Linear Programming and its Extensions

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Module No.# 01
Lecture No.# 17
Problems in Lecture 16, Starting Dual Feasible Solutions Shortest Path Problem

To be able to begin with the dual simplex algorithm or the primal dual we need the dual feasible solution, so I will tell you how to do it. Now, let us talk about how to obtain and we can begin from here only, how to obtain a starting dual feasible solution. So, suppose I have a basis, suppose B is a basis which is neither a primal feasible or dual feasible.

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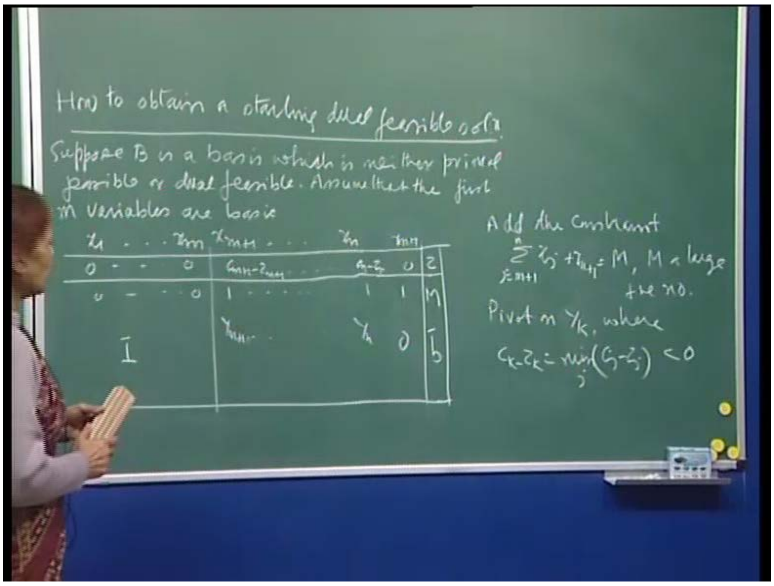

Let me make a tableau representation and assume - all it mean just an assume - that the first this, because it makes the representation of the tabular very simple, assume that the first m variables are basic, so then my tableau would be like x1 to xm, and xm plus 1 to xn which includes (( )) plus everything so here, you have its all 00's because the basic feasible there on the basis these m variables. When you have cm plus 1 minus zm plus 1 and so on up to cn minus zn, and this is your right hand side, whatever the value of z.

So, this is the matrix which is I here, here you have your Ym plus 1 and so on, Yn and you have your b bar. Now, since, the current basis is neither primal feasible nor dual feasible, there are some entries here which are less than 0 - strictly less than 0 - and there are some entries here which are also negative.

So, what we are saying is that, add the constraint summation xj; j varying from m plus 1 to n less than or equal to M, M a large positive number. So, sufficiently large number and it will be clear how sufficient, so if you add this constraint then you will add a slack variable here, so that becomes plus xn plus 1 is equal to m. So, let me add this constraint, so this will be all 0s here, this is 1, 1, 1, and then I am adding, so let me just make room for xn plus 1 right now, this is 0, so this is 1 and its all 0s.

This is M here, this is your z and this is your b bar, so this is my new tabular representation after adding that constraint, now what we will do is, I want to obtain dual feasibility, so what I will do is, pivot on k where or pivot on Ak, sorry, pivot on Yk, where ck minus zk is minimum of c j minus zj overall j and obviously, then negative ones of the lonely figure here, and this is less than 0.

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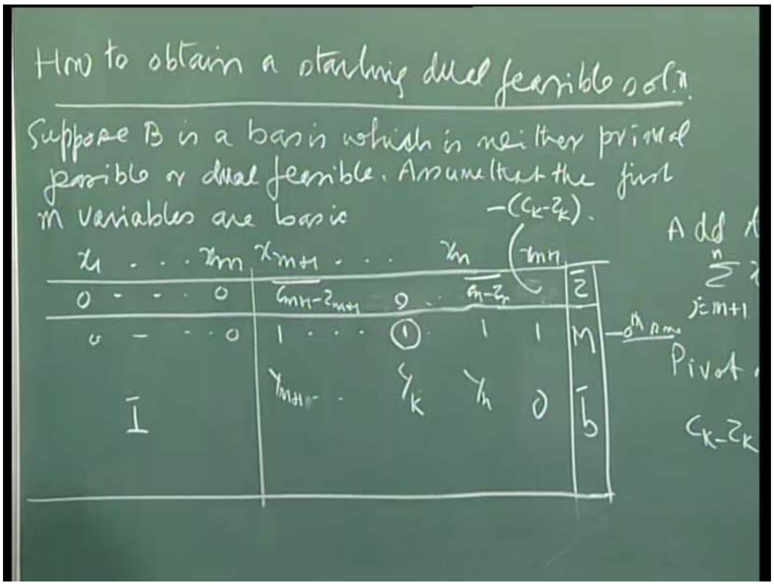

Since, there are at least one entry which is negative here, so the minimum will occur for negative one, so if you are more than 1, you select the one which the smallest. And then, so let us see here, you have ck minus zk and then, I will pivot on this entry here which is 1 and then this is Yk, because we want to remove this from the basis, so we will pivot out this. So, pivot on in fact, I should say pivot on let me make this more clear, pivot on Y0k here, I am calling this is the zeroth row, without any confusion just to write it there, but actually you are pivoting on this one. So, when you pivot on this one to make a 0 here, you will multiply this row by ck minus zk and subtract. So, because it is a smallest one when you subtract from here everything will become non-negative, because I am subtracting the smallest number from among all these numbers.

So, when I subtract it will make everything non-negative, so whatever the bars here, you can write them as bar, and this all become non-negative. So, this will be 0, and this is bar, this is bar here and then this will become minus of may be (( )) here, I will write it here this will be minus of ck minus zk, which is also non-negative, which is positive and then you will again do the same thing.

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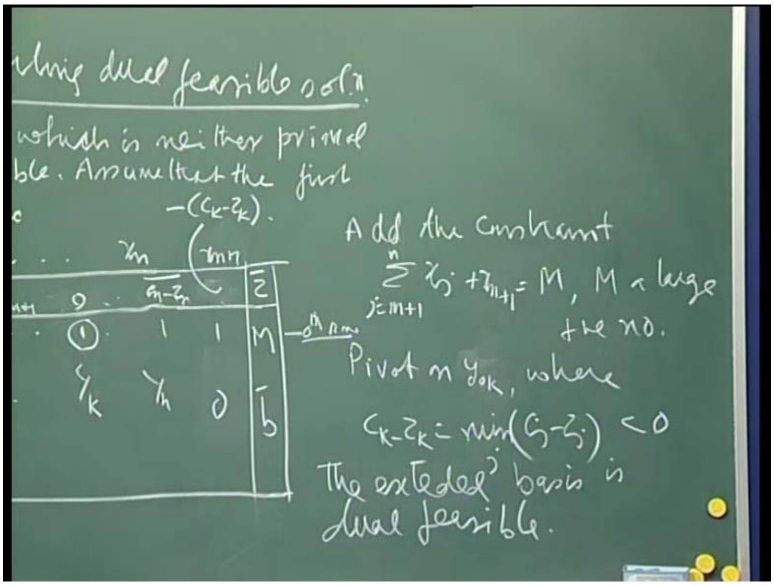

So, now, let me give you all the steps here, so actually you have dual feasibility, you do not have primal feasibility, so we have obtained a dual feasible solution. Now, new at the extended basis is dual feasible, because, you have non-negativity here, you have a basic solution primal feasible. So, once we can now proceed with either dual simplex algorithm or primal dual.

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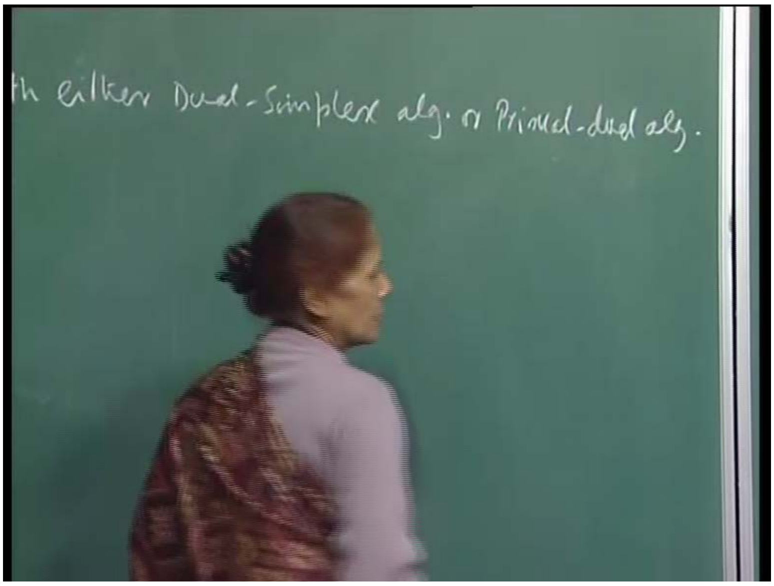

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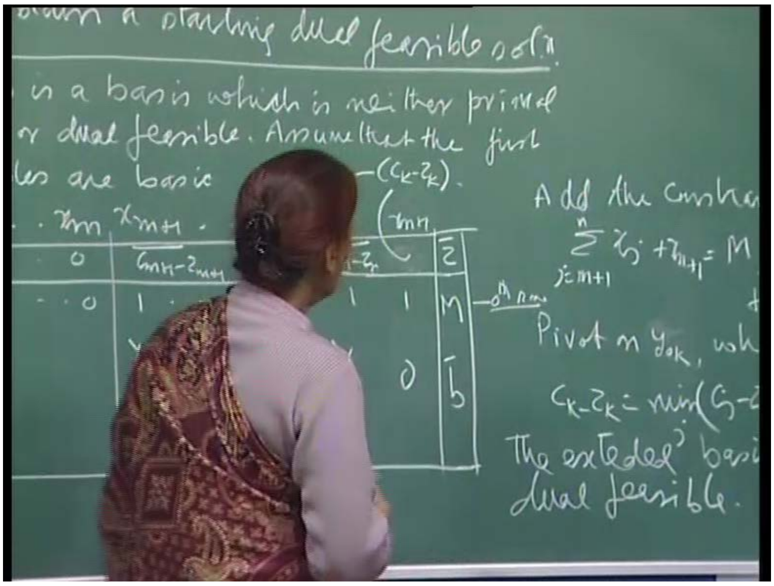

Once you have a starting dual feasible solution, you can begin either the primal dual or the dual simplex algorithm, but remember, you have a different problem, because you have added a constrained and so you have one more your basis size is extended.

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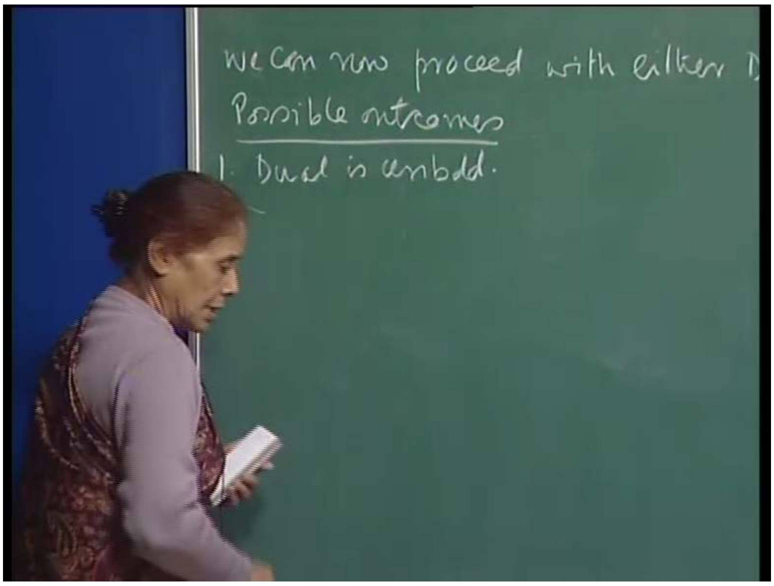

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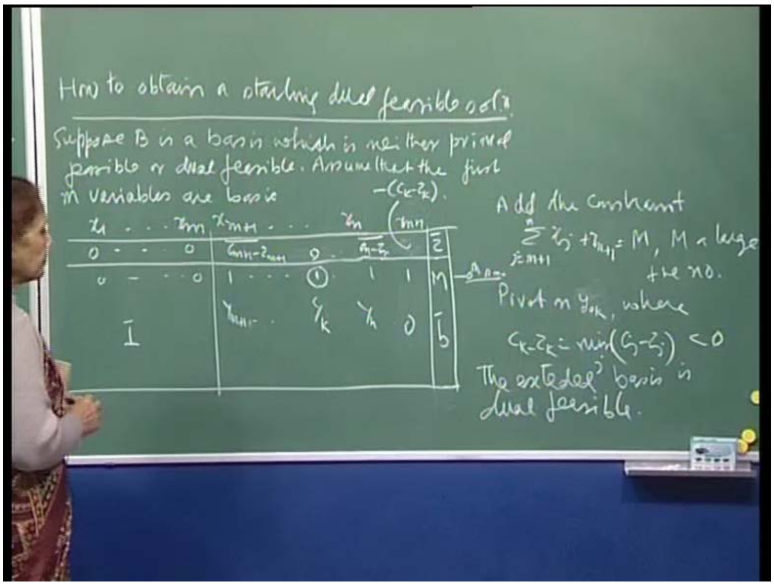

So, let us see, what are the possible outcomes, suppose, I start with dual simplex or primal dual whatever possible outcomes? First is, the dual is unbounded, by now if you have solved enough problems and you have understood the presentations here, you will understand that adding this constraint has nothing to do with dual unbounded condition, because the dual unboundedness will mean here, you have a negative entry and everything in this row is greater than or equal to 0. So, that you cannot proceed with the dual simplex algorithm for example, therefore, the dual problem can be unbounded, dual is unbounded which implies primal is infeasible.

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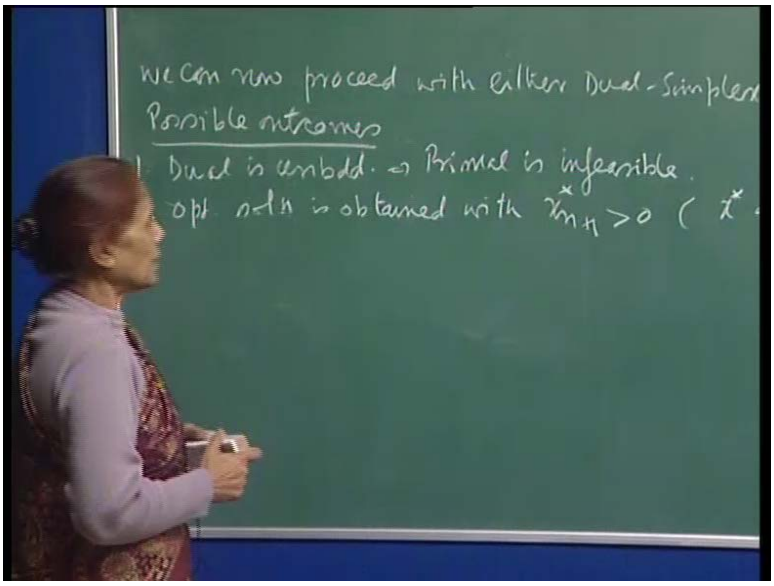

When you have obtained an optimal solution; that means, your algorithm has ended and what are the possibilities? Optimal solution is obtained with xn plus 1, and let me call it star greater than 0 where I am referring to this star as the optimal solution.

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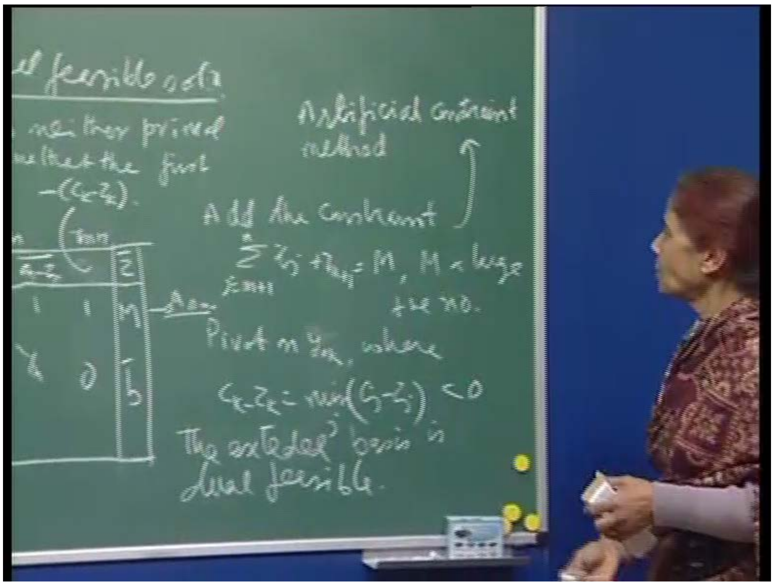

Here the stars, so suppose xn plus 1 star is greater than 0, what would that mean? That here, if this is positive, it is in the basis, so therefore, the constraint is not satisfied as equality - and so the new constraint is not binding - so it was just a device and by the way add the constraint when we say, this is also known as the artificial constraint method to obtain a starting feasible solution.

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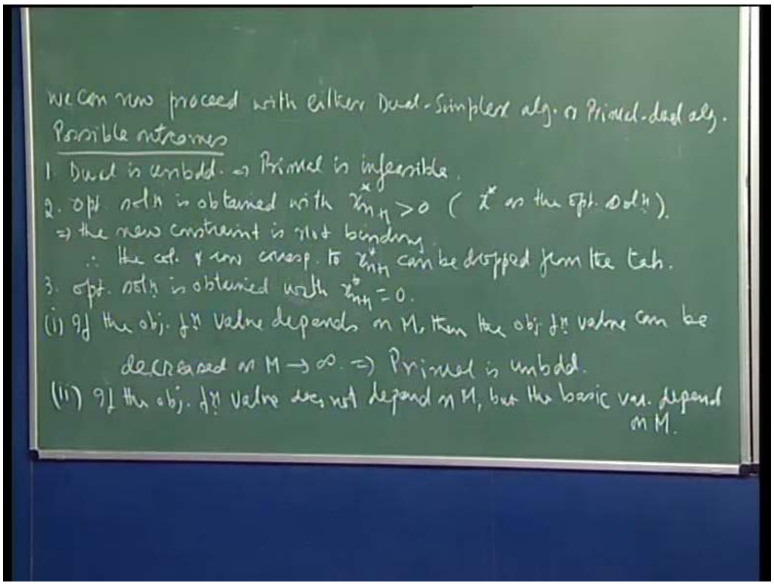

So, this constraint is not binding and therefore, I can drop, this implies that the new constraint is not binding. Therefore, the column and row corresponding to xn plus 1 star can be dropped from the tableau. And whatever you have is, the tableau corresponding to the original problem and so you have an optimal solution for the original problem. Now, what can be the third one; that optimal solution is obtained with xn plus 1 star equal to 0, so that means the status of xn plus 1 star is either basic or non basic but it at 0 level, and that is what we are concerned with, it is not important to know what the status of xn plus 1 star is, as long as it is 0, but the thing is that if it is 0, again 2 things are possible.

It is important that even this fashion you may not be able to do it right up to the end, because some loose and may be left out, that does not matter let see how long we you can follow, and I will just when as much times as I feel necessary and then you can always read up some more if all you can sit down and work out the detail. But anyway optimal solutions obtained with xn plus 1 star 0. So, this is case one, if the objective function value depends on M is a function of M. See, we do not know; that the final tableau this is equal to 0 and if the objective function value depends on n, then the objective function value can be increased. So, at optimality what is happening is that, the 2 objective function values are the same - the primal and the dual are the same.

So, you can, we will have to come back to it may be and the objective function value can be increased as M goes to infinity. So, I will tell you why did I argue that this should not be increased because you see, once we have obtain an optimal solution here, then my primal is feasible because xn plus 1 star is 0, so I will bet for decreased, as m goes to infinity, this will imply that primal is unbounded and so the second case would be that if the objective function value does not depend M.

But the thing is that since the constraint is binding, the objective function value does not depend on M, but the basic variables do, variables depend on M. And you might say that some, it is not very, very, clear, so we will come back to it sometime, but again you need a little more similarity with the simplex algorithm. In any case, see what is happening that, if the objective function value does not depend on M, but the basic variables may depend on M, because the constraint is binding. Therefore, what we can do is, we can reduce the value of M.

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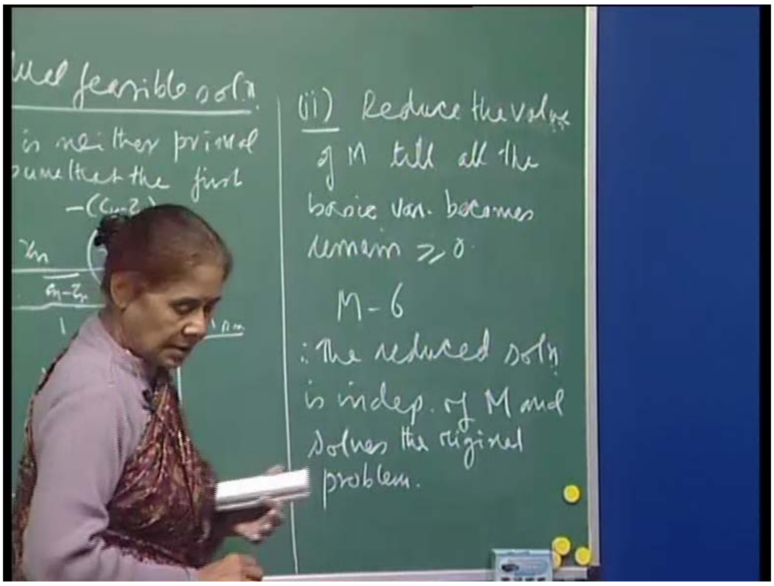

So, in this case, I am arguing about 2, so I am saying that reduces the value of M till one of the basic variables or may be more basic variable becomes 0, because the constraint is binding. So, till one of the basic variable becomes, you have to reduce the value of M till all the basic variables remain greater than 1 equal to 0, say for example, if you have the value M minus 6, then you can reduce M up to 6, because after that so obviously; that means, that one of the basic variables will become 0. So, this is smallest till you can go, and the new solutions you have and therefore the reduce solution.

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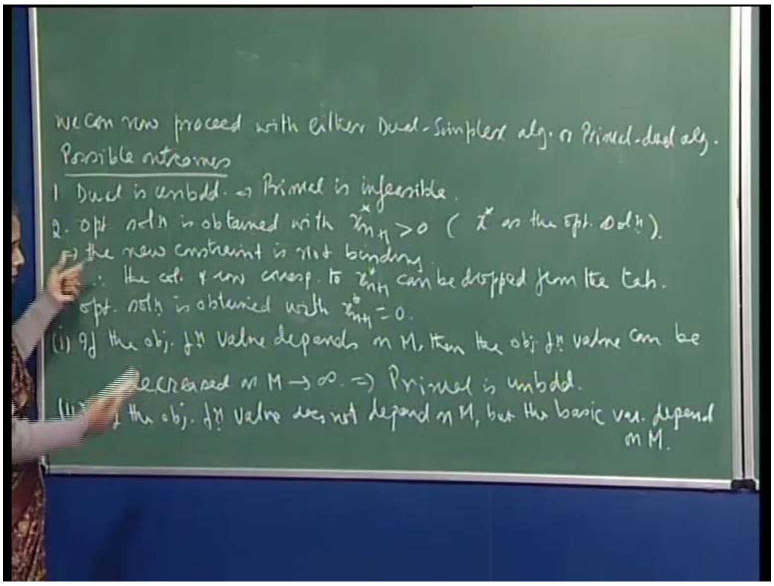

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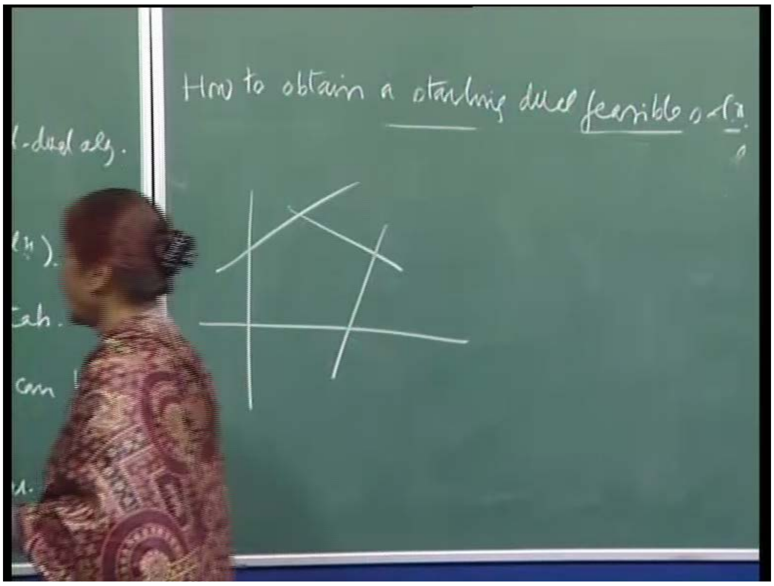

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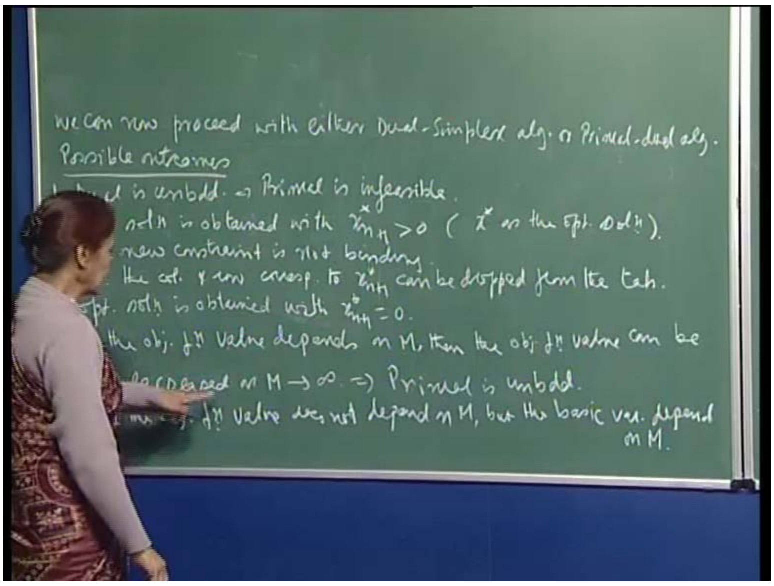

So, you will fix the value of M like that and then what you will have will be independent of M. So, the reduce solution is, independent of M and solves the original problem, so this is quite interesting and may be in the assignment sheet what I will do is, I will try to give you numerical problems which cover all these cases, I will be able to cover one right now, and then the other once I will give them as exercises, and hopefully and so by that time I will also have solve them and that we can check whether these outcomes become meaningful.

So, thirdly, let me just show you what we mean by this that the objective function value depends on M. And the objective function value can be decreased; the idea of here is, was a simple example, so here we are saying that the objective function value depends on M.

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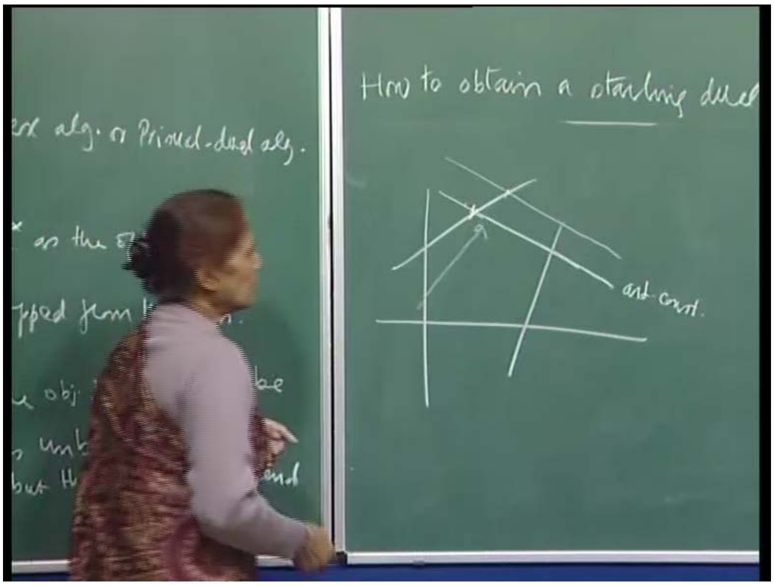

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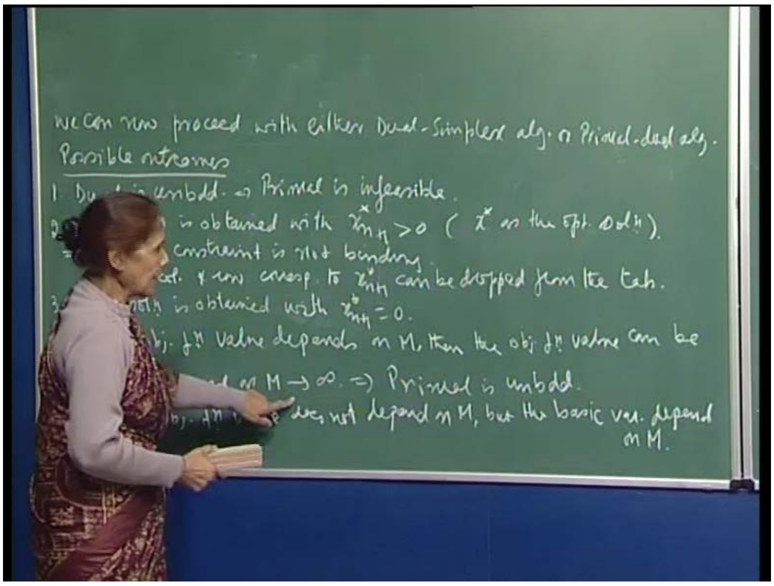

The objective function value can be decreased, the thing is that you may be at this point and this is your artificial constraint which corresponds to your M. So, this is your point, the thing is that since you are minimizing your this thing is decreasing in this direction. So, if I further increase the constraint that means, it will move parallel to itself, so then this will become the new optimal solution which will reduce the value of the basic function further and so on, so I can go on increasing by n and corresponding these objective function value will decrease, this is the idea and other things.

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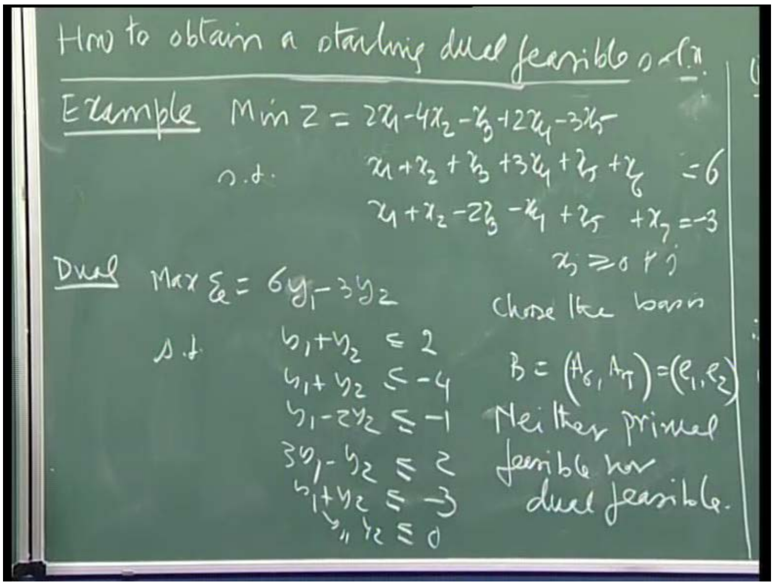

Hopefully, I should be able to give you some numerical examples on which you can work out. Here, I will just show you this one, so look at this example, the example is minimize Z equal to 2x1 minus 4x2 minus x3 plus 2x4 minus 3x5 subject to x1 plus x2 plus x3 plus 3 x4 plus x5 plus x6 is equal to 6. And the second constraint is x1 plus x2 minus 2x3 minus x4 plus x5 plus x7 is equal to minus 3, and xj greater than or equal to 0 for all j.

So, let us write the dual here, the dual would be maximize psi equal to 6y1 minus 3y2 subject to y1 plus y2 less than or equal to 2 y1 plus y2 less than or equal to minus 4, then y1 minus 2 y2 less than or equal to minus 13 y1 minus y2 less than or equal to 2 and y1 plus y2 less than or equal to minus 3. And finally, respect to this, you have y1, y2 less than or equal to 0, so in any case, if I choose the basis B as your A6 and A7 which is e1, e2, then that basis gives you the basic solution as x6 equal to 6, x7 equal to minus 3 and the you can see that from the top because the entries are not all non-negative, so it is also not dual feasible, so this basis is neither primal feasible nor dual feasible.

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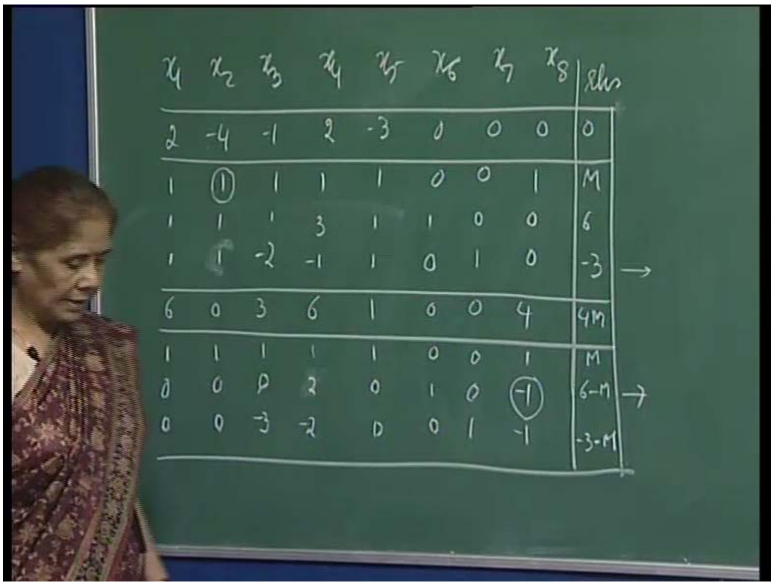

So, we need to have a starting dual feasible solution and we use the artificial constraint technique, I will have this as x1, x2, x3, x4, x5, x6, x7 and x8, so I am going to write that right away, and we choose x a 2 corresponding to the constraint, so here this is 2 minus 4, minus 1 2 minus 3, 0, 0 this is 0, so this is right now 0 and this is 1,1. So, because our basis corresponds to x6 and x7, so the artificial constraint that I will add will be 0 0 and 1 here. You are adding a slack variable x8 and then this is M very large number. Now, you write down the constraints which are 1, 1, 1, 1, 1, minus 2, 3, minus 1, 1, 1, 1, 0, 0, 1, 0, 0 and this is 6, and minus 3, so right now the basis is neither primal feasible nor dual feasible.

So, the first iteration here, we will look for the most negative because, I have been doing the dual simplex algorithm, you have see this basis shows you that it is neither old feasible nor primal feasible. So, you see that this is your outgoing variable and therefore, and then you choose, because this is the most negative have inserted this as the artificial constraint, so we will pivot on the most negative in this row, we will pivot on the element corresponding to the most negative cj minus zj, so that I first maintain dual feasibility, remember, the idea is to get the starting dual basic feasible solution. So, when I pivot on this; that means, I add 4 times this to the top row, and that will give you this will be 6, 0, 4 times this is 3, 4 times this is 6, minus 3 is 1 and then, this all 0’s this becomes 4, this was 0 here, and the objective function value is 4 n.

So, now, you have a basis which is consisting of x2 in the basis and then, this is, sorry, x6 is in the basis, I have started with x6 and x7 and that basis was neither primal feasible nor dual feasible. So, now, I added this artificial constraint, we remove x8, because x8 was the starting, so the initially your basis consisted of x8, x6 and x7. Now, I have replaced x8 by x2, so your current basis is consisting of the basic variables x2, x6 and x7 and you see that the basis; that means, the columns a2, a6 and a7, this basis is dual feasible but not primal feasible, because you have M is very large number.

So, all both these are negative and because we are using Bland's NT cycling rules, so let this variable be the first variable to leave the basis, and then you have to apply dual simplex, you look for a negative entry here, and this will be the pivoting element because this is the only negative entry, so you no need to take ratios and now when I pivot on this I have to make a 0 here, so 4 times we add this row to the top row and then, we will make 0’s here.

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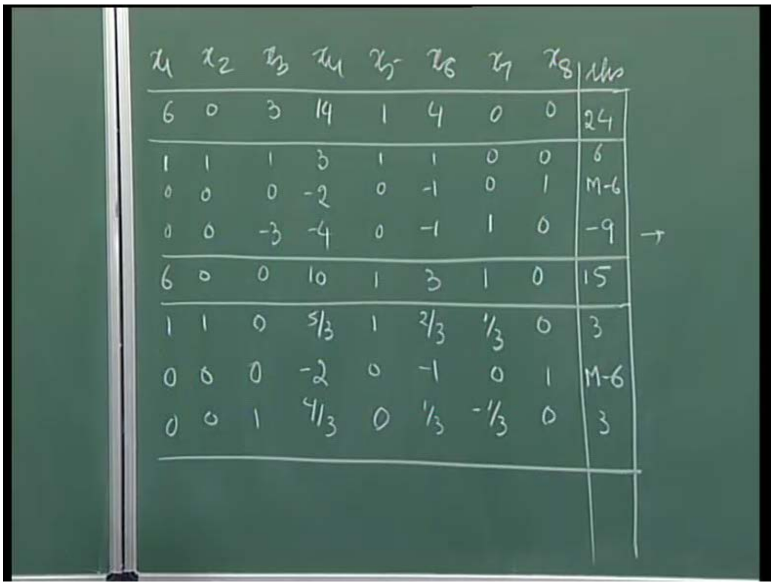

So, this table will show you the corresponding calculations, so here when you make 1 here, and then you are adding 4 times this to the top row, and so that becomes, you can just verify that these numbers do not change, but this is 4 times minus 8, so you are multiplying this by 4 and adding to the top row.

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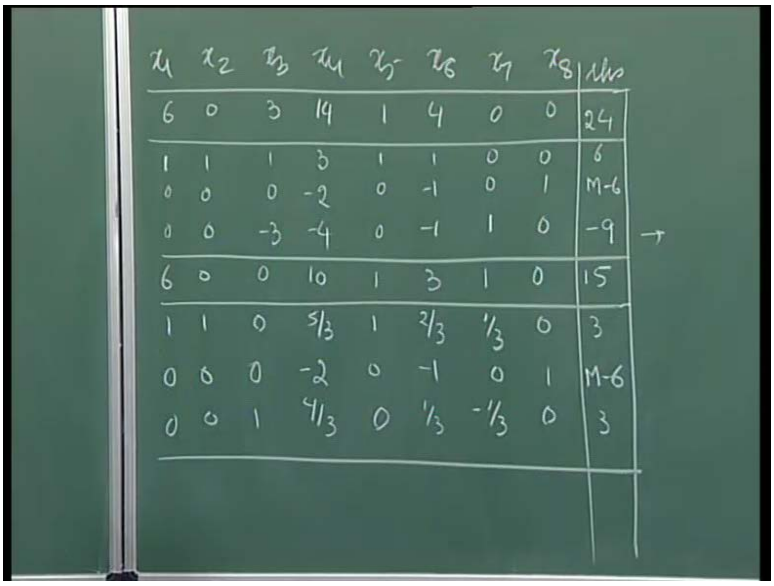

So, let us just make sure, yeah, this is a times plus 614, so the calculations are ok, and then I will multiply this row by minus 1, because it is a pivot entry, pivot element and then or you may be, you can first add this to this, and subtract this from here, and then we multiply the row by minus sign.

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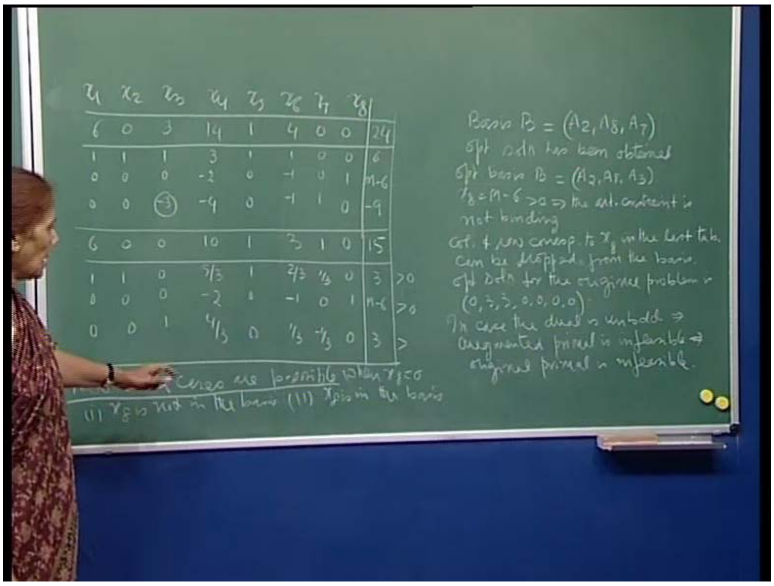

So, get the therefore, this shows you the next set of the next iteration tableau and you can see that here, now you have only one variable which is negative and the whole feasibility is maintain, because I am using dual simplex algorithm, so this is the outgoing variable and we look for negative entries here.

So, the ratio here is 1, here the ratio is 14 by minus 4 which is minus 7 by 2, and here it will be minus 4. So, we choose the maximum ratio and, therefore, this is your pivot element and you can see that in the next iteration, I am making this as 1, and all other entries 0 you will add this to the top row, so just to get the 0 here which will give you this again and you get the objective function value as 15 remember.

So, you just add this to the top row and then make a 0 or for then divide this by minus 3 to get a 1, and then, you subtract this row from here, to get the corresponding entries, and now, you see that at this point, so at this point, this is positive because M is a large number, this is positive and this is positive. So, we have obtained an optimal solution and the optimal basis is a2, and then a8 and a3, a2, a8 and a3. So, this is the optimal basis and x8 is equal to the value of x8 is M minus 6 which means it is positive, because M is being taken as a very large number.

So, that means when x8 is positive; that means, the artificial constraint is not binding, remember, we had added an artificial constraint to obtain a starting dual feasible solution, but now we realize that since in the optimal solution x8 is at positive level, the artificial constraint is not binding, it will satisfy the discrete inequality, if you remove x8. So, that means, therefore, we can drop the column and the row corresponding to x8 in the last tableau can be dropped from the basis, because we do not need it anymore, and so the solution to the original problem will therefore be with this a7 dimensional vector and this is your, this is obviously, you have.

Now, because you have two constraints in the original problem, therefore, this is a basic feasible solution and it is non-degenerative. Now, of course, see here, while because we were applying the dual simplex algorithm, it is possible that you may encounter at some intermediate situation that the dual is unbounded. And remember, the dual unbounded means, that the augmented problem, because you added an artificial constraint, the augmented primal is infeasible.

And now, I will like you to think about it, why? I can conclude therefore, that the original primal is also infeasible. And therefore, we will not proceed any further, because the movement while doing this you encounter that the dual problem is unbounded, you can then conclude that the original problem is infeasible. And therefore, stop there now there are two cases; so I have discussed the case when x8 is positive.

Now, suppose in the optimal solution 2 other cases are possible - I mean a sub case - which is when x8 is 0, so 1 possibility is that x8 is not in the basis - I mean - these are pathological cases, but one should know because once you have added a constraint you should know how to treat x8 in all possible situations, so either x8 is not in the basis because is that 0 level or x8 is in the basis.

So, now what I will do is, through the assignment sheet, I will try to set problems in the assignment, in which either this or this happens and then hopefully give you a hint, how to handle this, because we want to show that we know when either this case happens or this case happens, we know how to treat it - I mean - when x8 is not in the basis, and when x8 in the basis, so not difficult situations, but this now gives you one method to obtain a starting dual feasible solution, and then you can begin with the dual simplex algorithm and this will happen, because you need a dual starting dual feasible solution for the dual simplex algorithm and for the primal dual.

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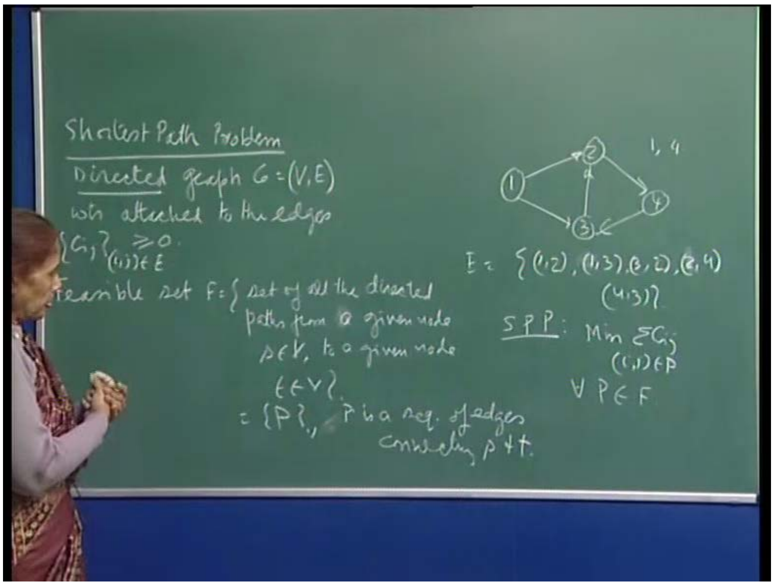

So, in either case, you now have a method for obtaining a starting dual feasible solution. Let me show you now interesting application of the primal dual algorithm or the primal dual formulation. See, this the shortest path problem, the idea here is, that you have given a directed graph and I will explain directed graph G V and E.

So, the usual notation is that you have a set of vertices or nodes and you have edges connecting them and when we say, that it is directed say for example, you take so they will be an arrow 1 to 2 you cannot go from 2 to 1 to this edge or the arc, we call it the arc. Similarly, you have like this then you may have this and you may have 4 and for example this.

So, this is a directed graph where the nodes are 1, 2, 3, 4 and the edges you will write down there is 1, 2, they can be many ways of writing down the edges, I do it through to the starting node and the ending node. So, this is for example this edge I will write as 1, 2 and similarly, you have the edges 1, 3, so your E consists of 1, 2, 1, 3 then you have 3, 2, and then you have 2, 4 you have 4, 3.

So, this is a set E, therefore, graph would be denoted as given set of nodes and given set of edges or arcs connection the nodes, so this is a directed graph; that means, that the direction from in which you can go from 1, 2 and so on. And then, you have a weights attached to the edge; that means you have weight cij - ij belonging to E.

So, for every edge, you have a weight and we are considering the case when the weights are non-negative, because the formulation and the application of this primal dual will be straight forward of course, you can handle other situations also but so weights attach to the edges are this. Now, your feasible set F is set of all the directed paths from a given nodes belonging to V, and to a given node E also belonging to V.

So, this is the feasible set all directed path for example here, a directed path can be 1, 2, 4 then it can be 1, 3, 2, 4, if my distinguish nodes are 1 and 4. Then all possible directed paths would be, so you have to specify s and t right in the beginning, to what is the pair of nodes to which you want a directed path. Once, you specify s and t then the directed paths 1 to 4, for example 1, 3, 2, 4 then that is it.

So, in this case, you have only 2 parts, but if you choose the edges as 1 and 3 then, you will have 1 to 3, you may have 1, 2, 4, 3 and so on, this is sub small graph. So, this is a feasible set, now then, so I can also say that this is all P, all paths. Let me denote this then, what is your problem? The problem is, so the shortest path problem is minimize summation cij; where ij belongs to P and for all P belonging to F this would be a concise formulation of a problem, so that means, we are adding up for directed path add up the weights of the edges which wake up the path ij is at P, then if you want to be more rigorously at I should have defined, say for example, I said that this is a path P which can this is actually would be so here I can say that, P is a sequence of edges connecting s and t nodes which (( )).

So, there will be a sequence of edges, for example here, if 1 to 4 then 1, 2, 2, 4, this is a path and this edges that make up the path are 1, 2 and 2, 4, so for the weight of this path it will be c1, 2 plus c 2, 4.

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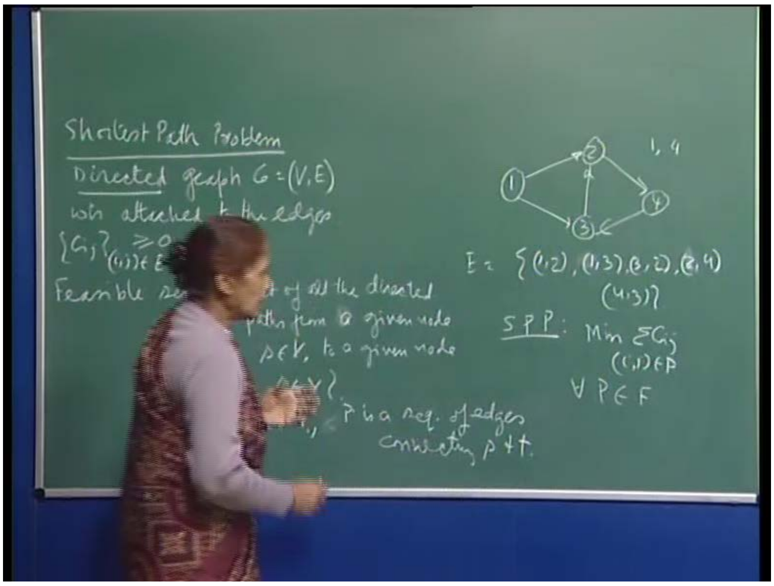

And so you would have all possible paths here, and then you want to choose the one which has minimum way, so this is your shortest path problem. And of course, this is one of them which have been solved quite efficiently, but the idea behind almost all algorithms for cij non-negative is some modification of the primal dual algorithm, I will show a straight forward application of the primal dual here.

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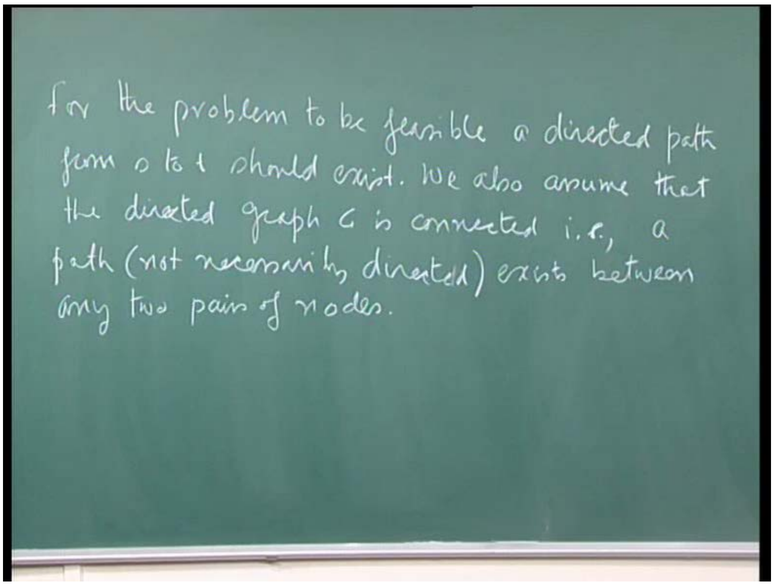

So, it is important while defining the shortest path problem that we should say that the problem is feasible and that means, that there should be a directed path from s to t, it should exists. Also, when I am talking of the rank of the node arc incidence matrix and so on, then the underlying assumption is that the graph is connected which means that, there is a path and not necessarily a directed path which exists between any 2 pairs of nodes. So, this underlying assumption then makes the other results valid, and so I should have pointed that out.

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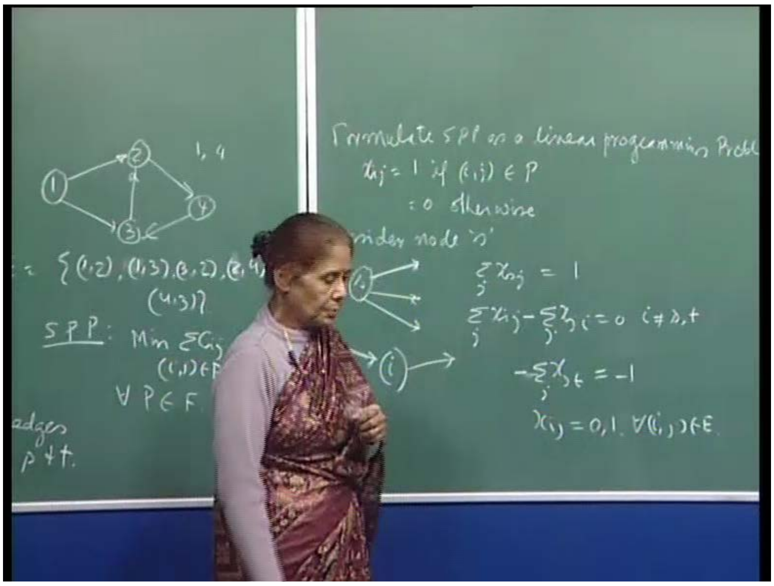

So, let us formulate this as a linear programming formula, formulate SPP as linear programming problem, so what we will define here, is as say for example, now so that means, you have to understand what a path would mean or how would you restrict your choice of the variables and such a way that you have path. So, for example, I will first define my variable xij to be 1, if ij belongs to P and 0 otherwise, if I do this, then for example, consider node s - the starting node - then from node s any path would consist of one of the edges which has leaving s.

So, therefore, the constraint for node s would be summation xsj summation over j, so I do not know whatever the arcs which are connecting it should be equal to 1. It is only one arc must leave node, the node s then for any intermediate node on the path, an edge will enter the node and an edge will leave the node, because if I reach up to, my objective is to go from s to t, so any intermediate node, if I reach an intermediate node I must leave it, and therefore, we can say that summation xij this minus summation xji should be 0. As long as i is not equal to s and t, for all other nodes I must have xij; that means, this is the node which are leaving node i this and then, so if in case this is 1, then this must be also 1, if this is 0 then this is 0, or either way if this is 1 then this must be 1; that means, if 1 arc has entered i then one arc must also leave I – that is clear.

And finally, summation xjt summation over j must be equal to minus 1; that means the last node t only path must enter the node t, there is no question of leaving the node t, because once you have reach node t, you are done, you have a path connecting s and t and then, you have your xij equal to 0, 1. So, this is not a linear programming formulation because, for all ij belonging to E, because I am forcing the variables to take the value 0,1.

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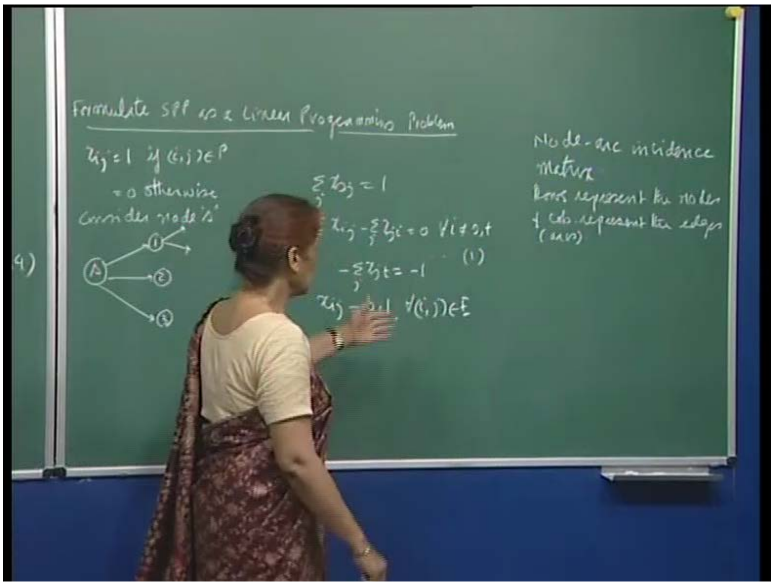

So, the constraints had been written out here, when I am trying to formulate this as a linear programming problem - the shortest path problem, these are the constraints, but you see that here. We require the variables to be integer value; so therefore, this is not in the strict sense linear programming formulation, so I will later on show you that we can do away with these constraints. Now, the objective function also has to be written down; that means, we will be adding up the weights of the path.

So, but right now, these are your flow conservation or you can say that these are the constraints with ensure that, you have a path as a feasible solution, because this says that at least one arc will leave node s and then, intermediate nodes i 0 equal to s, t if an arc enters the node, it must also leave it, and finally only one arc should enter node t, so these are the constraints. And to write the linear programming formulation let me now write down the node arc incidence matrix, so this is where the rows represent, the nodes and columns represent the edges or the arcs - I mean - see mostly people in the directed scenario, people prefer to use arcs instead of edges, but I have been using edge - the word edge does not matter.

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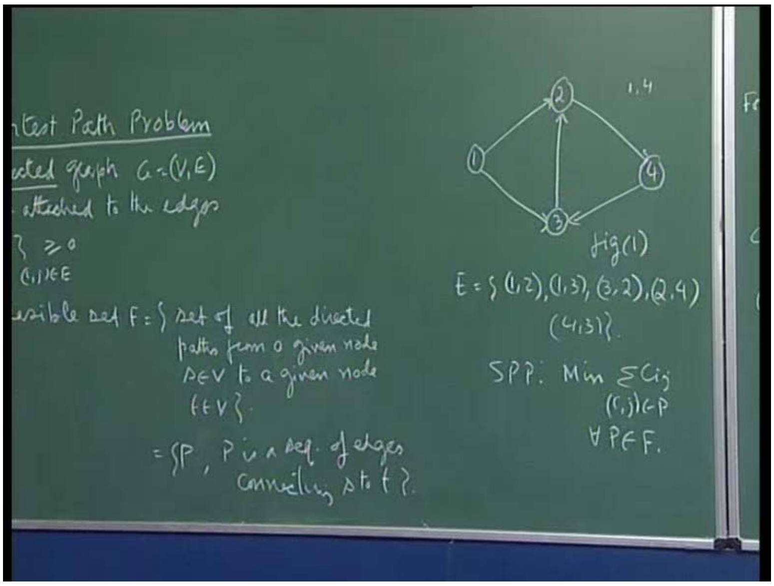

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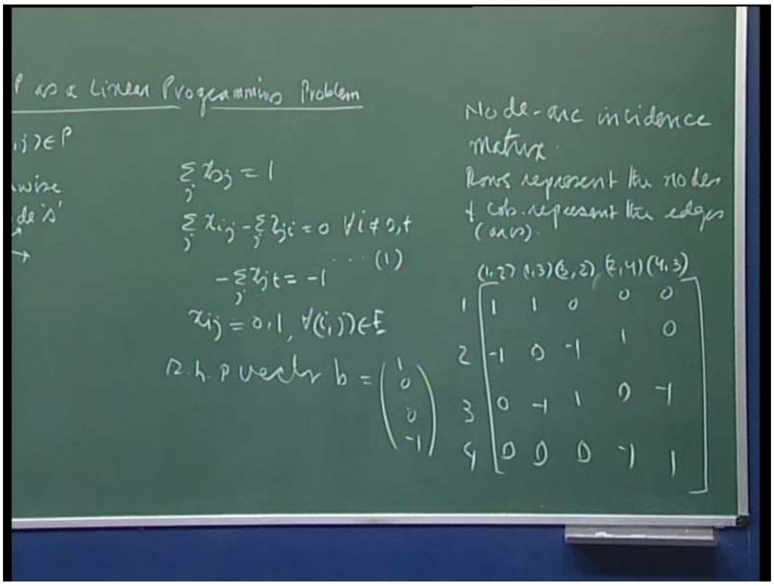

So, here, for example for this graph, when you write down the node arc incidence matrix, it will look like this. So, here, you had node 1, 2, 3 and 4 and the edges that you have are, 1 2, 1 3, 3 2 and 2 4 and 4 3. So, for 1 2, for example, this is leaving node 1 and entering node 2, so there will be a minus 1 0 0. Similarly, for this arc 1 3, it will be plus 1 here 0 minus 1 0, then 3 2 its leaving node 3 and entering node 2, so this will be 0 minus 110 and then 2 4 its going to be 0 1 0 minus 1 and 4 3 will be 0 0 minus 11.

So, this is your node arc incidence matrix, so if you write it, you take this as the matrix A the constraints here can be written as and your right hand side vector b will be as 1 0 0 and then minus 1, so this is for node 1, and this is for node 4 and then for 2 and 3 it is 0 the right hand side vector.

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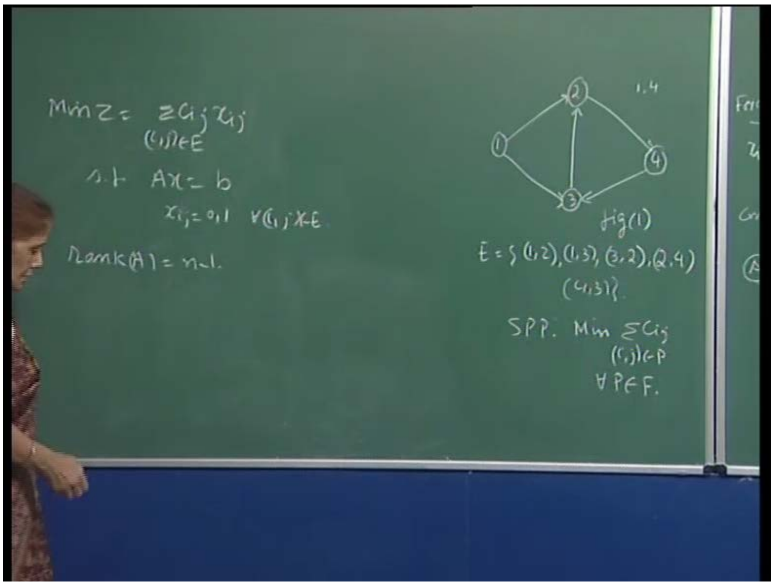

So, I can now write the linear programming formulation for the SPP with the objective function and everything, because the weights are so far this particular problem, the formulation will be now, you can say that minimize Z equal to summation cij xij ij belonging to your edge set E, subject to Ax equal to b, and right now, I am still saying that it is your this thing is, ij is 0 or 1, for all ij belonging to E, this is my formulation and now what I want to show is, that first of all of course, let us look at the matrix A, so what we are saying is that rank A is n minus 1.

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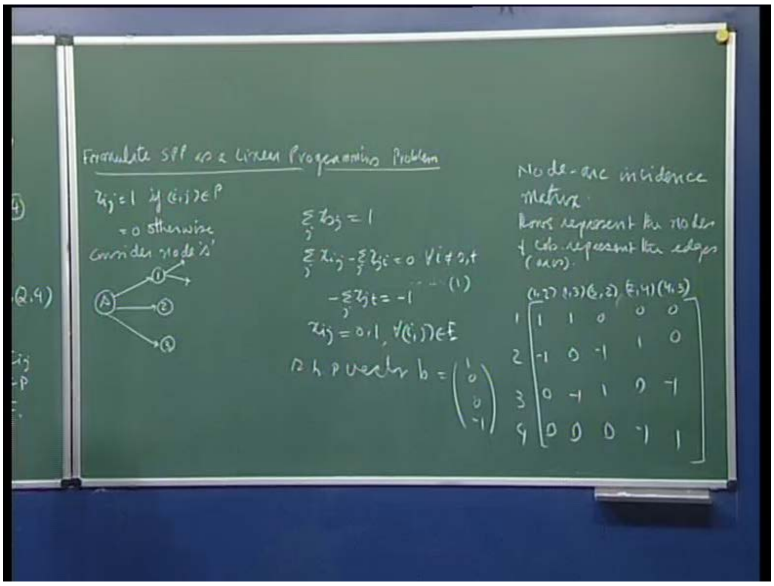

So, let me begin with all the properties of A, and then we will come back and show you that I can remove these integer constraints. So, the idea here is, you see, if you add up all this constraints, the right hand side will become 0 both side, because 1 minus 11 plus minus 1 is 0. And here, you see, if you have an sj here, xsj then, the for the node j xsj will appear with the minus sign.

So, this will cancel out, and so in pairs these variables will cancel out and so you will have 0 0 on either side, so when you add up all the constraints I have written this as 1, so add up all the constraints in 1 to obtain the 0 row. So, you can look at the matrix here, just at c, because every column has 2 non-zero entries and 1 is plus 1, the other one is minus 1. So, if you add up all the rows here, we add up to the 0 0 row, and so you know that the rank the matrix cannot be n, it has to be less than or equal to n minus 1. And then to actually show that it is n minus 1, we will require the little extra work, so you have to take my word for that the rank.

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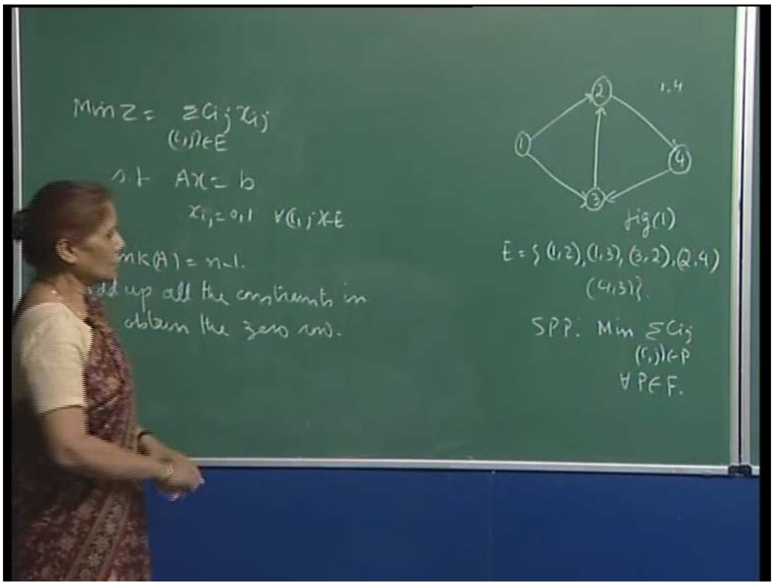

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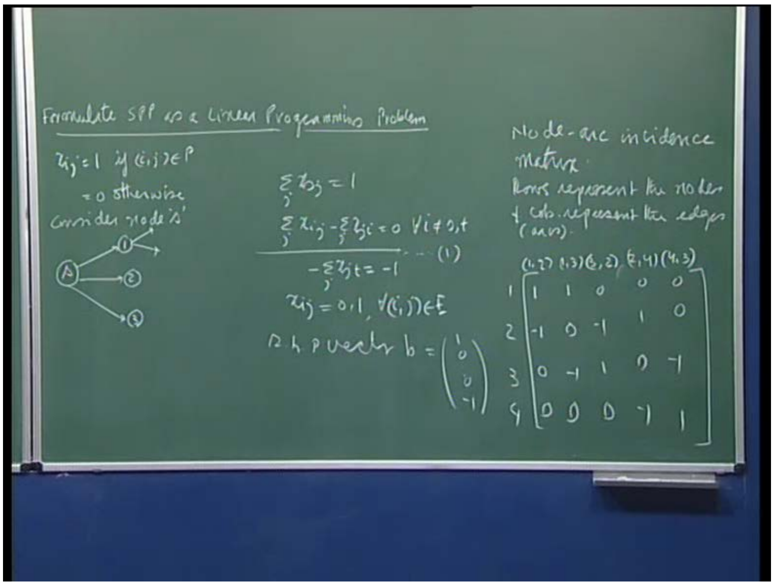

So, therefore, what I have shown you is, that the rank cannot be n, so but the rank is n minus 1 and just what we do, so therefore, that means, one constraint can always be expressed as a linear combination of the remaining constraints, or that means, I can decide that the last row can always be written as a linear combination of the first 3 rows here, so therefore, and that you can see, why because if you add up just these constraints; that means, add up the constraints.

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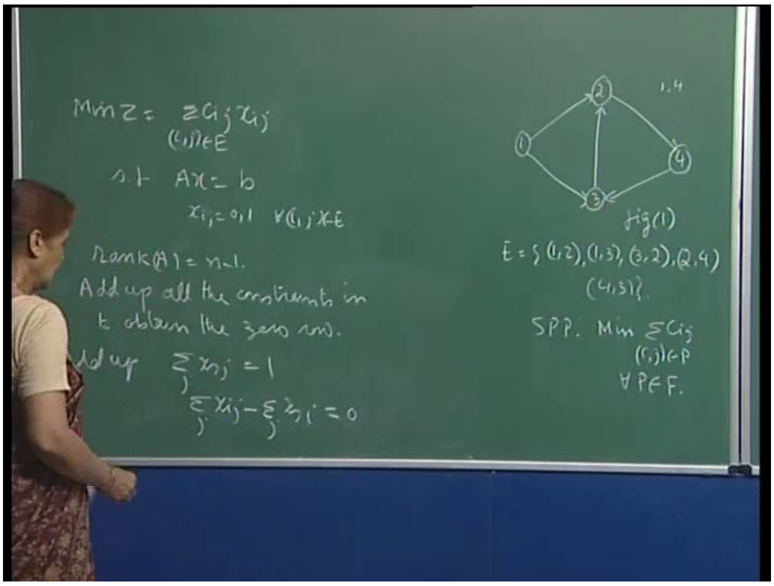

So, add up, which is again a repetition because if all adding up all the constraints gives you the 0 row and that means, one constraint can be written as a linear combination of the others that you see, add up the constraint summation xsj summation over j equal to 1, and summation xij - summation j over j minus xji summation over j is equal to 0, just add up these, leave out the last one.

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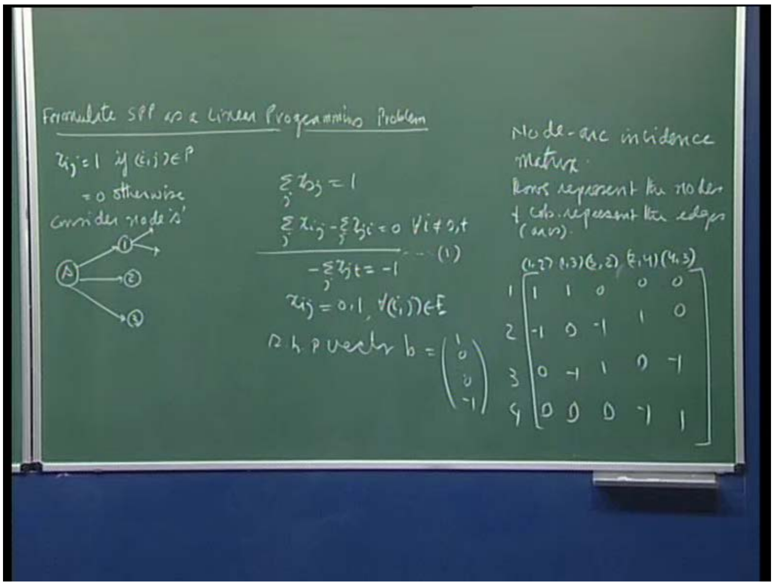

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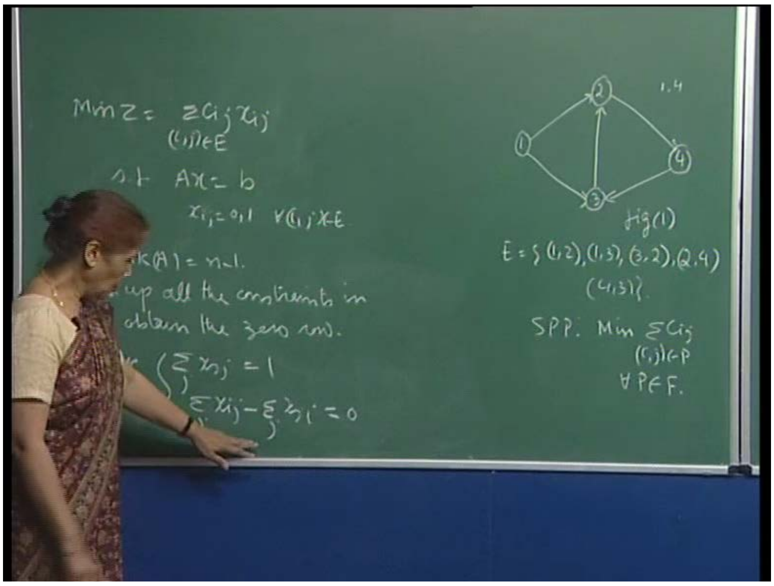

So, then again from here, you see that, all these will cancel out because, you do not have the constraint for the t th node, so you will be left out with xjt, because the variable for the node t, the arc which is coming in to t it is leaving the node j, so therefore, it will have a plus sign, so when you add up these constraints, you will be left out with xjt and the right hand side is 1.

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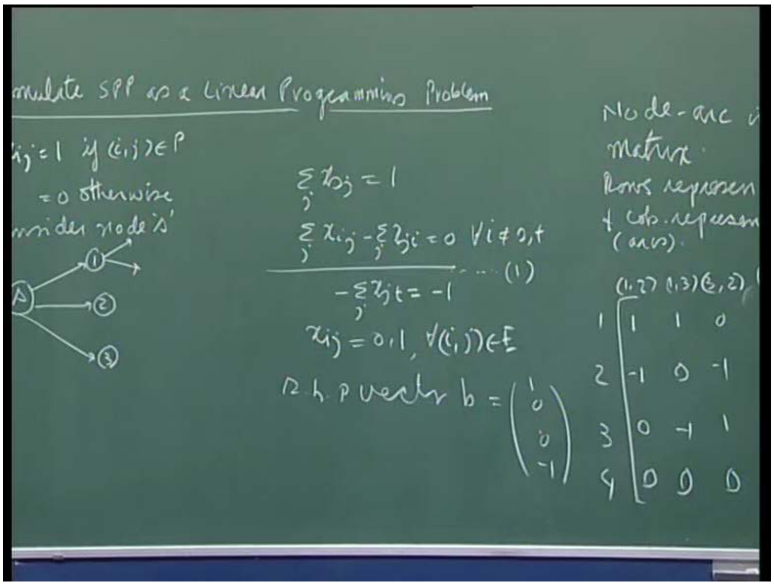

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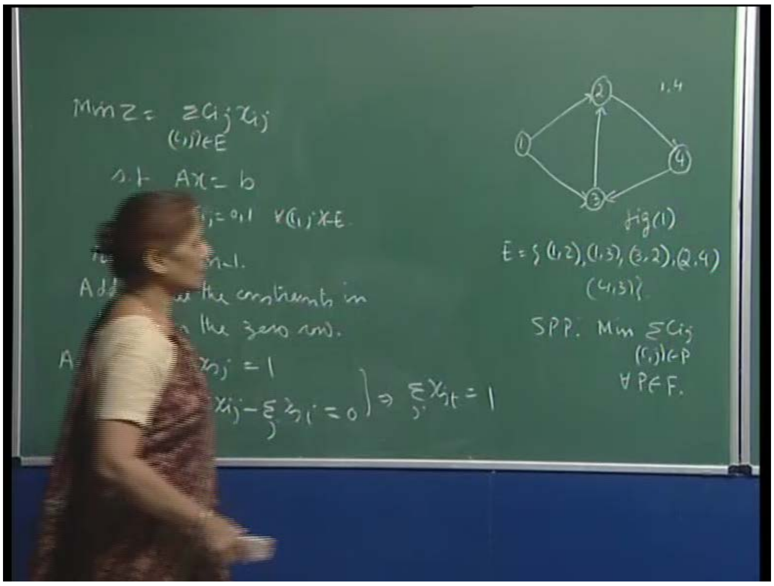

So, adding up this implies that summation xjt is equal to 1, summation over j, so we just multiply the minus sign because you want to keep this entire anyway.

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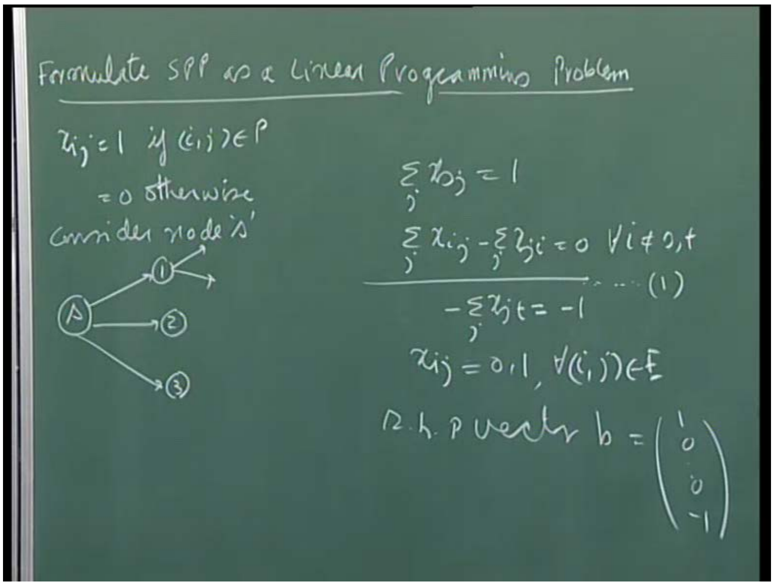

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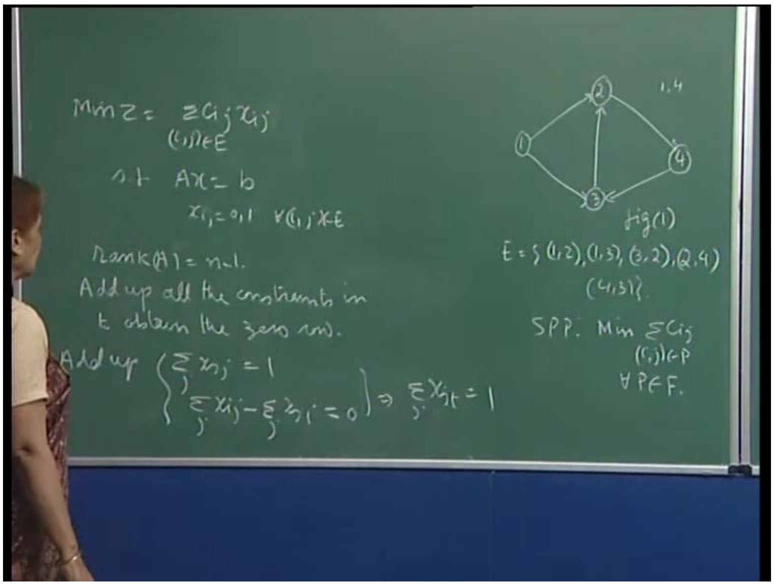

So, therefore, what we will decide is that, we can always, we can forget about the last constraint, because it is implied by the remaining ones and also, what we want to say is that here another property of matrix A is that the basic feasible solutions.

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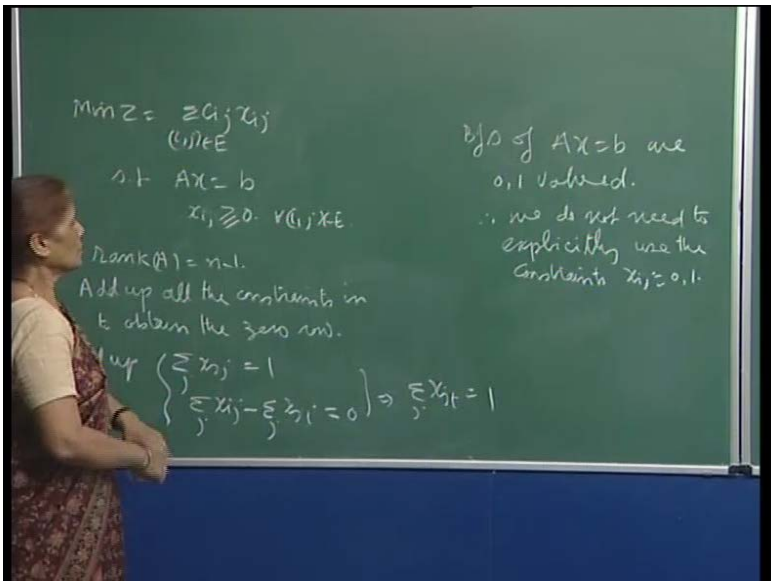

So, basic feasible solutions of Ax equal to b are 0, 1 valued, this is a property which is also known as uni-model active property again in this course we will not be able to actually show you that why this holds.

So, since this property is already there and linear programming formulation only works with basic feasible solutions, therefore, we do not need to explicitly use the constraints xij equal to 0, 1. So, therefore, I can remove these constraints and I can say that this is just greater than or equal to 0. So, this would be a linear programming formulation for the SPP, and you can also have a nice interpretation of the complementarily slackness condition. So, we are of course, you will have you write, so if you look at the dual, nice structure, so I would just write the dual here.

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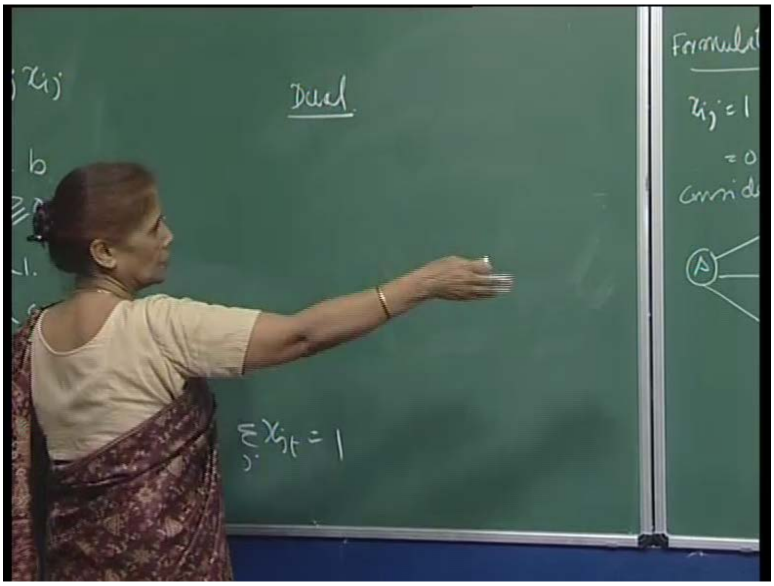

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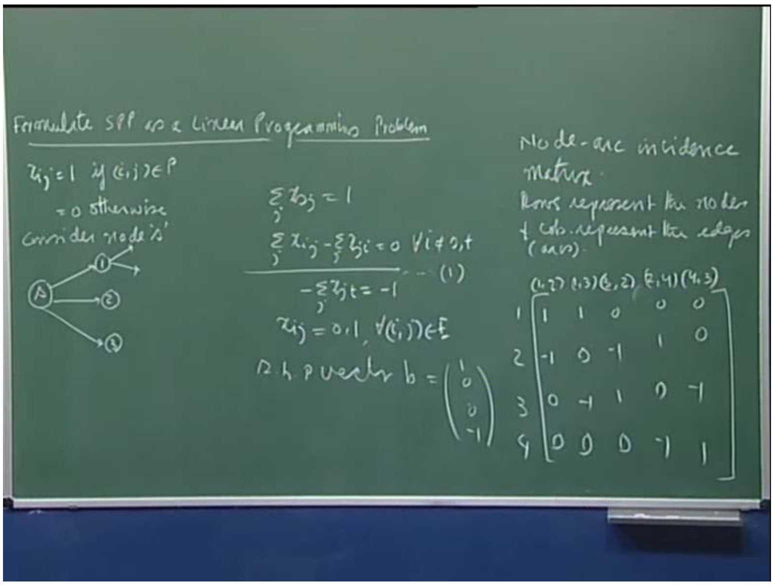

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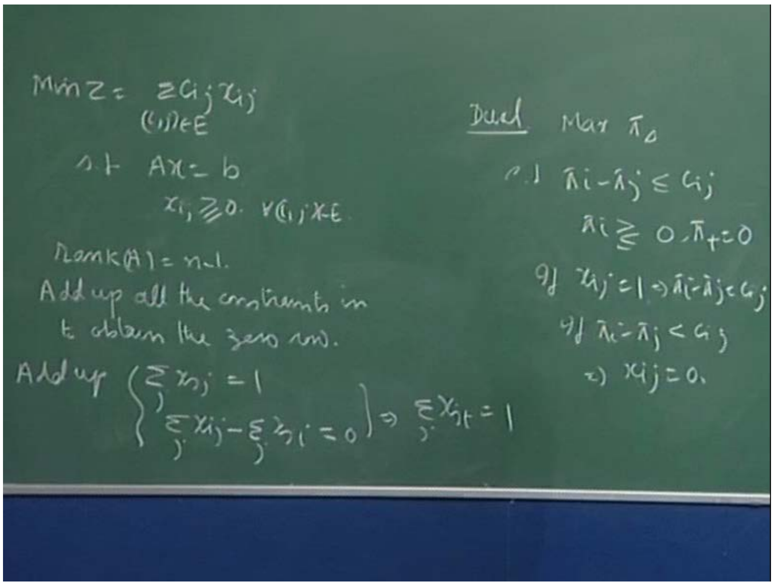

Because, you see that now the columns I have a very simple structure, so here the dual would be if I associate the variables pi i. So, this would be maximize pi s subject to pi i minus pi j and this is a maximization problem here, you have quality constraints and this is non- negativity.

So, this is less than or equal to cij and your pi i are unrestricted, and I will also use the convention of keeping pi t equal to 0, because we are going to not explicitly mention the last constraint. So, this is your dual problem and you see it has a nice structure, and later on we will be able to make use of this as special structure here. And then, the quick interpretation of the complementarily slackness conditions what it says is, that if xij is 1; that means, if it figures in the path, this implies that your pi i minus pi j is equal to cij, that the dual constraint will be satisfied as equality, and if pi i minus pi j is strictly less than cij this implies that xij has to be 0; that means, the corresponding arc will not figure in the shortest path. So, later on we will show you many more interesting interpretations and how the algorithms are developed using these formulations.