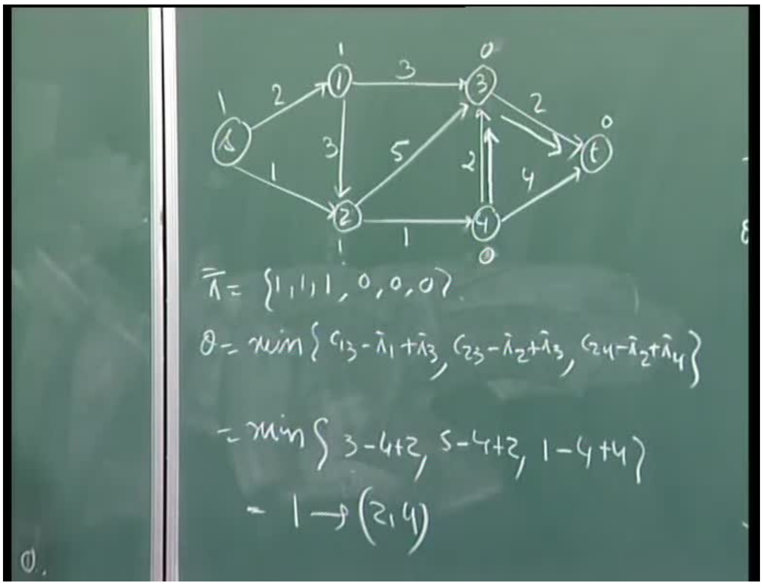

So, once you have your pi bar this is 1, 1, 1, 0, 0, 0, this is your restricted dual solution, and we need to compute the theta, so what will be theta now, I am writing all the step so that it becomes very clear and then we will stream line the algorithm. So, minimization now this is all 0, here you see, this will be you can start from there, this will be 1, 3, so we write c13 minus pi 1plus pi 3, then it will be 2 3 again, so c2 3 minus pi 2 plus pi 3 and then you have this 2, 4, so this will be c24 minus pi 2 plus pi 4, I think that is it, because these 2 are 0s, so you have this, this and this these 3.

So, let us quickly, because I have my dual solution here, so write down the values minimum c13; c13 is, 3 minus pi 1 - pi 1 is, 4 and pi 3 is how much? This is, I am just writing the values and then we can check. So, then c23; c23 is 5, minus2 which is again 4 plus 2 because pi3 is 2 and then, you have 2, 4, so 2, 4 is 1 minus this is 4, and what do you have for this plus 4, therefore, this is equal to 1 and this corresponds to what arc - 2, 4.

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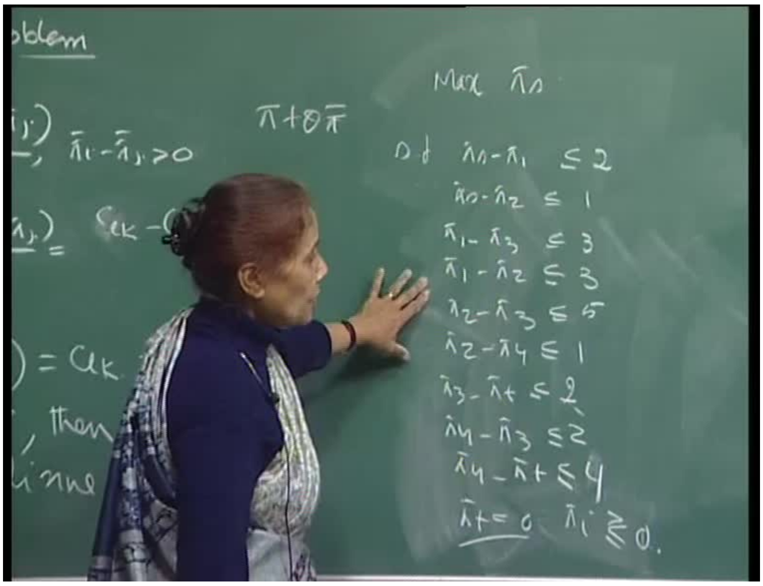

So, the new value; that means, theta is 1 and therefore, you have to update your dual solution, and yes in the meantime should have ask you to please verify that what you have is always the dual solution, so do this while you go through this example.

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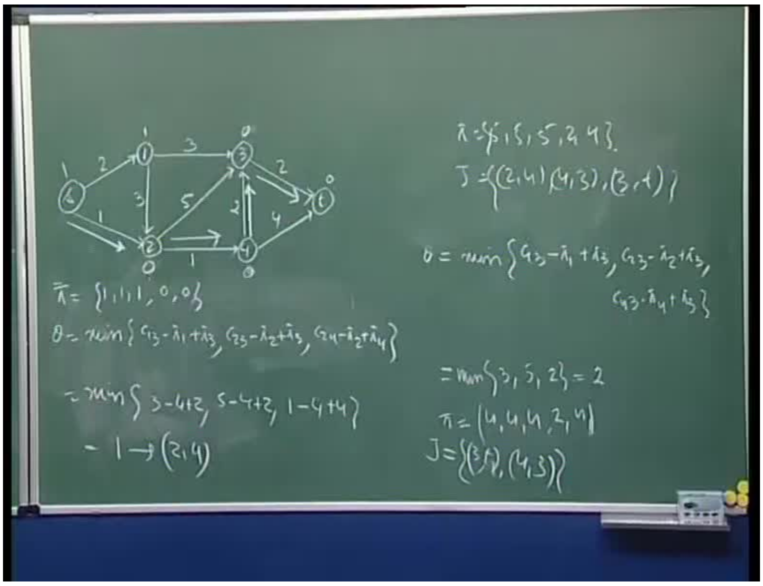

So, let us just compute our dual feasible solution, and this would be yes, so theta is 1, so 1 time you have to add this to this. So, what will it be, a new is 5, 5, 5 and then it is 2, 4, (()), so this is your new dual solution and you see that, this becomes 2, 4 gets added and your J, J gets updated to, this is a new arc that you have added here, so now you have 2, 4, 4, 3, 3, t, these are the arcs that you have, and so the new restricted dual solution I am showing here, will be this is 0.

And you see that now immediately, you can all these are 0s, so therefore, you have 2 arcs: 1, 2 and s 2 and that is it. So, now when you choose your theta that will be for arcs s2 and 1, 2 and you can immediately see that here, this will be the arc, I leave that to you sit down and work it out that your theta will occur for this - the minimum value and therefore, you now have a shortest path.

So, finally, when you do this, then because this becomes admissible, so your new restricting dual solution this will be 0 and you have done, because your pi bar s is 0, you have obtain an optimal solution to the restricted primal and so you have now an shortest path from s to t which consists of this - how we are going? s to 2, 2 to 4, 4 to 3 and 3 to t. And the total length would be here 1 get added, so I think it will be 6 - 1, 2, 4 and 6. So, the very interesting interpretations which I want to further show you that the dual variables at each iteration for certain nodes will always give you the distance of the node to that node t - the shortest distance.

So, interesting interpretation of the dual variable and stream line the algorithm, but you now get a feeling at each iteration the restricted primal or the restricted dual is nothing but building of the path from a node to the node t and that path is the shortest path. So, we will look at these, I will revisit this example and show you the stream line the algorithm.