Linear Programming and its Extensions

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Lecture No. # 11

Assignment 2 Progress of Simplex Algorithm on a Polytope Bounded Variables LPP

Let me begin today's lecture by discussing the assignment number 2. If problems are all given here, I will go through them one by one, and give you some hints and ideas, so that you can work them out yourselves.

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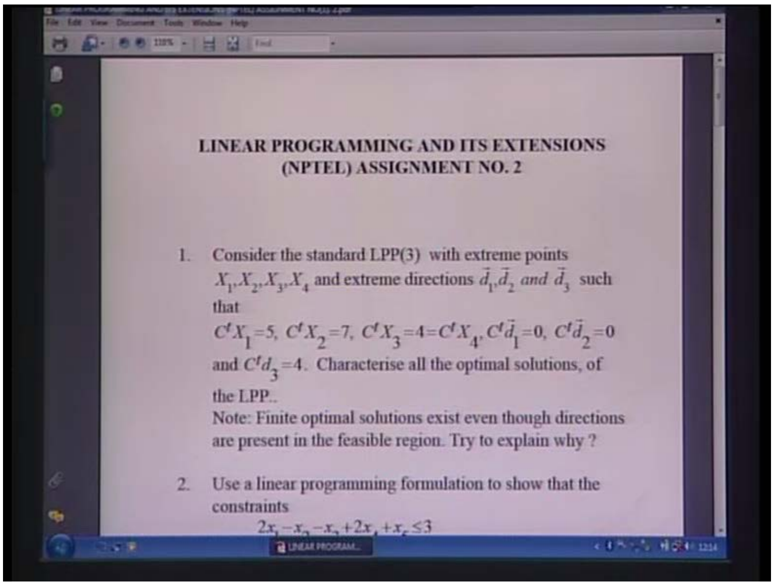

So, the question one you see, I have the standard LPP, remember I refer to the standard LPP which is minimization of C transpose X, subject to a X equal to b X non-negative. So, for a standard LPP, you are given the four extreme points, X 1, X 2, X 3 and X 4, and the directions, there are three directions in the feasible set, d 1, d 2 and d 3, and I have given you the values of the objective function at the four extreme points.

So the first one point was C transpose X 1 is 5 (Refer Time: 01:00), C transpose X 2 is 7, C transpose X 3 and C transpose X 4, both have the value 4. The directions of along d 1, there is no increase in the value of the objective function C transpose d 1 is 0. Similarly, C transpose d 2 is also 0, but C transpose d 3 is 4 characterize all the optimal solutions of the LPP.

So, I want you to, this information is enough for you to find out which is optimal solution for your given LPP. Remember the linear programming problem is a minimization problem; so, C transpose d 3 equal to 4 does not bother us, because the value is positive here and it will increase in this direction; if I move in this direction, the value will increase with a multiple of 4, and my objective function is a minimization function.

So I choose this example particularly, to show you that, even though your feasible region is not bounded, because it has directions present in it; it does not disturb the minimum value of the objective function, that we are considering here, because the direction that is present in the feasible region, the value of the objective function is increasing along the direction.

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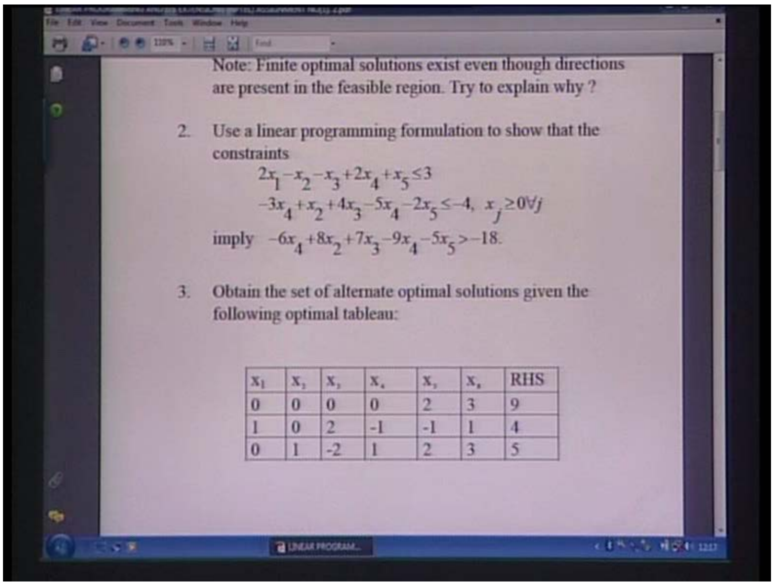

So, therefore, now you can tell me what the minimum solution would be; that is go on to the next one - number 2. Number 2, I am saying use a linear programming formulation to show that the constraints, I have given you two constrains, where the all non-negativity of constraints on the variables to imply that the third constraint will be satisfied.

So, I would like you to think here, what do I really mean. See, if I want the third constraint to be satisfied, that means, it should be greater than minus 18. What would you say the would be the value of the objective function, if you treat the left hand side, that is, minus 6x 4 plus 8x 2 plus 7x 3 minus 9x 4 minus 5x 5, if you treat this as the objective function and if you want that value must be greater than minus 18, you can now guess what should be the minimum, maximum, you have to guess. So, can formulate this problem, that means, you are trying to find out, if this constraint that the last one is implied by the first two, which then you can formulate this as a linear programming problem and then you can tell me that whatever your formulations.

So, whatever the answer of that was linear programming problem is final optimal solution value. Then, how will you, after having that optimal value, how would you answer this question? So, I want you to think about it. Now, problem 3, I have given you the optimal tableau for some linear programming problem; this is the right hand side and you can see that the basic variables are x 1 and x 2. So, for them, the top row C j minus Z j's are all 0, but you also have C j minus Z j 0 for x 3 and x 4; therefore, I am asking you to compute alternate optimal solutions, because x 3, for example, it can enter the basis and the value of the objective function will not change.

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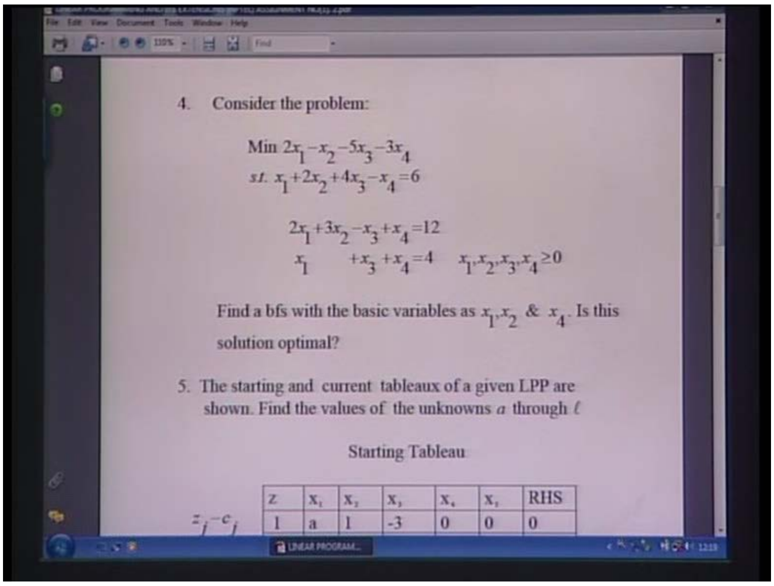

So, if by bringing an x 3, you have another basic feasible solution, that will also turn out to the an optimal solution; this is the idea behind, yes. Now, in the problem 4, I have stated a linear programming problem, and I am asking and find a basic feasible solution with the basic variables x 1, x 2 and x 3. So, basically I want you to be familiar with the pivoting rules, and then workout the, that you want to bring x 1; and of course, in this order, x 1 will be the first basic variable, x 2 will be the second basic variable and x 4 will be the third basic variable. And once you do the pivoting's, then at the end of this, when you get the table, you can check whether the optimality criteria is satisfied or not. If it is, then it is an optimal solution; otherwise, you will say that, no; this solution is not optimal; so, workout the fourth one.

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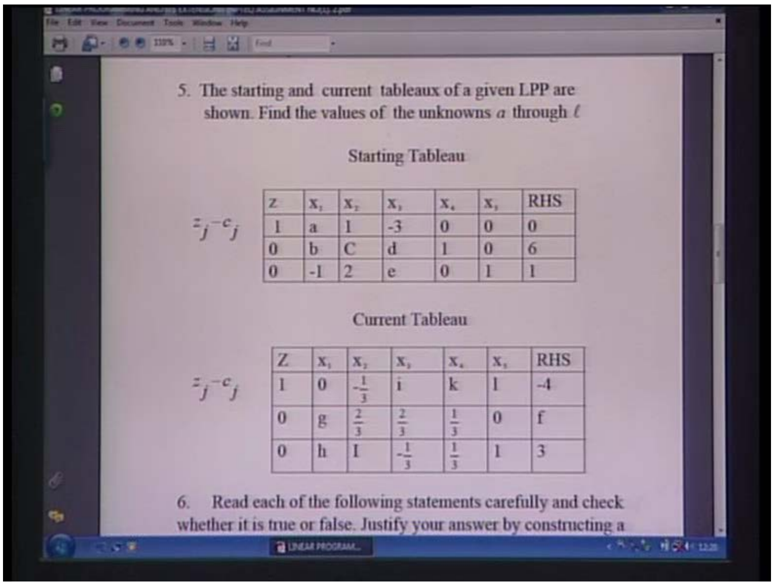

Now, fifth one, yes, let us just concentrates on the fifth one. So, the fifth one I have given you the starting tableau and I have given you the current tableau. And in this case, the problem I took from the book, that writes the top row as Z j minus C j; we have been using C j minus Z j. So, you just have to change the criteria; that means, when I say that for a minimization problem, the optimality criteria requires that, all C j minus Z j should be non-negative; in this case, it will be that Z j minus C j's are less than or equal to 0.

So, you can change the criteria, but the top row gives you the Z j minus C j here and you have all the information. Now, for example, I will just give so they are lot of unknowns here and I want you to obtain these; so, the 2 tables give you enough information, say for example, in this case, we have starting tableau X 4 and X 5 are the basic variables.

So, in the current tableau, under the columns a 4 and a 5, you will have your basis inverse, right, and remember your k and l would be minus C B B inverse. So, write that down on 1 side in the paper or a sheet of paper, then you can compute, say for example, here I am giving you that the value of that here the C j, I have not given you C 1, C 1 is an unknown.

Now, small a but in the well it is actually minus C a, because this is Z j minus C j; so, starting thing your C B's are all 0s; therefore, the top row actually is minus C j minus a is your C 1 (Refer Time: 07:00). Now, in the current tableau, this is 0; so, you know this C B B inverse. If you know the C B B inverse, then you can compute, you can find out what a would be. So, just remember what the formulae I have given you and what should be the current values starting from this tableau.

You can compute all the unknowns that I have asked you to compute; for example, let me see, if we can I check some more I can quickly give you, say for example, under column 2 you have a C. So, this is your a 2, and in the current tableau, what you have is b inverse a 2. So, since you have b inverse already with you, the b inverse is given here, 1 by 3 0 1 by 3 1; so, this times when you multiplied by C 2, you should get 2 by 3 and 1.

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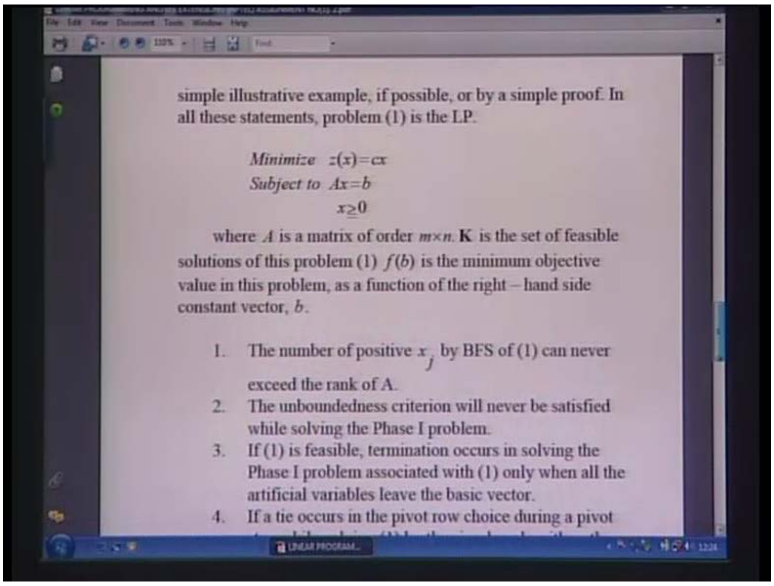

simple illustrative example, if possible, or by a simple proof. In all these statements, problem (1) is the LP.

$$\text{Minimize } z(x)=cx$$
$$\text{Subject to } Ax=b$$
$$x\geq 0$$

where A is a matrix of order $m\times n$. **K** is the set of feasible solutions of this problem (1) $f(b)$ is the minimum objective value in this problem, as a function of the right – hand side constant vector, b.

1. The number of positive x_j by BFS of (1) can never exceed the rank of A.
2. The unboundedness criterion will never be satisfied while solving the Phase I problem.
3. If (1) is feasible, termination occurs in solving the Phase I problem associated with (1) only when all the artificial variables leave the basic vector.
4. If a tie occurs in the pivot row choice during a pivot

So, you have the equation for finding out C, and this way, this will really help you to become very familiar with the pivoting rules and the working of the simplex algorithm. So, that is the idea behind problem 5. Now, problem 6 is, this is I have taken from K G

Murthi’s book, and here the statement, they do not write C transpose does not matter; it is understood that C therefore is a in a row vector.

So, minimize your zx equal to cx subject to Ax equal to dx greater than or equal to 0. So, a j matrix of order m by n, K is the set of here they are referring to the feasible region as K. We have been using the letter capital f, f b is the minimum objective value in the problem, as a function of the right hand side constant vector b.

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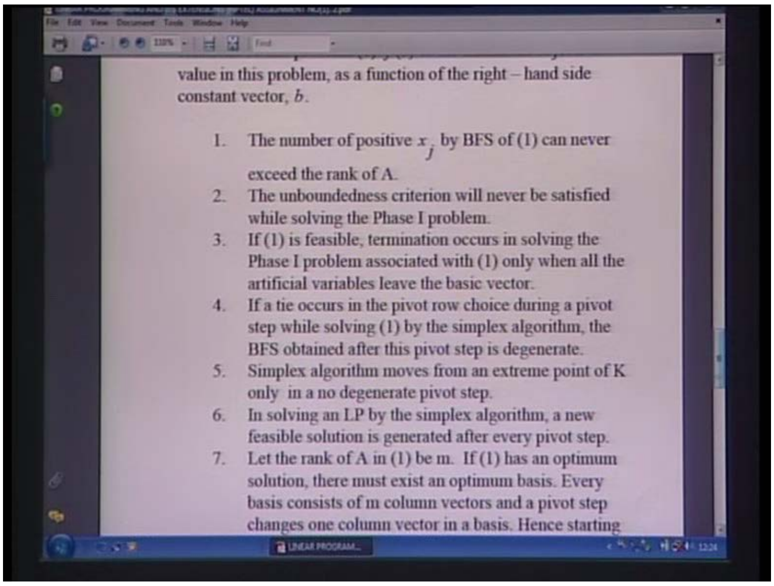

Let us just look at the, I will just try to give an idea, how to look at these problems and how to go out answering them. Let me take up number 2; number 2 says that the unboundedness (Refer Time: 09:00) criteria will be never be satisfied, while solving the phase 1 problem. Remember when I was discussing phase 1 and phase 2 with you, and I showed you that, since you are minimizing the sum of the artificial variables and the variables are required to be non-negative, therefore the minimum value - the values objective function is bounded from below by 0.

So, there is no question of phase unboundedness criteria being ever satisfied in phase 1; this is bounded from below; so, it can never go to minus infinity. So, the answer, so here, when you answer question number 2, you will say that, no, it will ever be satisfied; since the objective function value for phase 1 is bounded below by the quantity 0 by the number 0.

Now, look at number 4; if a tie occurs in the pivot row choice during pivot step while solving 1 by the simplex algorithm, the basic feasible solution obtained, after this pivots step is degenerate. So, tie means, what that, when you take the minimum ratio, so the minimum ratio is not unique. So, you are having two different r's for which the ratio is the same.

So, therefore, you have to decide which variable should leave the basis. Remember, in our case, we said that minimum ratio occurs for x b r upon y r j, and we said, we will choose that is so x b r will leave the basis; and instead of that, A j will the, x j will become a basic variable and the corresponding column will enter the basis.

But if there is a tie, then I told you that they can be problems, they can be cycling, unless we determine unless we make a choice in a unique way. And so, I gave you Bland's rule; the Bland's rule said, that you will decide on that basic variable to leave the basis, which has the smaller index. So, that means, if you have a tie for two rows, then you will choose the row which corresponds to the basic variable with the smaller index.

So, that would be the rule, and of course, that was not been asked in question 4; it says that, after this, the pivots step will be degenerated. So, obviously, because your theta value is the same for the two ratios, and so at the after pivoting, both x b r becomes 0; and suppose from x Bs was also having the same ratio, that will also become 0; it will restrain the basis. You would drive out only one basic variable from the basis, because you want to maintain basis size as m.

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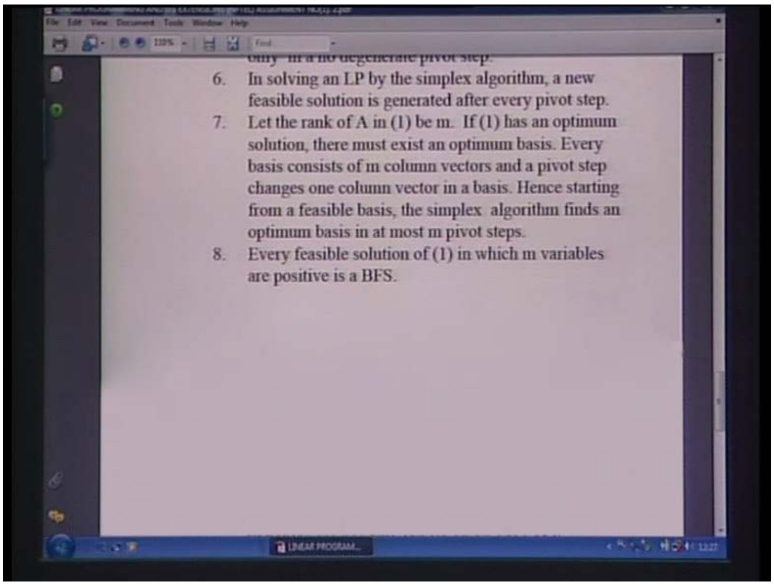

So, if two basic variables are becoming 0, only one becomes non basic, the others stays in the basic. So, the solution will be degenerate, whenever you have a tie in the minimum ratio rule, fine. Then, we can go to problem 7, yes 7; let us read the problem carefully. Rank of A in 1 be m - full rank - if 1 has an optimum solution, there must exist in optimum basis; this I proof to you, because I said that, it was any feasible solution and there is a basic feasible solution, and we have also shown you that a basic feasible solution will always be an optimal solution; so, that is that. Every basis consist of m column vectors and a pivot step changes 1 column vector in a base; hence, starting from a feasible basis, the simplex algorithm finds an optimum basis in at most m pivot steps.

In this question, they are asking you to comment. And do you agree with the statement? The statement is saying that, you start with m basis vectors, so you have m columns there, then you replacing 1 column at a time, and you are improving, you are trying to improve the value of the objective function.

So, he says that, once you have replaced, you know, at each iteration, you replacing one basis column. So, after you have replaced all the m columns, you should have reached the final solution; you should have reach the optimal solution. But we have seen in our own computations and so on, that it is not true; it is possible, that some column you replace, then it gets again replaced. So, it is not necessary that, once you change all the initial m columns in the basis, you will have an optimal solution. So, this statement is certainly not right. And what combination and what order you have your basis columns that matters, that gives you a different basis feasible solution; so, you cannot say anything here.

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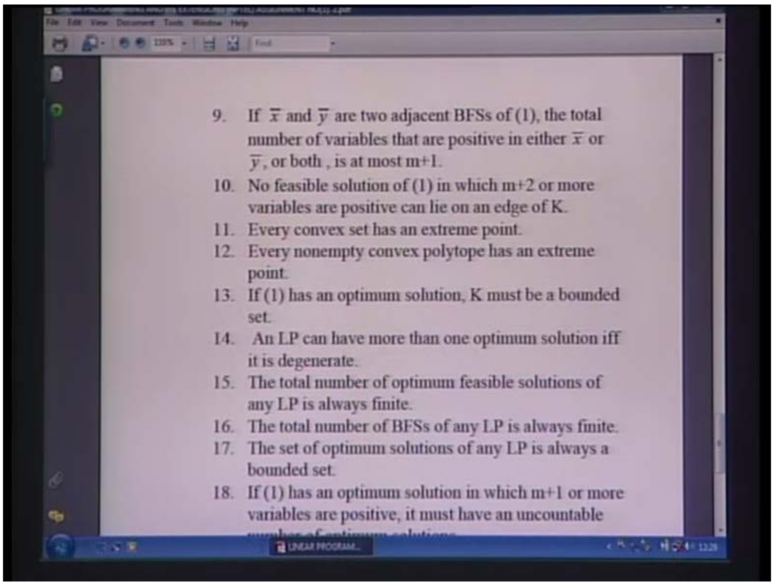

Then, I thought that, I mention problem 9 here; question 9 says, if x bar and y bar are two adjacent basic feasible solutions of 1, the total number of variables that are positive in either x bar or y bar or both is at most n plus 1, why? Because x bar may be degenerate, so it has at most m positive variables; y bar also has m at most m positive variables. But remember with they are adjacent and I discussed it with you in the earlier lecture, that the corresponding basis will differ in exactly one column.

So, that means, the m positive variables in y bar, the m minus 1 are the same as the 1 in X bar, only one differs. So, the when you add up the total number of positive variables in x bar and y bar together, they will add up to at most m plus 1, because you may have degenerate basis feasible solution, right. Then, 17 and 18 and 19, I want to discuss; let us put all of them together.

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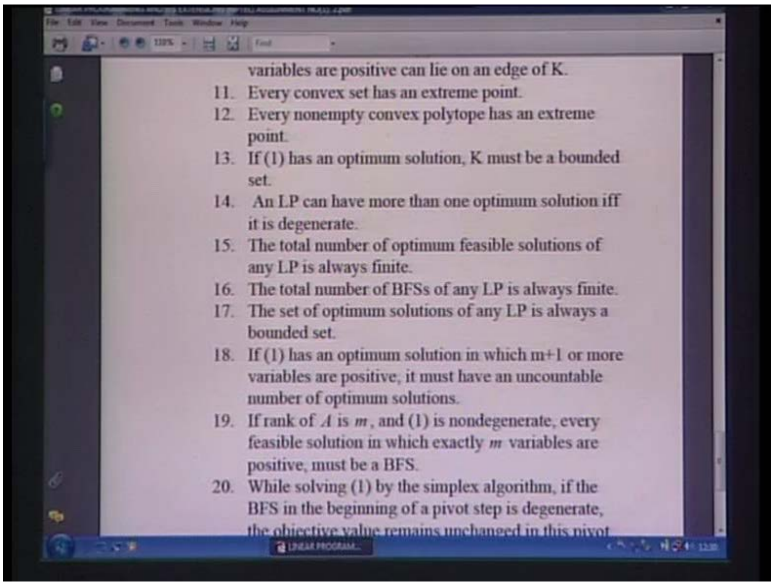

Now, look at 17, the set of optimum solutions of any LP is always a bounded set. The set of, see, if you have optimum solutions, then the value of the objective function is finite. So, obviously, you will not have optimum solution on a direction, because if you are on a direction, you remain feasible, that the value of the objective function keeps increasing. So, this is what is being said here, that if you have set of optimum solutions of any LP, and obviously, it have to be a bounded set. And 18, if one has an optimum solution in which m plus 1 or more variables are positive, it must have an uncountable number of optimum solutions.

See, now, if relate this to problem 9, see what we are saying is that, there is an optimum solution in which m plus 1 or more variables have positive; that means, there are at least two basic feasible solutions which correspond to an optimal solution, which gives you the optimal solution.

So, if you have two basic feasible solutions giving you the optimal solution, then every point on the edge joining x the two feasible solutions - optimum feasible solutions - will also should be optimal; so, you have infinite number of optimal solutions present. So, if you have more than 1, then you have infinite optimal solutions. I mention this also a time ago, when we were talking about... Now, question 19 is rank of a is m and 1 is non-degenerate, yes, I thought I should take up this, because I want to introduce you to the definition, that when do we say that an LPP is non-degenerate.

So, that is the when every basic feasible solution is non-degenerate. So, this is possible in lot of (( )) optimization problem also, you have a situation where you know that any basic feasible solution will be non-degenerate. So that is what we say; rank of A is m, and 1 is non-degenerate. And every feasible solution in which exactly m variables are positive must be a basic feasible solution.

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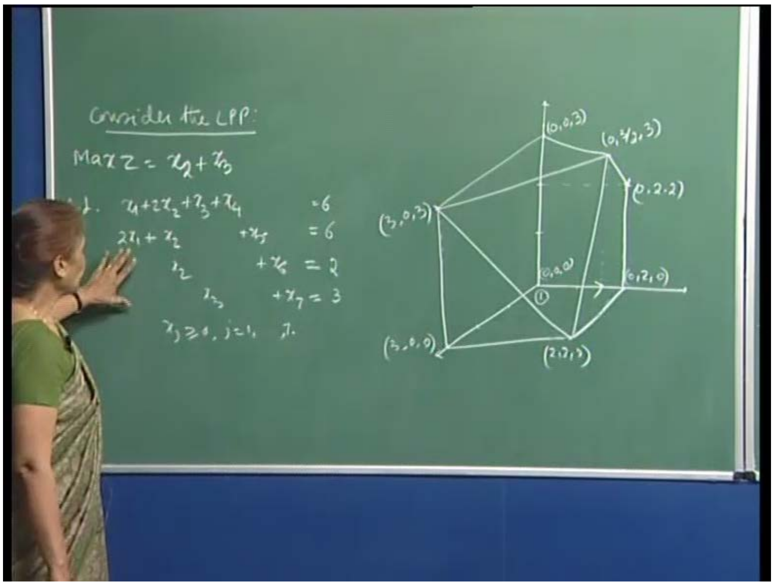

So, I would like you to think about it and answer the question. So, this takes care of the assignment sheet; so, I hope you enjoy doing the rest of the ones, and of course, these can always you are welcome to ask any queries. I will now like to show you the various aspect - geometrical aspects - of the simplex algorithm through this example. And I will trace the corresponding, see this called polytope I had discussed with you some time ago, when I was showing you the various faces of the polyhedron and so on. Now, if you take the same polyhedron, so these are the constraints by adding the slack variables here; so, that means, now this polyhedron describes the feasible set for this linear programming problem and I take the objective function to be x 2 plus x 3.

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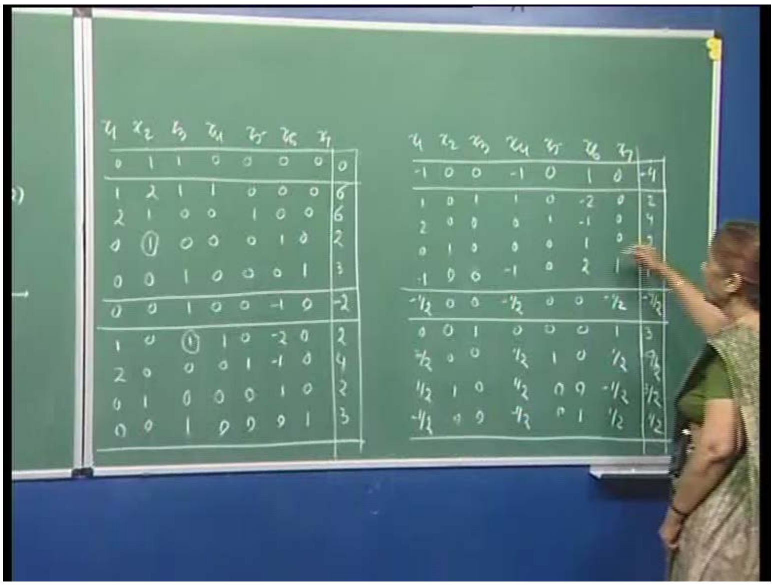

And so, if I want to continue, I mean now I apply the simplex algorithm to this problem with this linear programming problem, and here is the first tableau, so this is your top row, 0 1 1, 0 0 0, and then these are the constraints. So, you can see that, the starting solution to your basis consists of x 4, I mean, a 4, a 5, a 6 and a 7 - this is your identity matrix; so, your basic feasible solution reads as 6, 6, 2, 3.

But so regular variables x 1, x 2, x 3, are all 0. So, this is your starting point; this extreme point of the polytope is the starting point of your simplex algorithm. Then, because it is a maximization problem, I just take the first positive C j minus Z j, because that will improve the value of the objective function; so, then I have prefer the pivoting ratio is minimum here - 2 by 1. So, we will pivot on this element, and then, the after the pivoting, I show you this is the next tableau. And now, you see that, your x 2 has come in to the basis, it replaced x 6; so, essentially only out of x 1, x 2, x 3, only x 2 has become equal to 2, x 1 and x 3 are still 0.

So, the extreme point 0 to 0, therefore the simplest algorithm from here will go to on this edge of the polytope and come to this extreme point, where the value of the objective function now has gone up from 0 to 2. Remember this number shows you the minus of the objective function value.

So, we look at this then again looking at the top row; this is the variable, which become a basic variable and so you find out the minimum ratio here again is minimum for this; so, we pivot on this one. And once you pivot on this, this tableau shows you the next basic feasible solution. And, now you have x 2 and x 3 both in the basis, x 3 is at level 2 and x 2 is at level 2, the remaining positive ones are your slack variables.

So, in the value of the objective function has gone up to 4. So, 2 2, that means, 0 2 2 0 2 2 is this extreme point, and so, that means, along this edge. Since C from here you move on to this edge, come up to this extreme point, because beyond this if you go, the value of the… you become infeasible.

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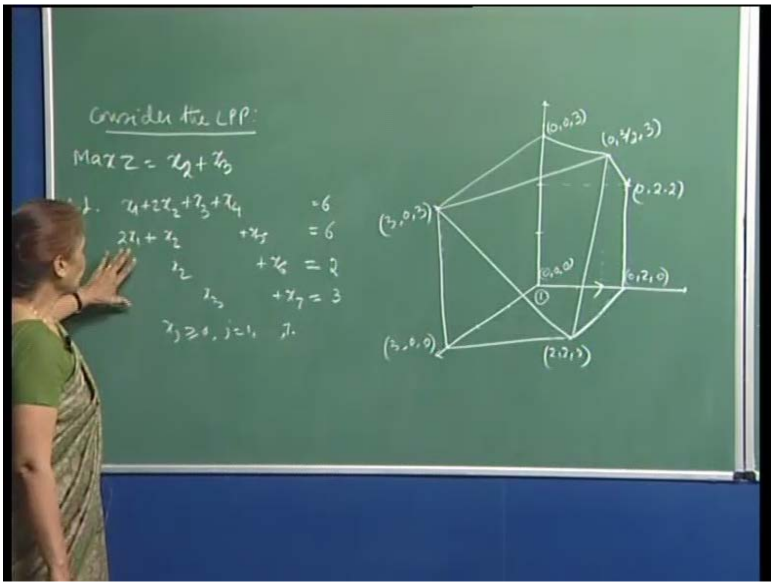

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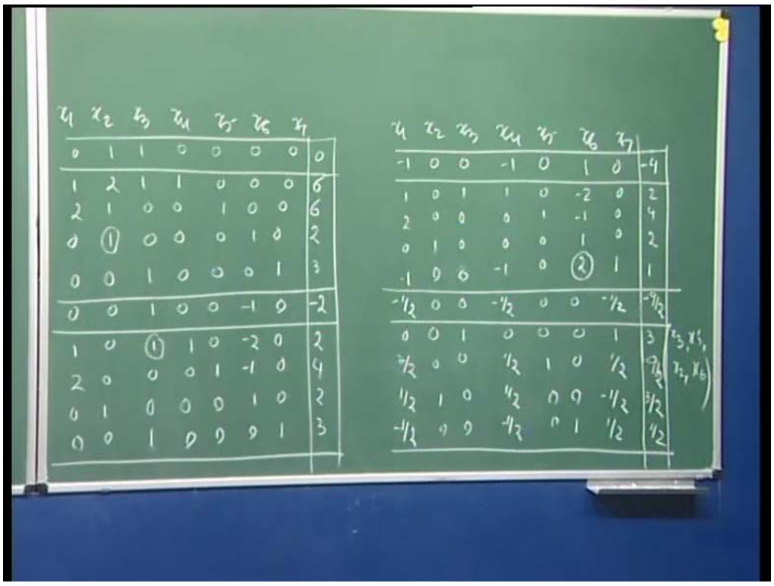

So, therefore, now from here, you proceed along this edge and the value the extreme point is 0 to 2, and the value of the objective function is 4. At this point, we were at these extreme points, and the corresponding tableau is here; then, since this is the only positive entry, this is now a candidate for coming into the basis this column. And we see that, the pivot element is this, because ratio here is 1 by 2 and this is 2 by 1; so, this is the minimum ratio; so, that means, now x 7 will become a non-basic variable and x 6 will become a basic variable in its place.

So, when after pivoting, this is the tableau we see here. And you see that the top row now has all 0 or minus negative entries; therefore, we have reached an optimal solution. The corresponding basic feasible solution is, so here, what is the basic feasible solution? This is given by x 3 and then x 5 and x 2 and x 6. But since we have only recording these polytope x 5 and x 6 are slack variables, so x 3 and x 2, what is x 3? x3 is 3, x 2 is 3 by 2.

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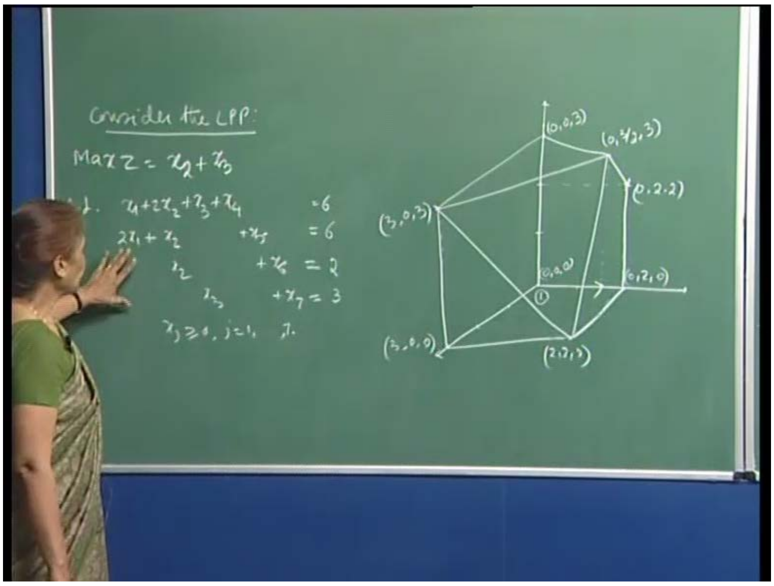

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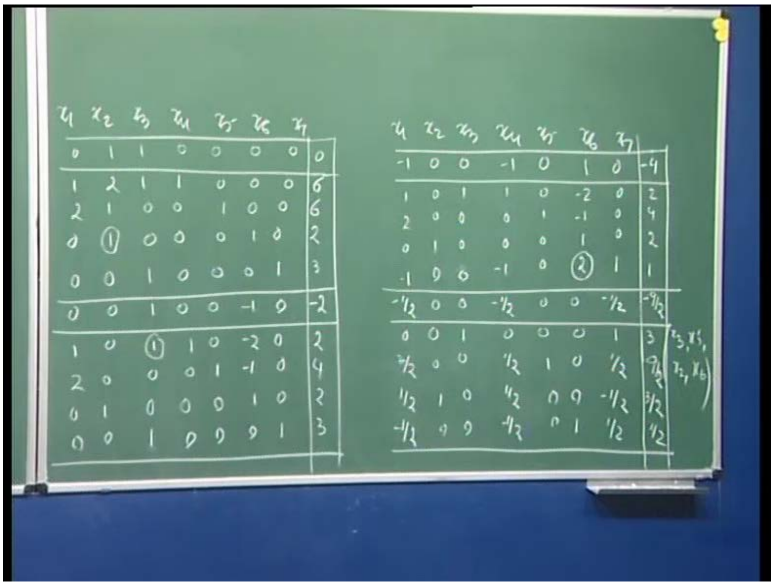

So, 0 3 by 2 and 3, so you move along this edge and this is the point where you reach, and this is now an optimal solution. So, the maximum value obtained is 9 by 2 and this is. So, this is the idea that the simplex algorithm starts from a basic feasible solution and then it moves along the edges of the polytope of the feasible corresponding to the feasible region and to an extreme point; and then, again change the direction wherever the value of the objective function will an improve. So, just see, this is moving to an adjacent extreme point here along this edge; then, for this extreme point, you move along this edge to the next adjacent extreme point, and from here, you finally come to the optimal solution.

So, the movement of the simplest algorithm is always along the edges of the corresponding polytope, and from the move along an edge up to an extreme point. Then, you go to an adjacent extreme point along another edge and so on; see, you keep on improving the value till you have reached optimal solution.

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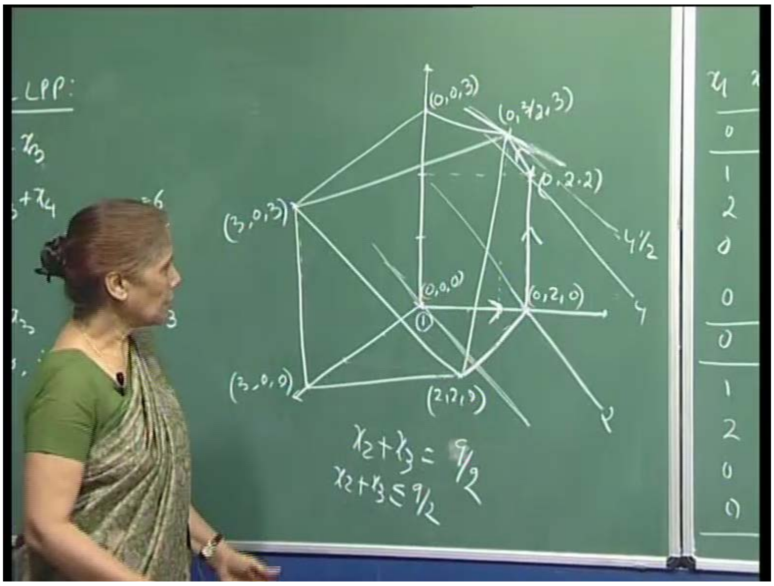

So, I hope this also gives you a feeling about, let me more a better feeling about the simplex algorithm. And then, see here, we have been also talking about the supporting hyper plane; so for example, see what was happening here is, if you look at this, this is not a very exact picture, because x 2 x 3 will have a slope of 45. Our objective function is x 2 plus x 3, so you see this is the maximum value; so for example, at this thing at point 0 0, if you take the x 2 plus x 3 here, the value was 0. So, you started with the 0 value of the objective function. Then, you move parallel to itself and when you came not a very good again a very good drawing.

But here, so here the value of the objective function became 2, which is given here and then you move to this extreme point. So, the movement of the objective function is parallel in the direction, in which the values in improving or increasing in this case; so, this became 4 here. And then, finally, this has to be parallel to this like this, fine, and then, it became 4 and half, and that was the best value that you can have.

So, you see now this becomes a piece x 2 plus x 3 has become a supporting hyperplane to this polytope at the extreme point 0 3 by 2 3, because now the whole of the polytope lies on one side of the or lies in one of the half spaces made by this, given to you by this hyper plane; x 2 plus x 3 equal to C, whatever you want to call it, so that hyper plane. So, in this case, of course, the hyper plane - the supporting hyper plane - is equal to 9 by 2; so, the half space x 2 plus x 3 less than or equal to 9 by 2.

So, this is the half space on this side, because every point of their polytope, every point of the feasible region has value less than 9 by 2 for the objective function. And the other half space which will be greater than 9 by 2 is on this side, but that is all infeasible.

So, therefore, the maximum value of the objective function is 9 by 2. So, I wanted to give you a concept about the supporting hyperplane also, and the hopefully, through this example, you now have a little better inside into the how the simplex algorithm works, and you know why we need this concept of adjacency moving along an edge, and then going up to the, that the objective function becomes a supporting hyperplane at the point which is the optimal point, because then you know that is the best value. And so, the whole of the polytope has to lie on one side of the half space, when the when you have the supporting hyperplane equal to the value of maximum value or maximum or minimum the optimum value of the objective function.

So, this was my concept; you could have so many other supporting hyperplanes here; this is not the only one. So, you can now in fact see that, you can sort of play around with this objective function, till you see that it continues to the supporting hyperplane for the polytope; and therefore, this point will continue to the point of maxima for that polytope.

For example, you can have this as a supporting hyper plane, you can have this as a supporting hyper plane; that means, the constraints itself this one or this one will continue to be so, yes, you can find out the edges correspondingly; you can have the supporting hyperplane, which now meets the polytope in an edge instead of at a point. So, then all these points infinite number of solutions will be an optimal solution, if you are supporting hyperplane was the meeting this polytope and this edge, and so on. So, you can just have a good feeling about the simplex algorithm.

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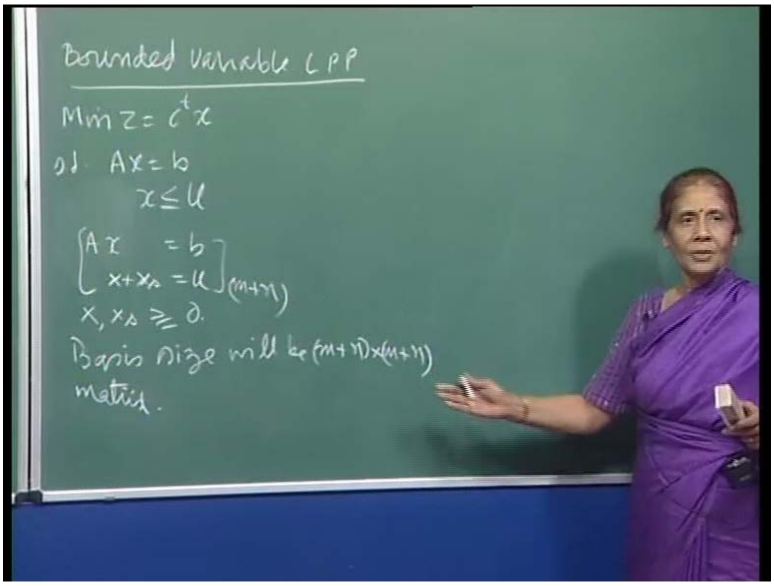

We start looking at special cases, then extending the theory of linear programming that I developed so far. So, first of all, let me consider the problems bounded variable linear programming problems. So, here, what happens if that apart from… so, the problem actually would be of the kind minimize Z equal to C transpose x, subject to A x equal to b. And then, we are saying that this is less than or equal to u, where u is a vector of scalars, u 1, u 2, u n. So, in other words, we do not want our value of x j to exceed u j and it is not necessary that all the variables in the problem are bounded; some may be free to take any non-negative value, but some may be bounded.

So, I will develop the algorithm for the case, when all variables are bounded, but then you can see that you can modify the pivot rules, whenever you have only a partial set of variables are bounded, fine. Now, you might say that what is the big deal. You can consider, because we had not been considering these constraints explicitly. Remember because we take care we when you go from 1 basic feasible solution to another, we make sure that we do not a violate the non-negativity constraints. So, that can be taken care of a n, but this can be part of this; so, essentially, you can rewrite this as, Ax equal to b x plus Xs equal to u, where Xn Xs, both are non-negative.

So, this constraint set becomes the enlarged constrain set here, but then the number of constraints now is m plus n, instead of m. So, if I have to choose a now my basis size, that means, the basis size will be m plus n by m plus n matrix m plus 1 this matri.x

So, just imagine, when the problems are small, it may not really matter. You can work with a basis size of m plus 1 by n plus 1, but if you have, let say few hundred constraints and thousand variables, the size is really enormous; it goes up by n. Secondly, the structure of these constraints is very simple; the matrix here is identity matrix - x is less than or equal u. So, let us try to exploit this structure; hence, see, we can modify our pivot rules. So, again you take care of these constraints implicitly, not explicitly; if I carry my basis size as m plus n by m plus n, then we are now looking at these constraints explicitly.

But suppose I can modify my pivot rules, such that, these constraints can be taken care of implicitly, then I will be very happy to be able to just work with the basis size of m by m matrix.

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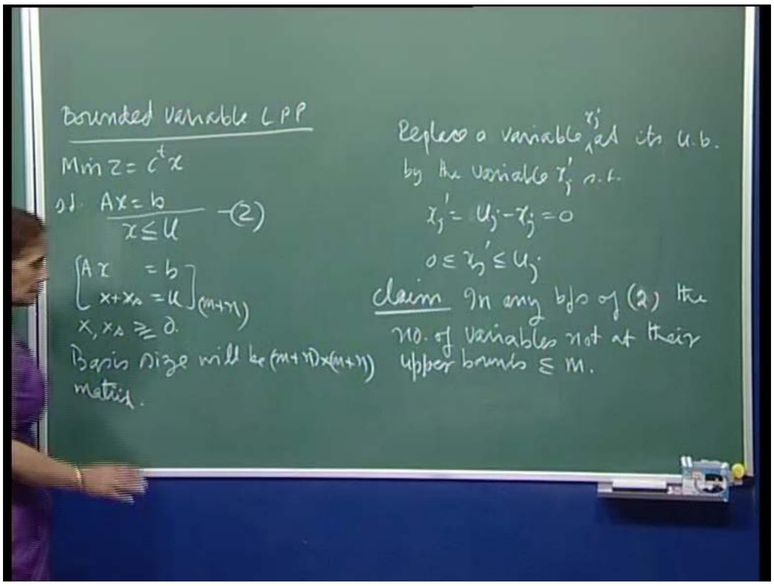

So, let us see how can we make sure. Now, so first of all, the idea is that, replace variable at its upper bound by the variable replace a variable x j at its upper bound by the variable x j prime, such that, x j prime is u j minus x j. So, currently x j is equal to u j; so, if I put it and I make the transformation x j prime is u j minus x j.

So, when x j is u j, currently this is 0, right; and you also see that, 0 less than or equal to x j prime is less than u j, because x j has to be non-negative; so, x j prime cannot exceed u j. So, what is the idea behind this is that, I do not want keep variables which are at their upper bounds, I will declare them as non-basic; this is the way I modifying, going to modify by pivot rule. So, variables which are at their upper bounds, I will try to use this transformation and reduce them to non-basic variables. So, that means, instead of x j, I will be carrying in my tableau x j prime, and x j prime currently is 0; it can then become later on positive, it becomes positive; then, obviously, you by this transformation, you know what the value of x j is; so, no problem.

So, therefore, what we are saying is that, we will not maintain, we will try not to maintain basic variables at their upper bounds, but then can I make sure, ensure that, I can have a basis size of m columns, in which the variables are not at their upper bounds; can I do that?

So, therefore, the idea here is ya so claim, yes. My claim is, in any let me call this problem with this constraint; so, this is let me call this as 2. In any basic feasible solution of 2, the number of variables not at their upper bounds is less than or equal to m.

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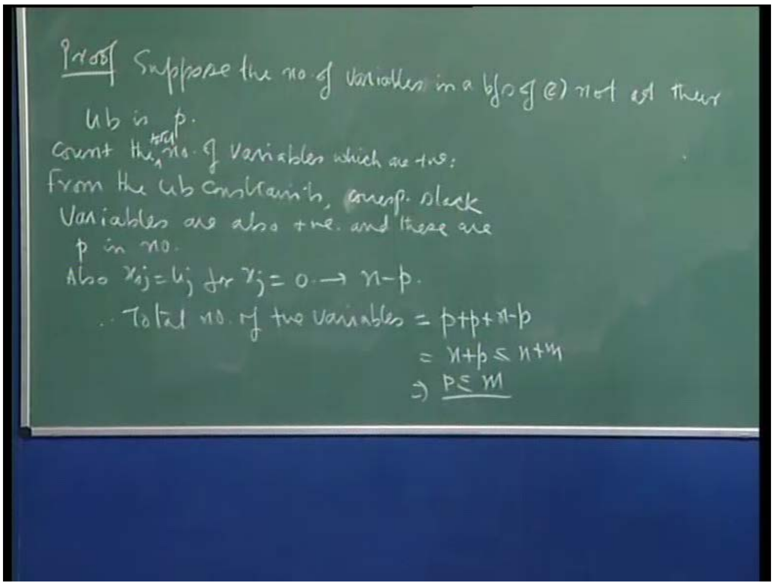

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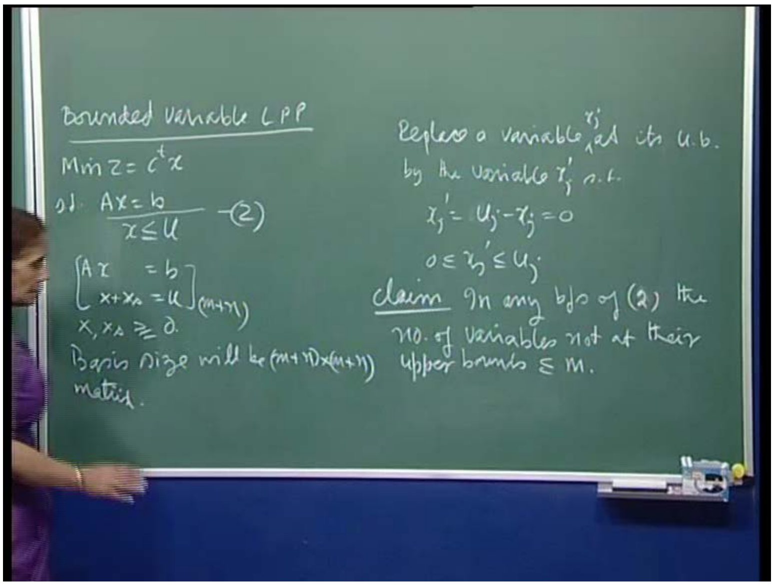

If I can ensure this, that I can always say that, I can have a basis for this new this bounded variable problem of size m by n. Suppose the number of variables in a basic feasible solution of 2 not at their upper bounds is p; so, I have a basis, that means, I have a basis of size m plus n by m plus 1; therefore, the number of basic variables is m plus n. I am saying that, suppose the number of basic variables not at their upper bounds is equal to p.

So, count the number of variables, count the total number I should say - count the total number of variables - which are positive. So, but how many? See, the variables when I say basic variables I am referring to x, remember because, finally I want to work with the basis for this set of constraints. And so, when I talk of variables which are not at their upper bounds, I talk of the variables from the set x.

So, yes, so p of these are not at their upper bounds. And a typical constraint here is x j plus x sj is equal to u j - a typical constraint. So, if x j is not at its upper bound, then x sj has to be positive, because they both must add up the u j; xj strictly less than u j, so x sj also has to be something positive, which is equal to u j minus x j. So, then from the upper bound constraints, corresponding slack variables are also positive.

Since p of them are positive, not at their upper bounds and minus p of these are 0. This from the upper bound constraints, corresponding slack variables are also positive and these are p in number. Also x sj is equal to u j, for x j equal to 0; so, non-basic variables… And these are how many? And these are n minus p in number.

Therefore, total number of positive variables is equal to p plus p plus n minus p, which is equal to n plus p, and this has to be less than or equal to n plus m, because any basic feasible solution cannot have more than… I am assuming that here rank of a is m, and of course, the rank of this matrix is what? Yes, I should have mention it, when I said basis size will be m plus 1 by m plus 1 matrix, why? Because the coefficient matrix is A 0, I and I; so, this is answer. Therefore, rank A 0, I I is m plus n, where rank A we are taking as m. So, that is no constraint, because you saw that we can, if there is a redundancy, the algorithm will find it for us.

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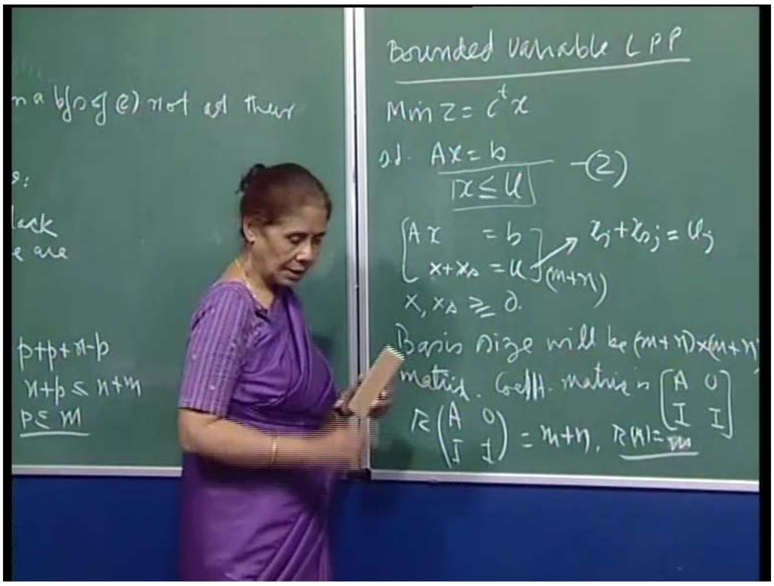

So, anyway, so this is what you have here and therefore the number of positive variables any basic feasible solution cannot exceed n plus m. Therefore, this implies that, p is less than or equal to m. So, in case p is less than m, we may either have a degenerate, we will have degenerate basic feasible solution or we may carry some variables from x at upper bounds. But we can suddenly make sure, that we do not have more than m positive variables in a basis. So, these variables will correspond to the x variables not to this; so, that shows you that, we can work with, we do not have to use these constraints explicitly; we will work with the original tableau and the original basis, but make sure when keep making the transformation, so that you do not carry too many variables at their upper bounds, fine.

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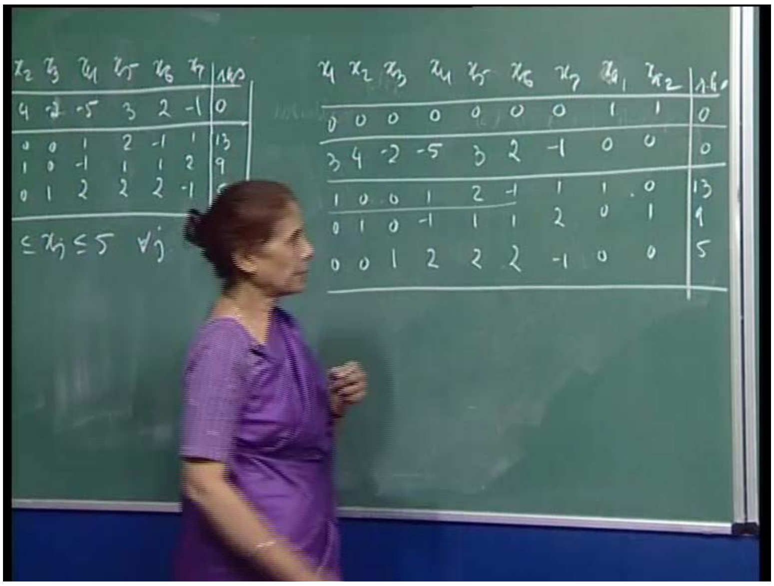

Now, let me just modify the pivot rules for you of it. So, let us now see, how the pivot rules get modified; so, let us look at, see remember, because we do the pivoting, we have multiplied by been while.

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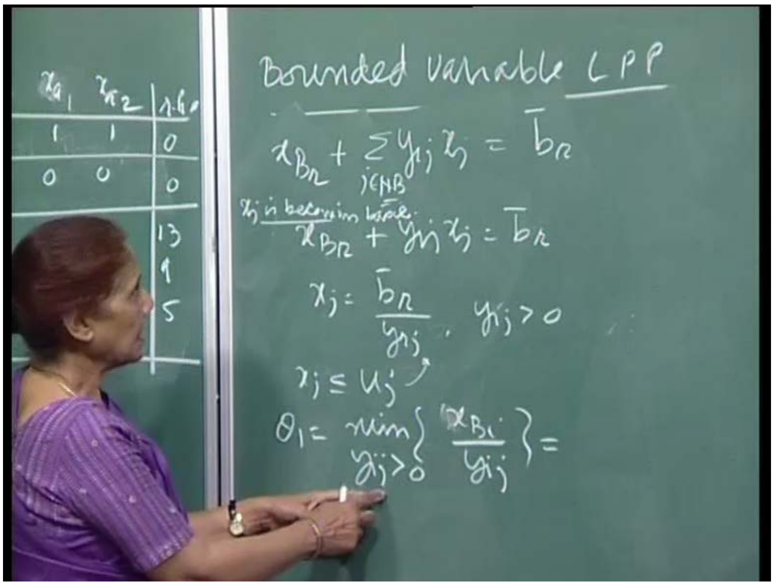

So, for example, here if you look at this constraint, now the transform constraint looks like x 1 or x b 1 is equal to, so they transformed is, so let me write the x B r plus summation y rj x j, j belonging to your non-basic set, yes; let me introduce this notation also, because we say B corresponds to basis set and non-basic, and B corresponds to…

So, this is equal to your b r bar, which we are also. So, currently all non-basic variables are 0, so your x Br is equal to b r bar. But now, we want to, you can introduce, suppose x j into the basis, and what would we doing? So, in that case, it will be x Br plus y ij x j is equal to b r bar. So, how does your this thing change? So, when j, when let say, when x j is becoming basic variable, so then the others are all 0. We are considering of how much x j can become; and we said that, x j therefore will be equal to, because x Br become 0, so we said that this will be b r bar upon y rj, where y rj is positive, remember; so, this is how you are computing, because when x j becomes b r bar upon y rj, then this is b r and so x Br becomes 0.

So, x Br becomes non-basic variable and x j becomes a basic variable, and it enters the basis at this level. But remember now, we also have this, that x j has to be less than or equal to u j. I mean, in case x j is required to be in a bounded from above, then we have to take care of this constraint also. So, then we cannot allow x j to enter at this level, if this number is bigger than u j. So, let us start defining this things; now, theta 1 is your minimum of number, this is minimum of x Bi upon y ij, where y ij was greater than 0. We took this, and that in this particular case, came out to be for i equal to r, the ratio was minimum.

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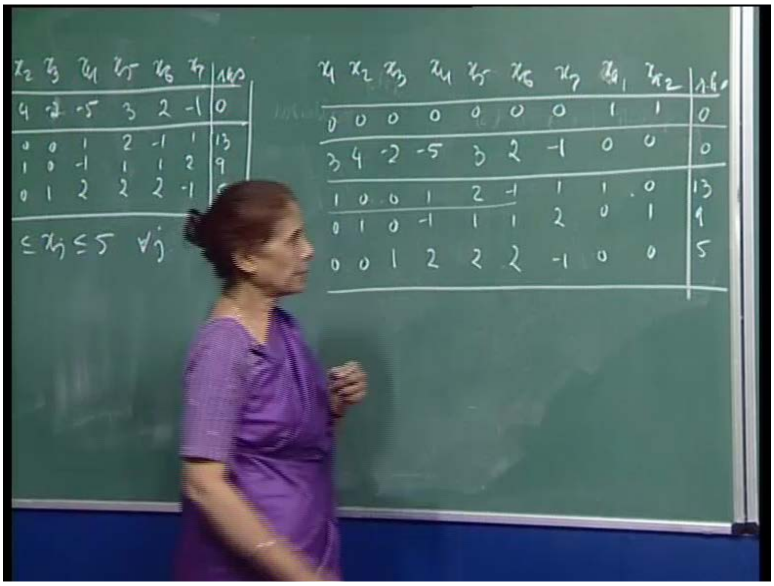

So, this is theta 1, but then we also have to take care that this theta 1 should not exceed u j. So, we will have to choose either theta 1 or u j, but there is another constraint, that is coming out, how at what level x j can enter the basis; see, what happens is that, we were only concerned with the positive y ij's here, but if you have all of the constraints, say for example here in this one, or let us say that suppose I want to suppose I am entering this column just for the argument sake, suppose I was entering this column here, then if we look at now x 2 the second basic variable, there is a minus 1 entry here.

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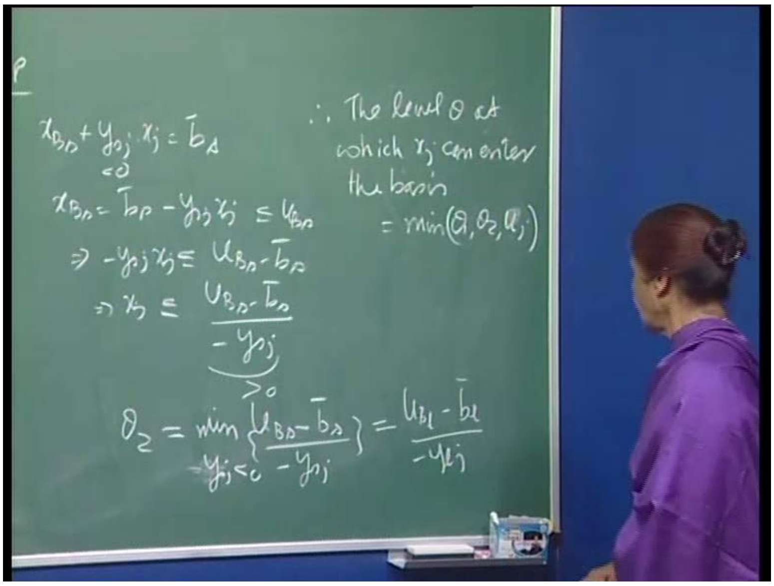

So, what happens? Your corresponding constraint is, x Bs plus y sj into x j, and this is your b s bar. Now, because y sj is less than 0 as x j increases, this value goes up, sorry y sj into x j this is become negative. So, for in order this therefore this equation to be satisfied, your x Bs will go up; the value of x Bs will go up, instead of becoming negative.

So, so far it did not bother us, because we did not take care of, we were not concerned with the entering column having negative entries, because if the value goes of fine, it stays non-negative; the value of x Bs, because this is an equation, so or right it this way, therefore, this means, at x Bs is b s bar minus y sj x j; so, y sj is negative, so this whole thing is positive, and therefore, if x j becomes positive, your value of x Bs is becoming more than b s bar.

But then, we want this to be less than or equal to u Bs; and that is also again, if the outgoing variable, if this particular variable for which the corresponding coefficient here is negative for the entering column, if that has an upper bound constraint, we will be bound by that also; we cannot violate this upper bound constraint. And therefore, this implies that, what is the restriction on x j. So, you will say that, minus y sj x j should be less than or equal to u Bs minus b s bar, which implies that your x j should be less than or equal to u Bs minus b s bar upon minus y sj, where this whole thing now is positive.

So, any quality does not change; so, this therefore, now that means, you will have to do this, you will have to compute this ratio for all those components of y j, for which the this thing is negative; so, that means, I will defined my theta 2 as minimum of u Bs minus b s bar on minus y sj, then where y sj is less than 0.

So, you will choose this theta 2, and therefore, we will finally say, that the level theta at which x j can enter the basis is equal to minimum of theta 1, theta 2 and u j. So, three numbers, that means, when it enters the basis, it should not violate its own upper bound constraint, it should not violate another basic variables upper bound constraint; and of course, this is the normal pivot rule; that means, so here, what do we do? If theta is equal to…

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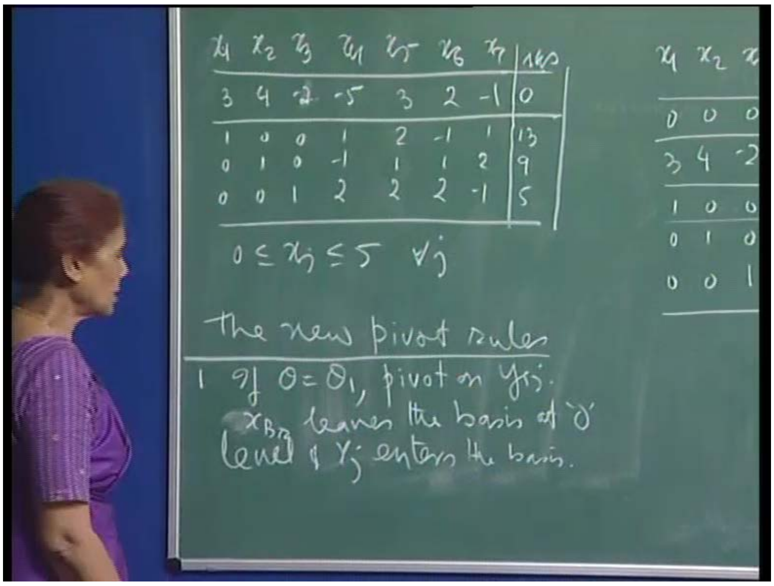

So, let me write out these three cases. Now, so once we have this computation clear, now I have these three. So, I need yes, so let me discuss the cases here and then we can start working or (( )) example. So, the pivot rules - the new pivot rules; so, first, if theta is theta 1, pivot on y rj and x br leaves the basis at 0 level, and x j enters the basis; this is your normal pivot.

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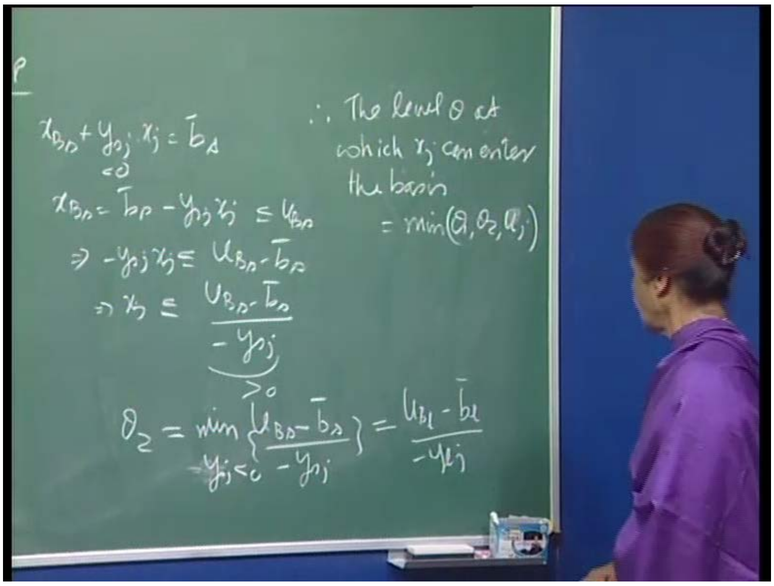

So, we will pivot on y rj and continue with the simplex algorithm. Second, if theta is theta 2, then again we will allow, if theta is theta 2, then we will pivot on y sj over whatever the number of which, yes. So, we will, if theta is theta 2, pivot on the… you want me to write it someway something, may be let me say that, this minimum is for u Bl minus b l bar upon minus y lj; this minimum ratio occurs for x equal to l, pivot on minus… what can happen is, because if I pivot on y sj, then see the thing is that y lj is negative.

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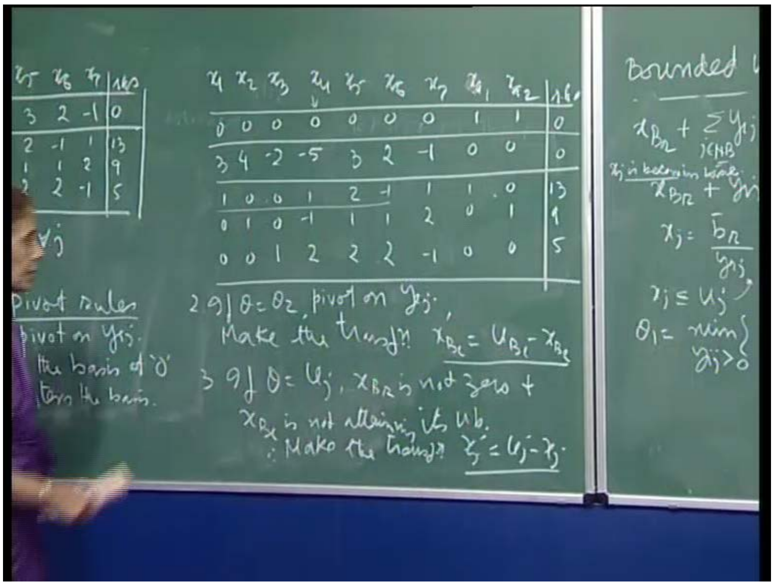

So, we can pivot on this one, so pivot on pivot on y lj; therefore, that means, A j will enter the basic set of columns, but then make the transformation for, because the corresponding variable which is reaching its upper bound, the corresponding basic variable which is reaching its upper bound is x Bl, make the transformation x Bl prime is is u Bl minus x Bl; so, because x Bl would have reached its upper bound, so then this will become 0.

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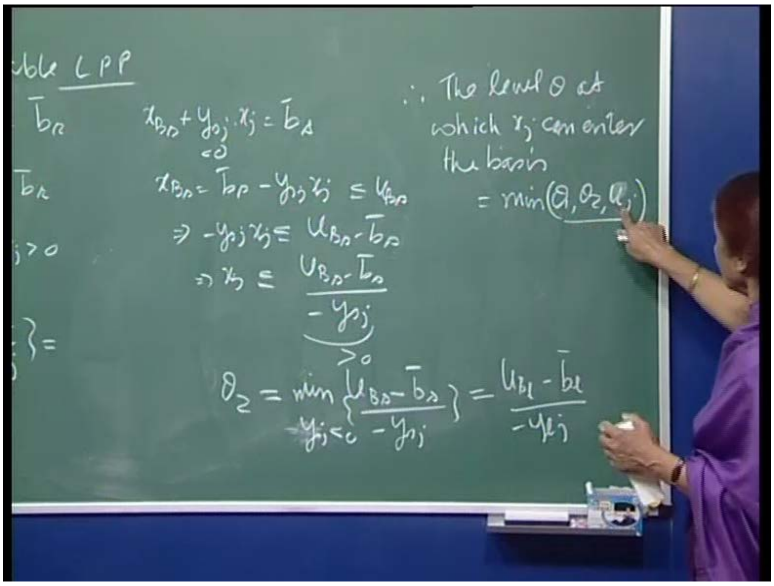

Currently, in my tableau, the variable that I will carry will be x prime Bl, which will be at 0 level. So, if therefore it will not be a basic variable, and so my basis will still consist of m positive variables from our at most m positive variables, from among the regular variables, not the fine. And what will be the third one? If theta is u j, that means, u j is a smallest number out of the 3. So, u j is the smallest, that means, when x j is equal to u j, when u j is the minimum number here, it means that your variable x Br for which this ratio was minimum is not becoming 0; this one because theta 2 is bigger than u j, that means, your corresponding x Bl is also not reaching its upper bound.

So, they both remain positive and not at their upper bounds; therefore, I can only bring, I can pivot on or I do not need to pivot on A j, because pivoting implies that you are entering it into the basis, but I am not pivoting on A j; I will simply make the transformation x j prime equal to u j minus x j; you get the point, so that I still maintain the same basis, but the value of course the your testing would have change, how?

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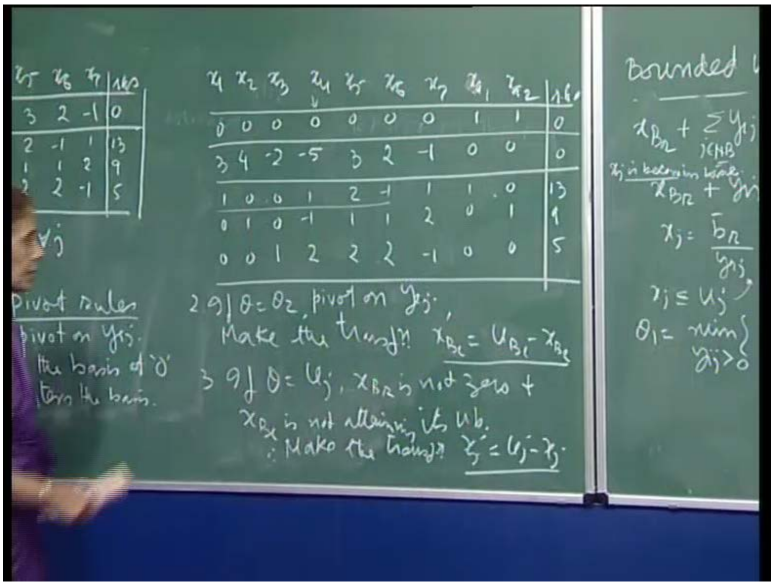

So, what if theta is u j, we want say x Br is not 0 and x Bl is not attaining its upper bound; therefore, make the transformation x j prime is equal to u j minus x j. I should have made a note here also; that you make the whenever you make this transformation, how do you effect this transformation on the tableau? So, these two cases I shall have here, also I should have made that point, that if you pivot on y lj, then because you are current basic variable x, Bl has reached its upper bound; I need to make this transformation.

So, how do I do this? See, remember, in this thing, its y Bl times u Bl minus x Bl. But currently, right now the tableau, say for example, I want to this has reach its upper bound, I want to make the transformation that this should be l minus x Bl. So, then, in this case, you see, a y Bl times this - the new y Bl; the new y Bl times u Bl is a constant thing, which you will go to the right hand side, and this will read as minus, because x prime Bl. So, the corresponding column here will become minus of the column for x Bl, and the y Bl times u Bl will get transformed here; so, my right hand side will change. So, the objective function my basic feasible solution has changed, because I am making this transformation, I am making this transformation or this transformation in any case.

So, in this case, we will not be doing at the pivoting; we will simply say that the corresponding column, say for example, as I was saying that, if this is to enter the basis and what will happen is, I will take this column - this whole column - times u j, write it on the right hand side, right. And then, multiply the whole column by for minus sign, because the column for x j prime is now minus times the column for x j.

So, this is the change; so, whenever you make these two transformations, you have to, of course, in this case, we do the pivoting. Then, we transform this the current y column times u b l to the right hand side and multiplied by minus sign to maintain it here; and we will also, if you are making this transformation for example, then I will indicate this by a minus sign. To say that, when you finally get the, when you reach the optimal solution, you would have you want to read the solution in them for the original variables. So, you can then use this transformation and get back to the original feasible solution, which is optimal.

So, this chooses you all the pivot rules. So, I will keep these here and let me show you the steps according to this example. So, what is happening here is, the given problem is like this; I have given it in tableau form.

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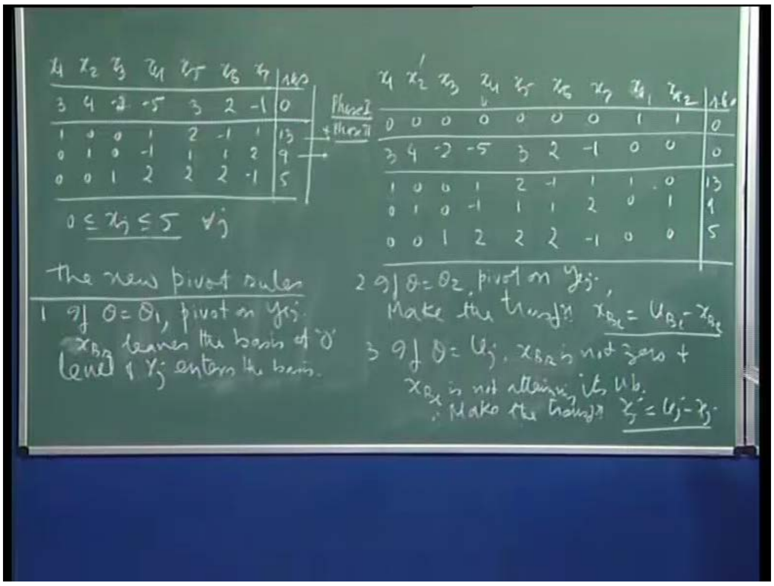

So, the upper bound the restrictions on the variables are each variable must be less than or equal to 5. If we look at this tableau, the current basic feasible solution for this… if I want to take this as the basic feasible solution consisting of x 1 x 2 x 3 are the basic variables, then you see that these two numbers are beyond 5. I can keep this at level 5, but these two numbers are more than 5; therefore, I cannot immediately obtain starting feasible solution to my bounded variable problem, since all of have to be; therefore, I will now use phase 1 phase 2; so, this is phase 1 and phase 2.

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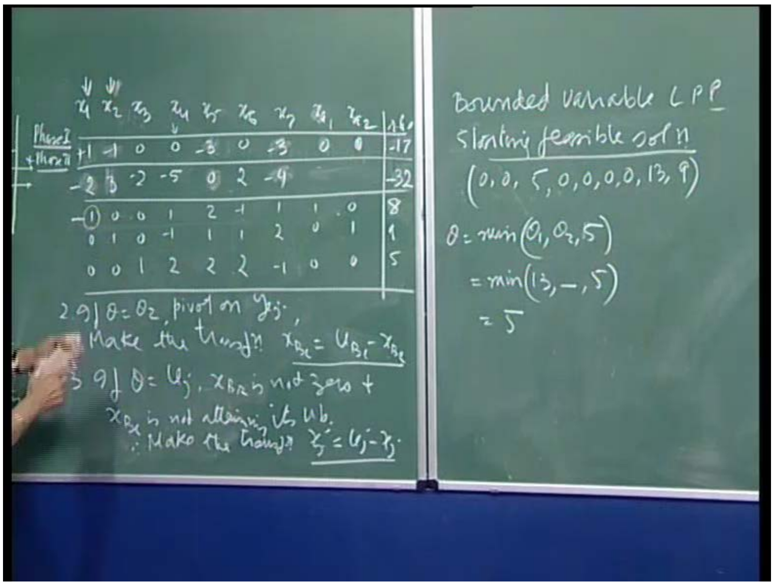

So, phase 1 and phase 2; I add the artificial variables x a 1 and x a 2, and I did add them only to the first two constraints, because of the third constraint satisfy; so, that means, my starting basis is let me write down here. So, starting feasible solution, starting feasible solution is, for example here, in this case, it is 0, and reading this as 3, 0, 0, 5, then 0, 0, 0, 3 4 5 6 7 and then, it is 13 and 9; is it with you? x 1, x 2, x 3, x 4, x 5, x 6, x 7, x a 1, x a 2; this is my starting basic feasible solution. Remember there has no upper bound constraint on the artificial variables, fine.

Now, in order to have this, so I am using, I am keeping the carrying the original objective function also, and we will do keep the operations - we will do the operations on this row also. So, here, of any way, you have to add these two rows and subtract from the top 1; so, add these two rows that gives n subtract.

So, minus 1 minus 1, this is 0, this is 0, minus 3, this remains the same, this is minus 3, 0, 0 and this is minus 22, you do the same here; add this and subtract, so this gives you 2, this gives you 3, minus 2, 0, 3, this gives you 0, 0, 3, minus 4; this I do not have to, in fact, I can leave this like this and then this also minus 22, fine.

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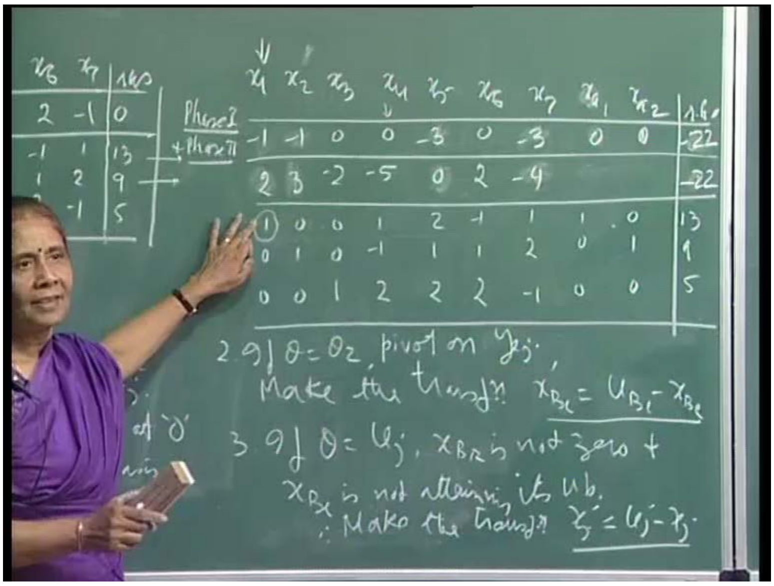

So, now, this this is the starting tableau for your phase 2, phase 1 problem for this bounded variable LPP. And look at the first one; so, this is the one to enter the basis. So, here, of course no other entries negative, so only one positive entry, and see what happens; if you pivot on this one, then the entering variable will be at level 13.

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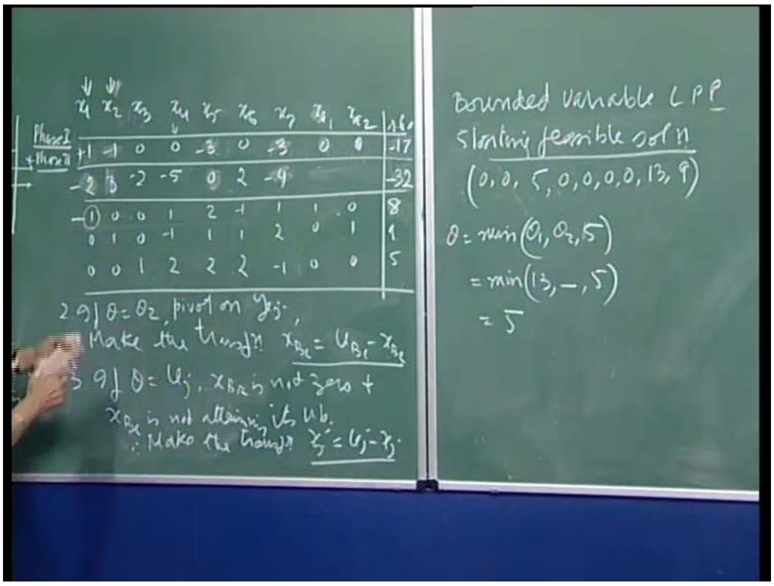

So, remember what was the rule; so, here, you see your theta had to be minimum of theta 1, theta 2 and here its 5 - the upper bound. So, theta 1 is coming out to be how much? 13, ratio is 13, 13 by 1 which is 13. So, in this case, it is coming out to be minimum of 13; theta 2, there is no theta 2, so it is nill and this is 5. And therefore, what do we do? We simply make the transformation, as I said here, you simply make this transformation, and what would that mean? It would mean that, 5 times this row gets transformed gets taken to the right hand side and then I multiply this by minus sign.

So, 5 times I do this plus 5; so, this will become plus 5 and that will be minus 17. Then, 5 times is subtract; so, this becomes 32; actually this has no meaning now. Then, 5 times you do it, this becomes 8 and that is it; and then, I multiply this by minus sign, because it has to be minus y j of x j, but it has to show here. So, now, you would show you one more and let us just see that quickly. So, this is the next negative one; remember, I do not have to do any calculations here, because no pivoting has been done.

So, when I enter this, this is the only positive entry here, and again the same thing follows, right; so, I do not pivot on this one, because this more than 5. So, again I will do the same thing, I will make the transformation, 5 times this whole thing gets transform to that side; so, 5 times when you take it, this will become 12; I will not compute this right now, and then this becomes 4, you see this becomes 4; and then, you minus, minus, minus.

So, I will continue with this problem. I will come back in the next lecture, I will write it again and to show you two more iteration, so that I can sort of demonstrate to you all possible pivot rules that we have to. So, once you have modified you see, you can very nicely take care of the upper bound constraints in an implicit way and maintain the same basis size. And so, this gives you a little more insight into how this simplex algorithm works. So, I will continue with this in the next class.