Linear Programming and its Extensions

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Lecture No. # 01

Introduction to Linear Programming Problems

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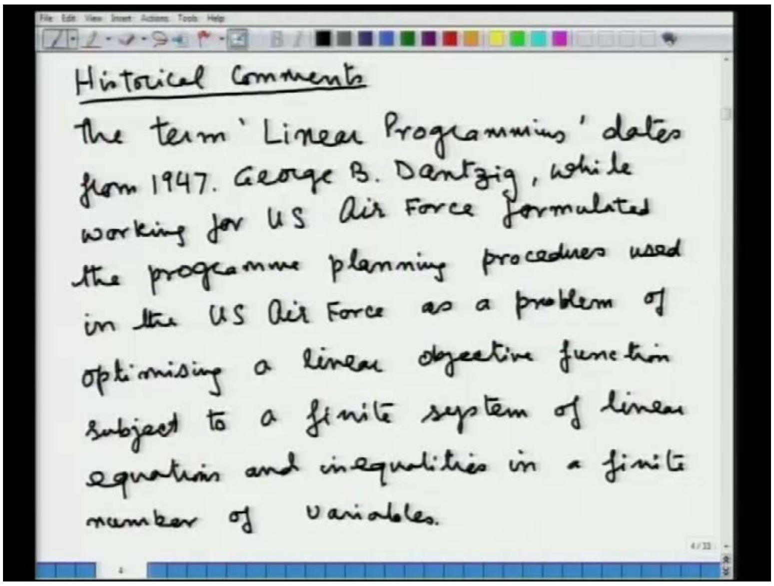

In this lecture, I will introduce you to a class of optimization problems known as linear programming problems. Let me begin, I giving you a few historical facts about this subject of linear programming; the term linear programming dates from now 1947, I may not be the exact date, but anyway let us it was around that time a from 1947, George B Dantzig, while working for US air force formulated the program planning procedures used in the US air force as a problem of optimizing a linear objective function subject to a finite system of linear equations and inequalities in a finite number of variables.

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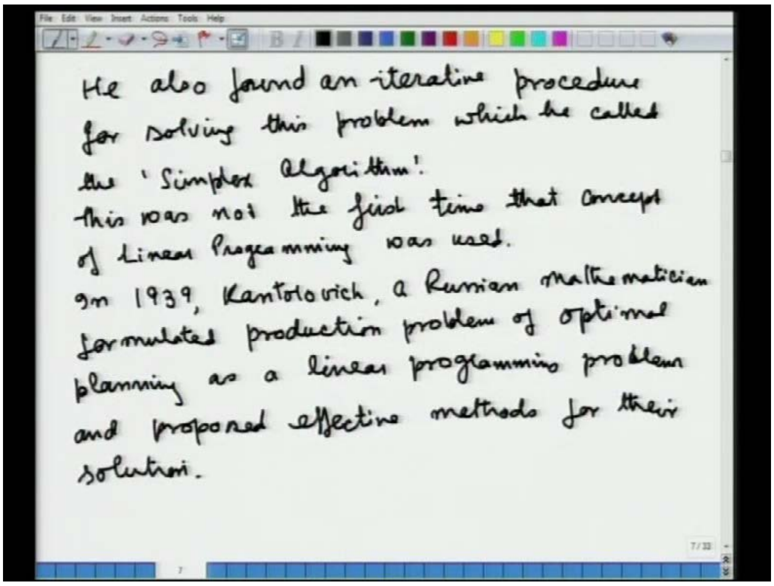

He also found an iterative procedure for solving this problem, which he called the simplex algorithm. This was not the first time that concept of linear programming was used. In 1939, Kantorovich, a Russian mathematician formulated production problems of optimal planning as a linear programming problem and proposed effective methods for their solution.

In fact, there is an interesting story that Dantzig went to Kantorovich and very proudly try to show that he had come up with this simplex algorithm, and he had formulated this problem is a linear programming problem; Kantorovich, open his drawer and took out his own formulation of this is the production problems, you know, planning production planning problem as a linear programming problem and he showed his own algorithm for solving it.

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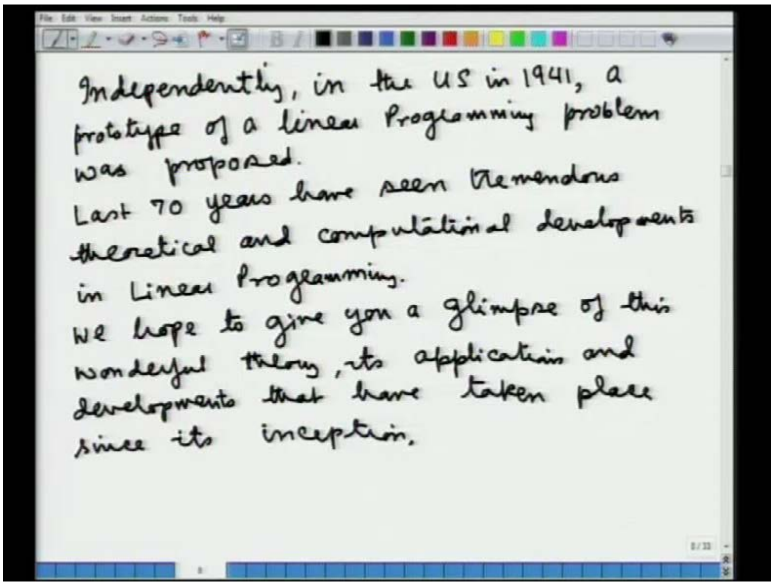

Independently, in the US in 1941, a prototype of a linear programming problem was proposed. So, last 70 years have seen tremendous theoretical and computational developments in linear programming. We hope to give you a glimpse of this wonderful theory, its applications and developments that have taken place since its inception.

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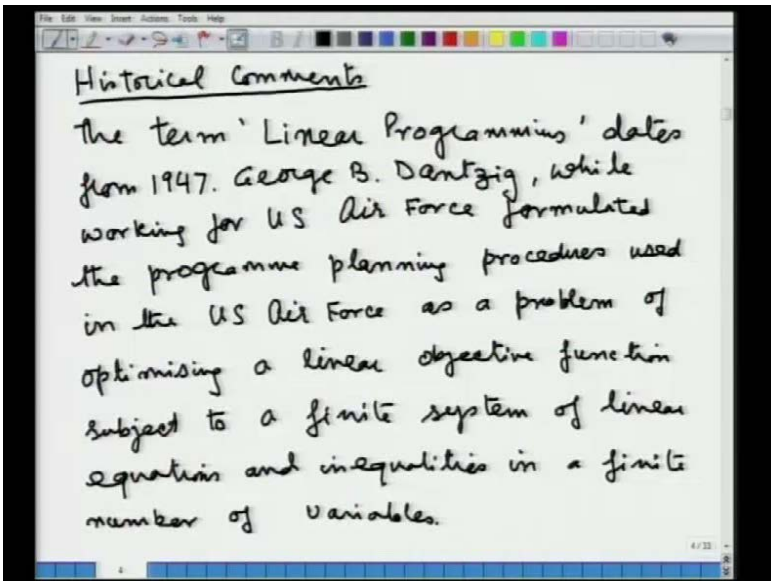

Just on a, you know, if I go back to the first slide, I would say that the date was 1944 or 45 in Dantzig actually proposed the linear programming problem and method of solving it. These classes of problems follow axioms of linearity, which I will try to introduce you, and they have been very successful in formulating and solving a large class and variety of problems through these linear models.

So, I will take up an example to demonstrate to you how these axioms of linearity work, and this problem would as known as the blending problem; it was initially introduced by professor KG Murthy, he formulated the problem; situation is that, there is a gasoline company or maybe we can say petrol and diesel making company, which has three raw fuels, it gets a raw material, and it wants to blend it into three fuels of different octane ratings, the fuel that we use for different vehicles have an octane rating at we try to determine the price and the market value through this octane ratings.

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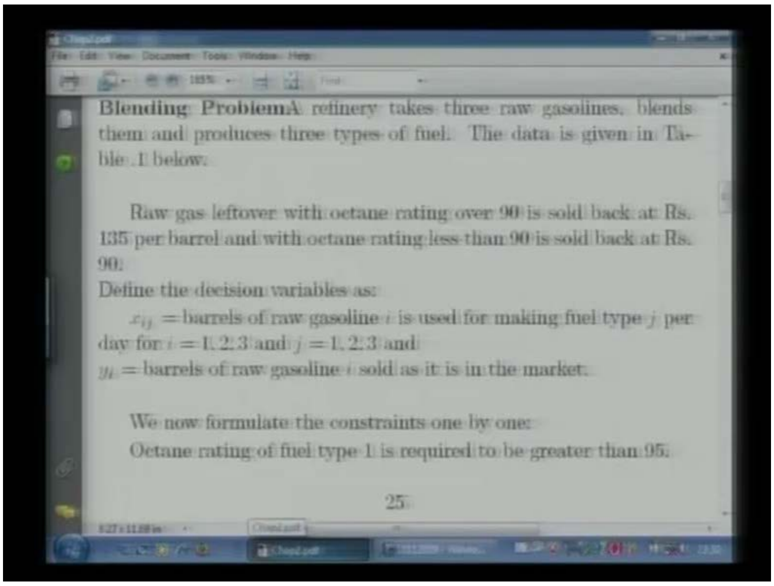

So, now, I would like to show you while introducing the problem, the data that we will use to formulate this problem as a linear programming problem; let say that the number of barrels we will say that the our decision variable, which I will defined as x i j is the number of barrels of raw gasoline i used for making fuel type j, so x i j, i j are the two suffixes; so, the first suffix relates to the raw gasoline that is used; and then j is the a suffix used for the fuel type j, which is used for making the final product that is that goes to the market right.

And then y i will be the barrels of raw gasoline i sold as it is in the market; that means, the company has some - raw materials - raw gasoline, it wants to blend it into with different fuel types; then whatever is left over that is also sold back, because some people can use some chemical companies or some people can use this raw gasoline also.

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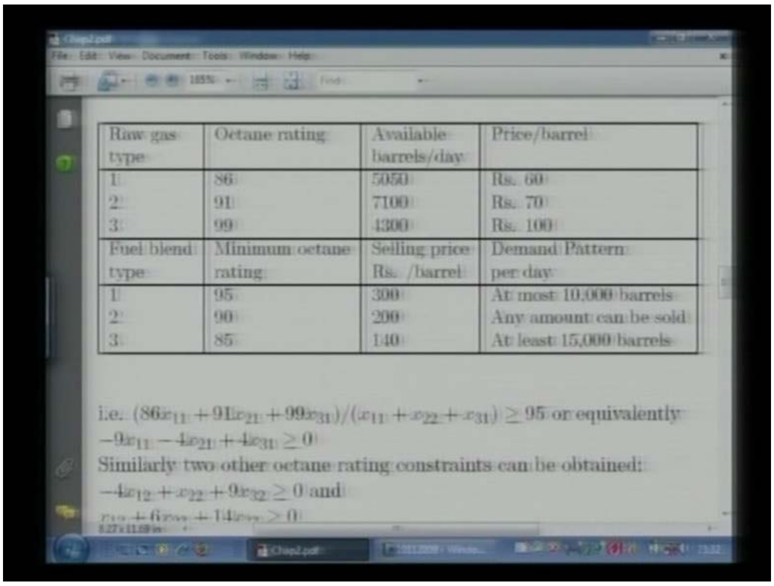

Raw gas type	Octane rating	Available barrels/day	Price/barrel
1	86	5050	Rs. 60
2	91	7100	Rs. 70
3	99	4300	Rs. 100
Fuel blend type	Minimum octane rating	Selling price Rs. /barrel	Demand Pattern per day
1	95	300	At most 10,000 barrels
2	90	200	Any amount can be sold
3	85	140	At least 15,000 barrels

i.e. $(86x_{11}+91x_{21}+99x_{31})/(x_{11}+x_{22}+x_{31}) \geq 95$ or equivalently $-9x_{11}-4x_{21}+4x_{31} \geq 0$

Similarly two other octane rating constraints can be obtained:

$-4x_{12}+x_{22}+9x_{32} \geq 0$ and

$x_{13}+6x_{23}+14x_{33} \geq 0$

So, I will first give show you the data to…; so, now, here if you look at this table, so raw gas type we show under 1, 2, 3, then they have different octane ratings; so, like for example, raw gas of type 1 has octane rating of 86, and the number of barrels available per day is 5050, and the price for barrel is rupees 60; then a raw gas of type 2 has an octane rating of 91, and the number of barrels available is 7100, and cost is rupees 70; so, similarly, for the raw gas of type 3, then the fuel blend that means when you blend these three types and then come up with the three different kinds of fuels.

So, the first fuel, for example, has a minimum octane rating of 95, and this will be the selling price will be 300 rupees for barrel, and the demand for this fuel is at most 10000 barrels; so, you know, we have to could stress on the word at most, that means, you cannot sell more than 10000 barrels in the market, so the company has to keep this in mind that you cannot you should not in manufacturing more than 10000 barrels per day; then for fuel type 2, the minimum octane rating required is 90, that means, it can be more than 90, and the selling price will be rupees 200, obviously because the higher the octane rating, the higher the selling price, and for this the demand can be anything any amount can be sold.

For the fuel type 3, the octane rating the minimum octane rating is 85, and the selling price will be rupees 140 for barrel, and at least 15000 barrels are required, so the demand is at least 15000, but it can be more; so, let us try to formulate this problem and let see the look at the constrains.

For example, we when we blend three raw gasoline, so my variables are…, see if I for making the first type of fuel I will use all the three raw gasoline, and so the my variables are x 1 1, x 1 1 says that is the first type of raw gas and using for making the first fuel, then x21 also says that, it is the second raw gas that I am using for making the first fuel, so for making my first fuel type of gas I will be using the variables x 1 1, x 2 1, x 3 1. So, therefore, the then I am saying that the rating for this blend would be 86 x 1 1 plus 91 x 2 1 and 99 x 3 1 divided by the total number of barrels, that will give me the octane rating for per barrel, and this should be greater than or equal to 95.

We have to again see this thing, because when we specified the octane ratings we said the minimum octane rating is 95, it can be more than 95; therefore, the constraint would be greater than or equal to 95 right; and then when you rearrange the variables, and bring the more to the right inside, the final constraint is minus 9 x 1 1 minus 4 x 2 1 plus 4 x 3 1 greater than or equal to 0.

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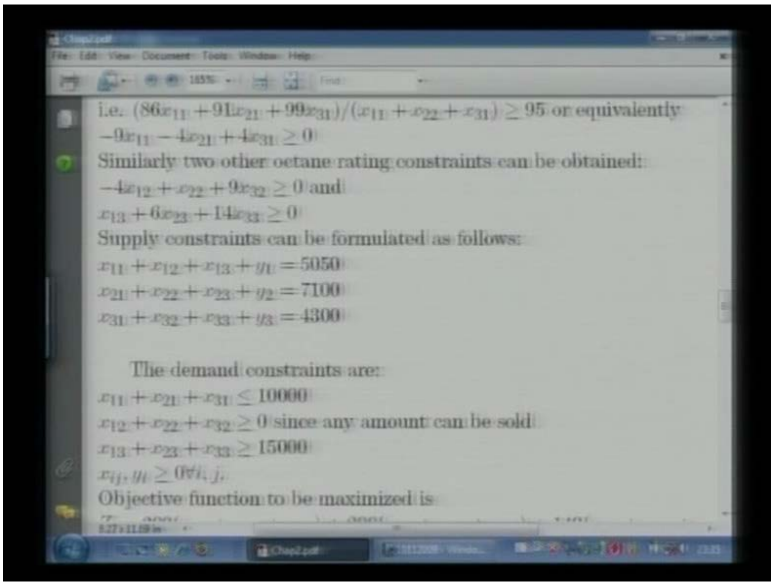

So, this is of course, constraint which specifies that the fuel blend of type 1 must have minimum octane rating of 95. Similarly, if you see that the other two constraints, which for the other two fuels the octane ratings, so the other two constraints to the same way we can derive and we get the other two constraints; so, this these three constraints now say that whatever blend I make has to be has to satisfy this octane ratings, when we also had constraints on the amount of barrels which are available.

So, this apply constraints..., here for example, x 1 1and x 1 2 and x 1 3 tell you the number of barrels that I used for the first raw gasoline to make up all the three blends; so, the total consumption of first raw gasoline you can say is x 1 1 plus x 2 1 plus x 1 3 and y 1 is the left over, so the total thing should not be more than 5050, whatever we have available I can only use up that many barrels; similarly, for the other two types of raw gasoline, whatever barrels I have available, I can use them up, and so that gives me the supply constraints.

Now, comes the demand constraints; so, the total amount that I have taken the total number of barrels that I manufacture for the first type of fuel add up to x 1 1 plus x 2 1 plus x 3 1, and this should not exceed 10000, so that tell us that the demand for first fuel type cannot exceed should not exceed 10000, therefore we cannot manufacture more than10000.

Similarly, for this second demand, for the second fuel it is no amount is specified, that means, the demand can be anything, so we will simply say that x 1 2 plus x 2 2 plus x 3 2 should not should be just non-negative, because whatever amount we manufacture that can be sold right; and now, this comes the third constraint which says that the fuel of type three they must be at least 15000 barrels that should be manufacture; so, therefore, x 1 3 plus x 2 3 plus x 3 3 should be greater than or equal to 15000.

So, through this example I want to show you that, how various situations can be easily modeled through this linear optimization model as we call them, any optimization problems where we can apply and so now before I say anything more I would like to come and so.

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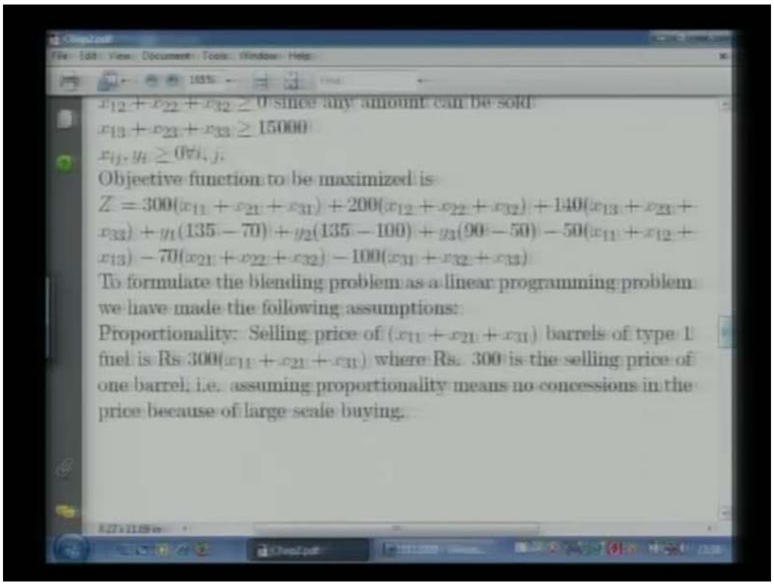

Now, let me just spend a little bit of time on the how we formulate the objective function. So, the objective function remember x 1 1 and x 2 1 and x 3 1 are the number of barrels, which were manufactured for the first fuel for the first type, so I will restate an x 1 1 plus x 2 1 plus x 3 1, total is the number of barrels that I manufacture for type 1, so and the selling price is 300 rupees.

Therefore, I am saying that my cost function which would I would like to maximize that is the company, would like to maximize the profits that it earns, so then the selling through selling price it for the type 1 fuel, it earns 300 into the number of barrels that it manufacture for fuel type 1, then simulate 200 times the number of barrels for fuel type 2 and 140 times the number of barrels for fuel type 3.

Then remember for raw gasoline which is more than 91 octane, the it was a we the selling price for that is 135, and since the cost for raw gasoline of type 1 was 70 rupees, so the total earning from selling y 1 barrels of raw gasoline of type 1 is 135 minus 70; similarly, I have the terms containing y 2 and y 3; and then finally, the total…, for example, raw gasoline of type 1, this should be 60, we can make the correction here, but anyway by mistake this is type as 56 should be 60 times x 1 1 plus x 1 2 plus x 1 3, total number of barrels that I used for manufacturing that I used from the first raw gasoline is x 1 1 plus x 1 2 plus x 1 3.

So, similarly, minus 70 times; so, the cost of the barrels that I use up I subtract that, and the first portion or the first few terms give you the profit or the earnings from selling the fuels that company manufacture, and the second last three terms give you the cost price or the money that you spent in buying the raw gasoline.

So, the difference gives you the final earnings of the company; so, this is your objective function. This we would define as a linear optimization model or a broad term is linear programming problem; now, here if you look at this the axiom that we have used the first one is proportionality; 300 rupees is the price of 1 barrel for selling or the type fuel 1, but when I manufacture x 1 1 plus x 2 1 plus x 3 1 number of barrels for fuel type 1, I multiplied by 300, that means, no concessions for large scale buying; it happens that, when you go for a large scale buying, then you expect and the trader also gives you concessional rates, but here we are assuming proportionality, that means, whatever the cost for 1 unit, no matter how many units you sell you earn the same income, so it is 300 times.

So, this is what is known as proportionality. So, wherever we find a situation where you can sort of approximated by proportionality, we would say that we can formulate this problem as a linear programming problem. Then the other one is additivity; axiom that I am using here is then we blend different raw gasoline, it is possible that some chemical reaction may takes place, and so the volume may go down, that means, it is possible; but here we are assuming that no additivity is valid in the sense that, if I use 10 barrels of one type and 20 barrels of the second kind and we may be 50 barrels of the other, then the total number of barrels that I finally blend is just hiding up all these numbers.

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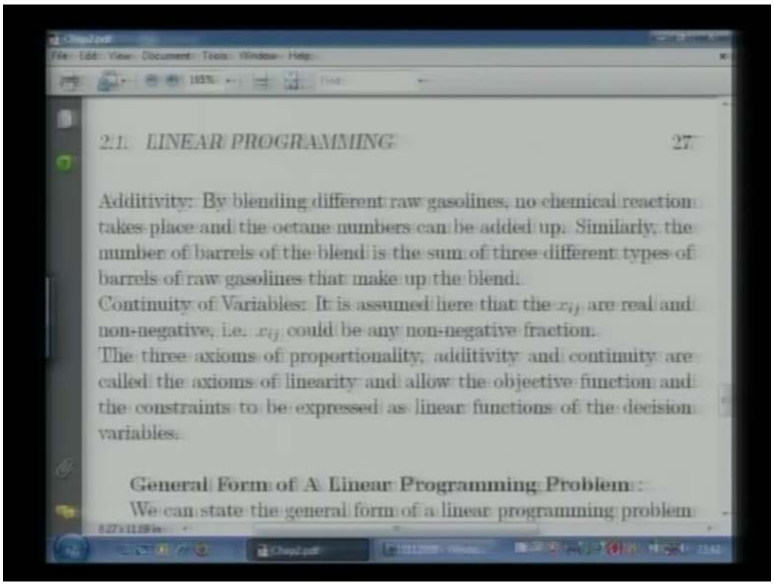

So, no loss, no chemical reaction is taking place, and no loss is taking place, so additivity is assumed to be valid here; and that is why I could formulate the constraints and the objective function using this axiom. And the third one is continuity of variables. Now, here of course, in this particular example my variables x i j give you the number of barrels of type i used for making type j fuel; in linear programming again we assume continuity of variables, that means, it x i j could be any fraction.

So, here also I can say that it can be half a barrel or two thirds of a barrel, so x i j can be a fraction. So, these are three basic axioms of linearity, which are proportionality, additivity, and continuity, and these allow us to write down the formulate the problem as a linear programming problems; and as I will want to repeated again that, in most of the situations it has been very…, it has been demonstrated very successfully that this linear modules are very helpful in solving everyday life problems and may be any more complex problem also.

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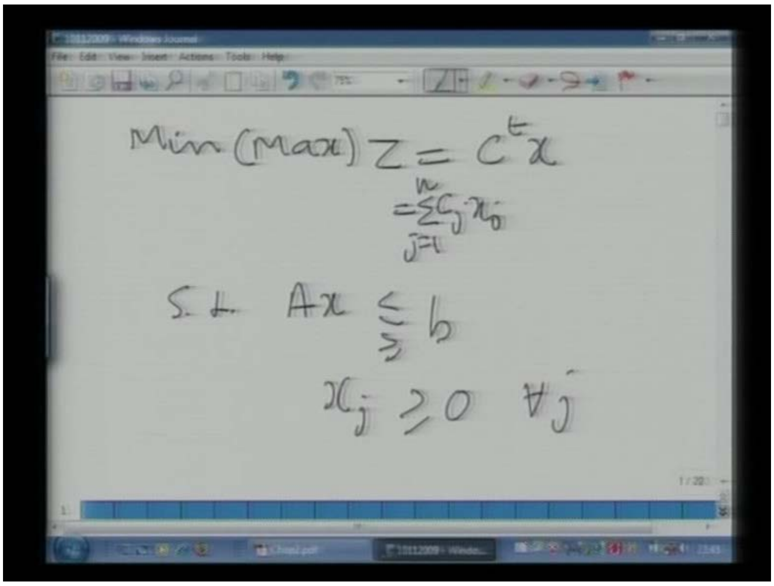

So, therefore, if you use the matrix notation I can formulate a linear programming problem as I can write minimize short for minimization; if I write min or your objective function can be a maximization problem, so this would be equal to c transpose x, so here for example, this is the short form for I am writing c j x j summation j varying from 1 to n.

If I am assuming that I have n variables n decision variables c(s) are my cost coefficients they could be profit coefficients or cost coefficients depending on whether I am minimizing an objective function or maximizing it, so this will be my objective function, and subject to in a precise form I can write the constraints as Ax less or greater this equal to b write; and then here you see I should have mentioned this also in the beginning when I introduced the decision variables x i j that either we will use a particular raw gasoline or a particular blend or I do not.

So, therefore, it is very natural that, my variables will be x j greater than or equal to 0 or all j; so, here I have used only one index, but it can be two indices for variable, also as we saw in the earlier example just to simplify the presentation I am saying that a linear programming problem can be formulated as z equal to c transpose x subject to Ax less than greater than equal to b which is my right hand side numbers and then my variables are non-negative.

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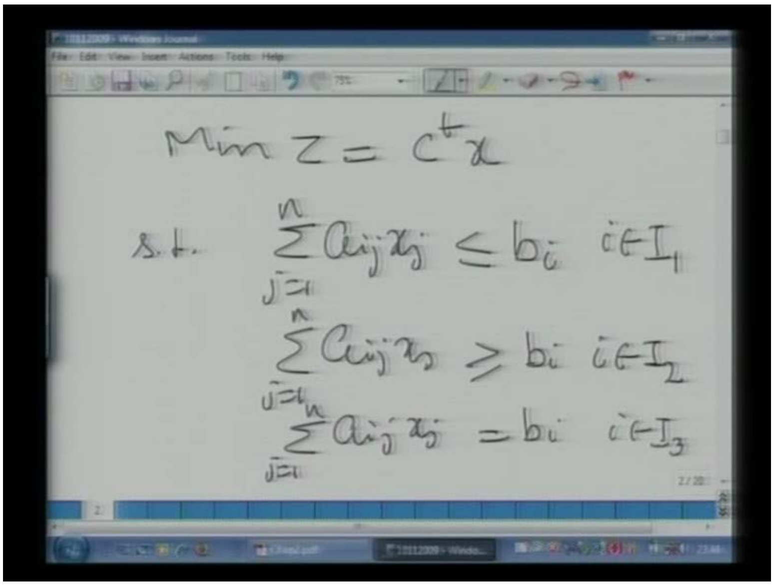

Now, in fact, one can write a more general form of here, I can write a more general form, and as we saw in this problem in the bending problem I can write it as…; and let me just take to minimization, because you see that you can always make it a maximization problem by multiplying all the cost coefficients by minus sign.

So, this is minimize, this is c transpose x subject to j varying from 1 to n which is less than or equal to b i, i belonging to I 1, this is my index set; so, it is my constraints may be less kind then the other set of constraints 1 to n, where greater than or equal to kind; and finally, I also had some constraints which were my supply constraints, which were equality kind, so j varying from 1 to n, this equal to b i, i belonging to I 3.

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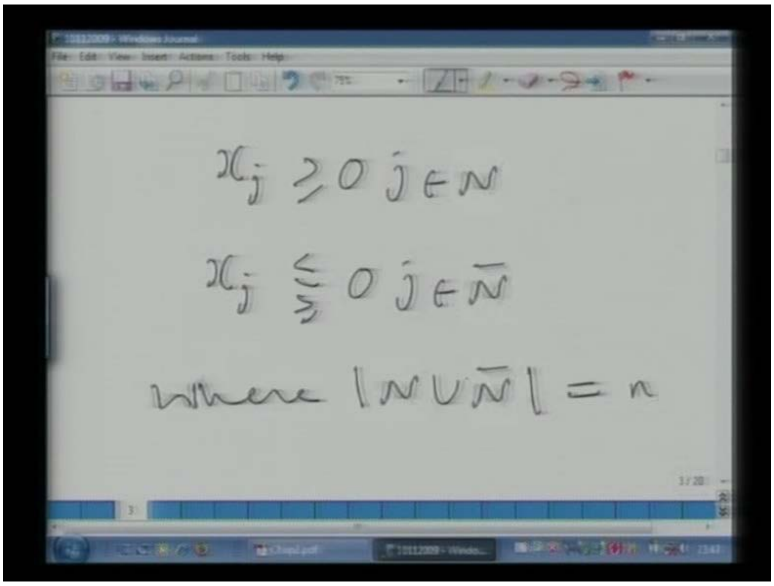

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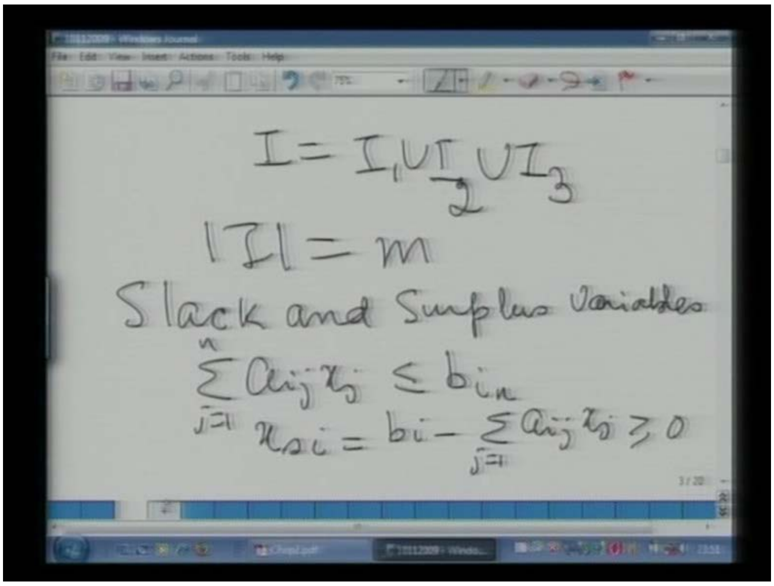

And here, then my total the index set, so that means, I which is the addition of all the index set is I 3, and the cardinality of I, that means, the number of total number of testing can be m, this common notation, and everywhere we just say that the number of constraints is m the number of variables is n. So, this will be my general statement of a linear programming problems and I will now show you how to…, maybe we can give it some more this thing I want to talk about. Let me now introduce the concept of slack and surplus variables.

The names are quite suggestive, let me explain; if you have a constraint of the kind a i j x j j varying from 1 to n less than or equal to b i; just as suppose, we had a demand there was a demand constraint and it set that the number of barrels produced for fuel type 1 should not exceed 10000. So, let us say this is by demand constraint for the fuel type 1; then here I will introduce a variable x s i which is equal to b i minus summation a i j x j j varying from 1 to n; so, this is the difference between the right hand side number and the left hand side expression.

So, this tells you that, when I actually come up with my schedule of how many barrels to use in what fuel blend, so what is my final production of each fuel type; then it is possible that, I may have manufactured more than 10000 barrels for the fuel type 1, and so x s i measures the difference between sorry I should say the other way that the left hand side tells you the number of barrels that you are manufactured for fuel type 1 and the and it should not exceed 10000.

So, the difference x s i gives you the difference between the 10000 barrels that is required for fuel type 1, and the actual number of barrels that you produce for fuel type 1, so this difference is the slack, is the slack in your first kind of demand; and so, and of course, this will be non-negative, because you are not going to produce, if you want the constraint to be satisfied, it is not going to exceed 10000 which is, and therefore so we will say that x s i non-negative.

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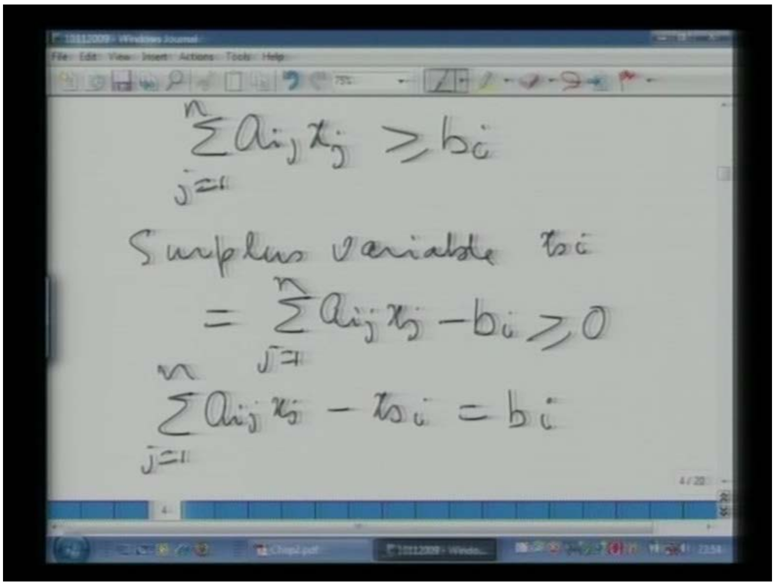

So, this is the slack in your in the constraint in the less than or equal to (( )), so the name is quite suggestive. Now, let me go to…, let me handle constraint of the type greater than (( )); so, this will be the a i j x j j varying from 1 to n which is greater than or equal to b i. Now, in our fuel type 3, it said that the number of barrels produce should be at least 15000 that mean, the demand can be more than you the supplied for that fuel type 3 can be more than 15000 but, it should not be less than 15000.

So, here if you few talk of the surplus variable; surplus variable x s i, I will write it as equal to the left hand side expression that means the total number of barrels that you actually produce and the least requirement the minimum requirement; so, this will tell me though, so the surplus variable x s i in this case measures the number of barrels that you have produced over the minimum demand right, and this also we required to be non-negative, so that the constraint will also be satisfied then.

So, now, you see that if I going this to this side, then this can be written as summation a i j x j j varying from 1 to n minus x s i is equal to b i, so by forcing if I ensure that my x s i also maintenance the non-negative sign that you see that the original constraint sigma a i j x j varying greater than or equal to b i will be satisfied, that means, if I have a set of variables x j(s) j varying from 1 to n and a corresponding z plus variable x s i, which satisfy this constraint, then you see that variables 1 to n x 1 x 2 x n will satisfy the original constraint.

So, I ensure that by keeping a sign of the slack of surplus variables, non-negative my original constraints will continued to be satisfied, so this is the idea; and of course, here what we are saying is that, there is no penalty attach to either not making…, see in the first case the demand was 10000 barrels at most, so if you produce less than that, there is no penalty; and in the third case, in the third fuel the demand is 15000 barrels at least, so we produce more than that, and whatever we produce we are able to sell it, so then there is no penalty attach to how many by producing excessive barrels of the fuel type 3.

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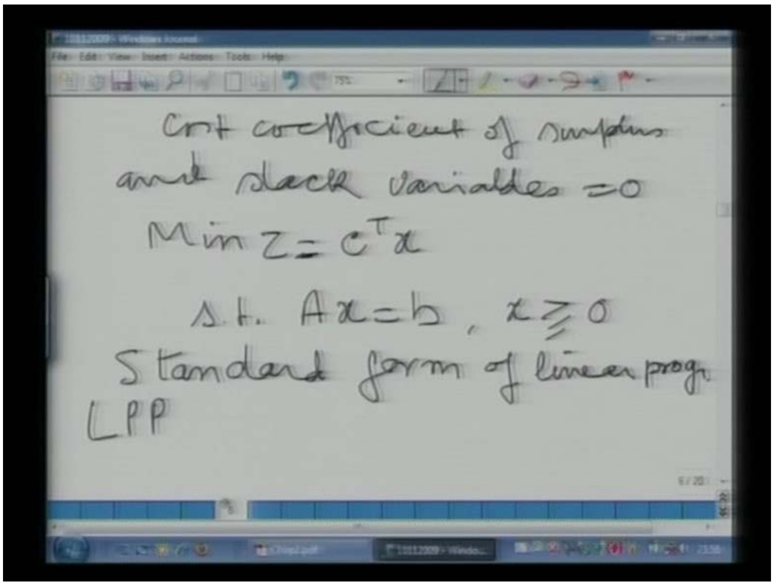

So therefore, we say that the prices…, so then we say that the cost coefficient, cost coefficient of surplus and this thing is equal to 0; so, therefore, we start it out with the general form and I am able to reduce it to…; now, the after adding surplus and slack variables all the constraints reduce to a equality form and so my linear programming problem would be minimize z equal to c transpose x subject to Ax equal to b, x greater than or equal to 0.

Now, here the variables includes slack and surplus variables also right and so this is known as the standard form standard form of linear programming problem linear program; for short, I will all sometimes we using the abbreviated form LPP to denote linear programming problems, so this is known as the standard form

And now, you see that the way I constructed, the way I converted the equality constraints to in a quality constraints to equality constraints; you saw that and by putting the cost coefficients of the surplus and slacks variables as 0, there is 1-1 correspondence.

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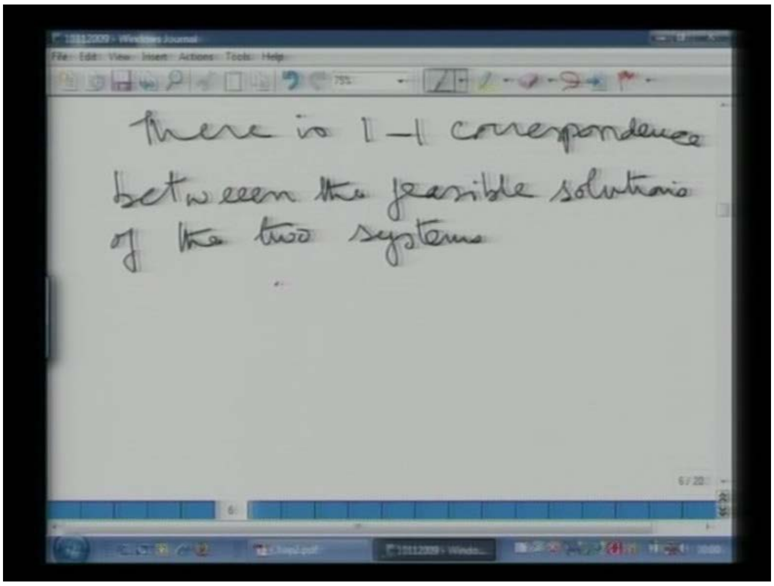

So, what I want to essentially point out is that, the original problem which I stated it as a general form of whatever, it is of a linear programming problem had less than greater than constraints, in the standard form all are equality constraints; and you see that, there is 1-1 correspondence, 1-1 correspondence between the feasible solutions of the two systems.

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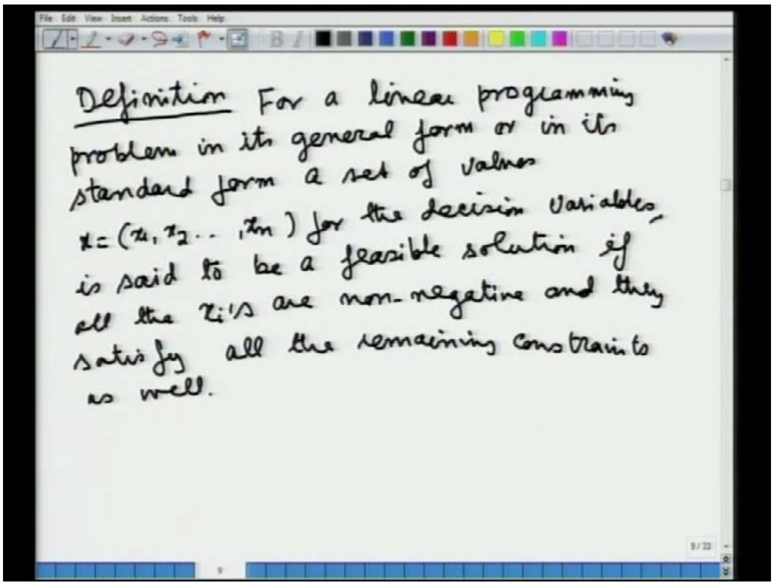

So, let me give you definition of a feasible solution for the linear programming problem that we have defined; so, we say that, if for a linear programming problem in its general form or in its standard form a set of values x equal to x 1, x 2, up to x n for the decision variables is set to be a feasible solution, if all the x i(s) are non-negative and they satisfy all the remaining constraints the linear inequalities equalities whatever the constraints are also satisfied them as well; so, all the constraints are satisfied, and the values are the non-negative, then we say that it is a feasible solution.

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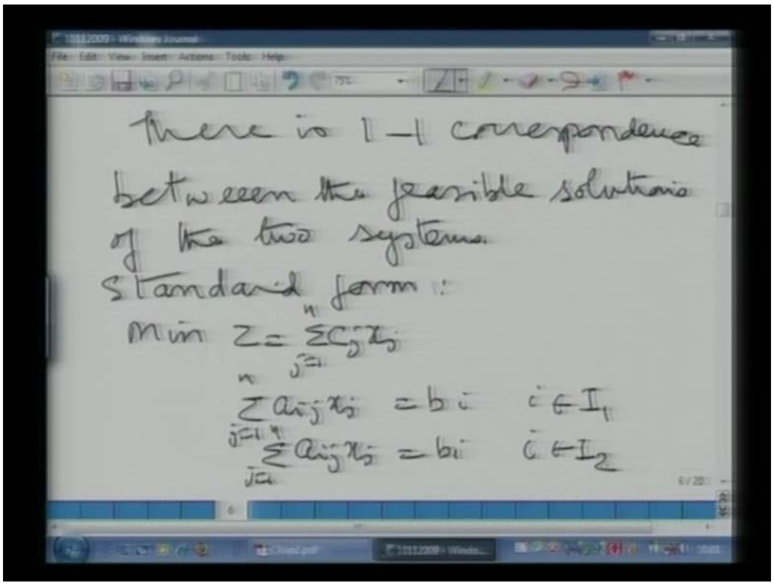

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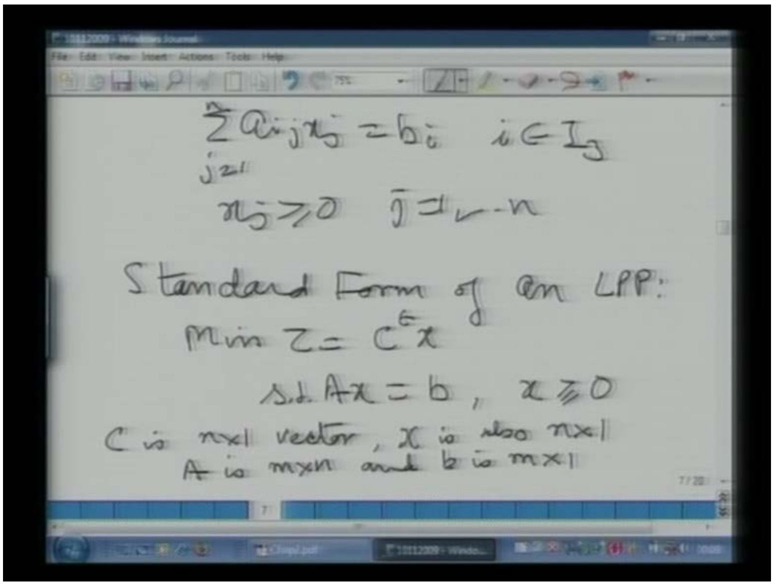

Earlier, I had stated the general form of a linear programming problem; so, this standard form I can write out in detail, the standard form will be minimize z equal to c j x j summation j varying from 1 to n, then summation a i j x j j varying from 1 to n this equal to b i, i belonging to I 1, I have been able to add slack variables here, and convert it to equality system; then the second set of constraints also reduced to equality system, this will be j from 1 to n is equal to b i, i belonging to I 2; this will already equality from i belonging to I 3, and I require all variables to be non-negative j varying from 1 to n.

So, this is my standard form; and what I want to say is that, there is 1-1 correspondence between the feasible solutions to this system and feasible solutions to the original system; if I have a feasible solution for the original system, then I can get a feasible solution for this system by substituting the values of the original variables in this constraints in the equality constraints and the differences will give me the slack and surplus variables; if I have a feasible solution for the equality system that is the standard form in equality constraints, then by dropping this slack and surplus variables the remaining variables will be the original variables and that they will constitute a feasible solution for the original system.

So, there is 1-1 correspondence, and since the objective function cost coefficients for slack and surplus variables are 0s, the two objective functions values are the same; that means, for a feasible solution for the original problem I have a corresponding feasible solution for the new system with the quality constraints and the two objective function values are the same.

So, this is now I am introducing concept of equivalence, that is we have a concept of two problems being equivalent if there is 1-1 correspondence between the feasible solutions; and for optimal solution of one problem, I can derive an optimal solution for the other; this is a very useful concept and you can see that why it is needed; right now I can as we have seen that, it is always you may have already experienced that handling linear constraints is definitely easier than handling in equality constraints.

So, by reducing to an equivalent system, by reducing my original linear programming problem to system in which all the constraints are equality form, it will be easier for me to get a solution procedure for the second for the for the system with all equality constraints, so the advantage is there; and as we go on in this course we will see that this concept of equivalence is use very often, and of course, the objective functions for the two problems in this case are the same, but sometimes it is possible that when you have two equivalent systems or two equivalent problems you can the relationship between the two objective functions may not be so direct, you may have to derive optimal solution of one from the other, but since there is 1-1 correspondence between the feasible solutions, this is not difficult, we can do it all the time right.

So therefore, by adding slack and surplus variables we have reduce the system of constraints to equality constraints; and we will now try to obtain solution procedure for the standard form of a linear programming problem; so, in matrix notation I will again now write the standard form of an LPP.

So let me say that standard form of an LPP will be minimize z equal to c transpose x subject to Ax equal to b x greater than or equal to 0, where c transpose when I say so originally c is n by 1 vector, the decision variable vector x is also n by 1 right, and your a the cost coefficient matrix is n by n and the right hand side vector, and b is m by 1, that means, the number of total number of constraints is m, the number of decision variables is n.

And we can also again revisit the 12 blending problem and say that in that case, of course, it was my maximization problem, so the c coefficients of c were your total earnings from selling the manufactured fuel, and then a the matrix a is also known as the technology matrix, because depending on the depending on the technology that you have you generate or you obtain by blending different gasoline, you octane fuel of a particular blend of a particular octane rating.

So, the coefficients in A are…, the they indicate the technology that is being used in your manufacturing process or even many different situations you can use them in some other ways, and b of course is your right hand side constants which are in our case; if they represented the number of barrels that were available, the supply this is supply amounts then some of the components of p also represented your demand maximum minimum whatever it was.

So, the therefore, in an actual the linear programming problem requires you to specify the cost coefficients; there is a given technology through which whatever you manufacturing process your using, you say that you put in some raw materials, and you come out with a final product; and of course b can be…, and one can there is no end to the number of situations one can describe, where these axioms of linearity can be considered to be valid; you see one cannot say that very exactly this axioms of linearity would be valid for this situation.

But if we find that the error that one encounters is negligible then and the benefits, that means, you may not get a very exact schedule of…; for example, in this case how many barrels of each fuel kind to produce, but one gets a fair enough idea, you know, is to how much of this much and how much of that and one can worked around is, so it can be treated as a guiding tool whatever the outcome that you get from; suppose, we manage to solve this problem is fuel blending problem.

So, it will tell the manufacture the decision variables will tell you how much of each type of gasoline raw gasoline to use to manufacture the final product; and then the manufacturer kind, of course, may be not use the numbers exactly, because the axioms of linearity that we are assuming may not be exactly valid for the system, so that goes on; but as I said that, over the last 50 years is linear optimization models have been very successful, and it has helped people to get reasonably good answers. So, now, let me start reviewing a few things before, and of course, I will try to discuss lot of pathological cases that can occur, but just let me review a few situations.

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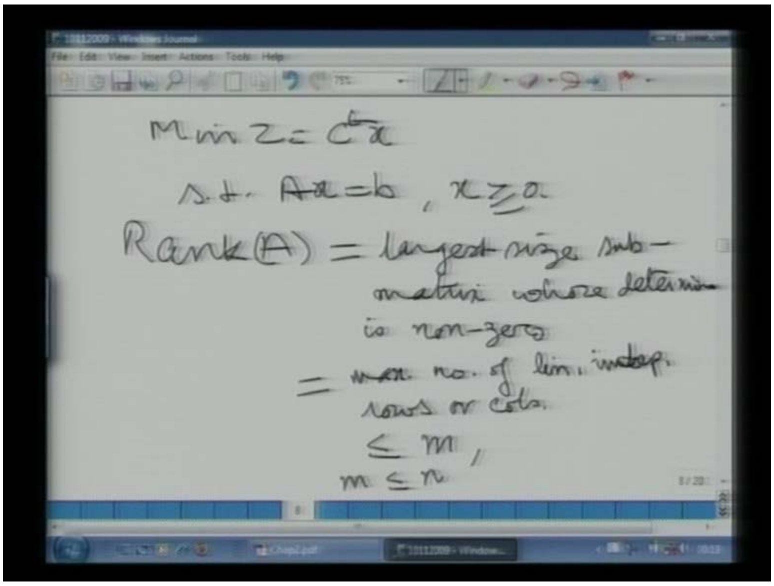

So if my problem as I again have to…, I will write it here again which is minimize z equal to c transpose x subject to Ax equal to b x greater than or equal to 0; so, if you simply look at, see I will now require, so in another way of speaking this problem is that, you have a number of combination, that means, your variables 1 to n, the variables x 1 to x n, these are your decision variables the constraints are Ax equal to b.

So you need to find out the values of these x 1 to x n which satisfy this constraints and the non-negative values only because Ax equal to b the system of equations has large number of solutions; and in fact, later on they will also actually count the number of solution that are possible.

So, this may have Ax equal to b as a large number of solution, but all solutions may not have all the components non-negative; so, I need out of the feasible solutions Ax equal to b, I need those which have all components non-negative, that means, the components can be add a positive or negative or 0 cannot be negative, so the number becomes smaller.

And then what I am ask to do is to out of all these solutions which satisfy these two sets of constraints, I have to select the one which gives me the minimum value here, so this become my optimization problem; because it is possible that for the manufacturer of the gasoline there may be many different combinations in which the manufacturer can blend to the three raw gasoline available to him to manufacture the three types of fuels.

So, they these constraints in the demand and the supply constraints all satisfied, but out of all these possible combinations they will be may be one or two which gives you the best profit; so, as I am saying that I am referring to the problem, whereas minimization problem, but if I multiply this by minus sign, it becomes a maximization problem; so, in general, we just write to state a problem is a minimization problem, but there is no loss of generality here, it can be a maximization problem which can be stated also as a minimization problem.

So, the idea here is that, out of all the feasible solutions we want to find one that minimizes or maximizes my objective function; somebody may say that what a big deal is it; you have a finite number, for example, if we now try to recall, let me, just talk to you as in your school level, you have looked at…, you know, the rule for finding out, for example, if you want to find a solution to Ax equal to b; before we talk of finding a solution, let me first say that what is the condition under which, so I need to in order to we able to say that the system itself has a feasible solution; before we want to say that we want to find out the best solution out of all possible solutions here, first I have to know that whether this system itself has a feasible solution or not, and so the use need the concept of rank.

So, rank of a matrix A, the many ways of defining the rank, this you can say is the whatever may be if you have the definition that you may have learnt is the size of the largest size, largest size sub matrix whose determinant is non-zero, may be this is not very clear, so determinant and a and t largest size sub matrix whose determinant is non-zero; this can also be defined as maximum number of linearly independent rows or columns; so, if some of you are familiar with the concept of linear independence, then you know that the maximum number of rows or maximum number of columns which are linearly independent in the matrix A, that also constitutes the rank of a matrix, and of course, the two definitions are equivalent, I am using the word equivalent here again.

So, you can say that the corresponding sub matrix with the larger…; so, if this is the maximum number whatever, it is then you can find corresponding sub determinant of that size whose sub matrix of a of that size whose determinant is non-zero; so, once if you know that the rank A is whatever it is here and we also know that this has to be less than or equal to m, because if it is a maximum number of linearly independent rows or columns or the larger size some matrix, then this cannot…; here I am assuming, see again the size should have pointed out earlier and that is that your m is always considered to the less than n, that means, we say that the systems we handle here under linear optimization modules or under determine systems, that is the number of constraints is always less than or equal to number of variables.

Because if you have large number of constraints, then it is possible that you have restricted the system so much that you are not able to find a feasible solution; so, by practical experience and also it has been notice that in a situation practical situations we are trying to formulate optimization problem; the number of constraints always is less than or equal to the number of variables; and, so under determine system…, so here if I say that the rank is less than or equal to m, I am assuming that m is less than or equal to m.

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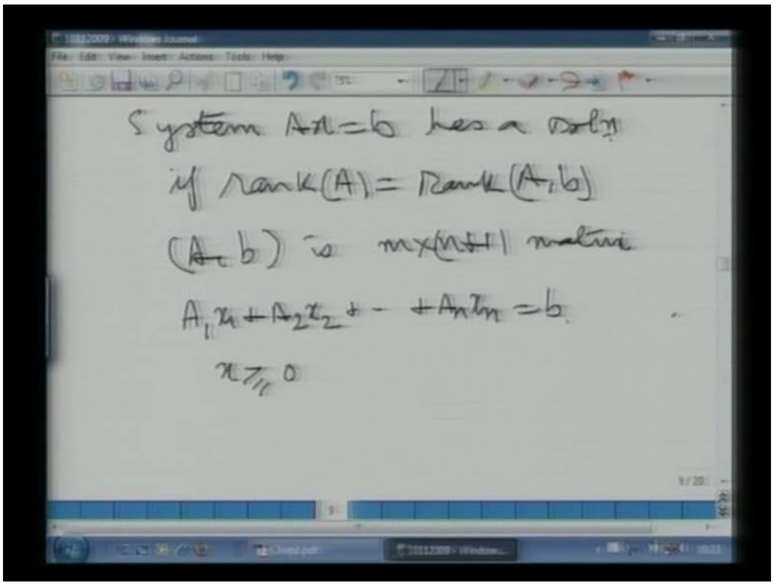

So, then this gives you the rank; and we say that the system Ax equal to b as a solution; if rank A is rank A, b, that means, if I comes, and this is called as the augmented matrix. So, here I have added an extra column, that means, this new matrix is my…, that means, you're a, b is m by n plus 1 matrix.

So, this is the augmented matrix, so we say that the if the 2 ranks are equal, then this system will always have a solution and vice versa, and which again if you want to…, if I can use the language of linear independence all it says that if rank A is rank A, b then b, that means, can be written as a linear combination of the columns of A; and this is what we are saying we are saying that if this system has a solution, then you are able to write b as a combination, because you are able to write…, if I breakup my matrix A into columns and this is A 1 x 1 plus A 2 x 2 so A 1 A 2 A n are the columns right n columns, so then this x n is equal to b.

So, when I say that this system has a solution, that means, I can point x 1 x 2 x n such that this linear combination of the columns of A is equal to b; and so, obviously, since I am saying that the rank of a matrix is equal to the maximum number of linearly independent columns, therefore b cannot be independent of the columns of A, because if the system has a solution, then b should be expressible as a combination of the columns of A.

But in this case we want more; therefore, this is the condition. So, when you formulate a problem in a linear programming problem; if you are looking for…, if you want the system, if it is a feasible formulation, then this should happen that rank A is equal to rank A, b right; and then we want something more, what are we saying, we are wanting a solution for which x is also non-negative, that means, we are not simply saying that b should be expressible as a linear combination of the columns of A, we want the b should be a non-negative linear combination; that means, the scalars x 1 x 2 x n that I use for writing b as a combination of the columns A 1 A 2 A n these should also be non-negative, that will constitute a feasible solution for my linear programming problem.

So, this is what we are looking for; and as I said that here the task that lies ahead of first, that means, we have to design an algorithm to locate feasible solutions to the system Ax equal to b, which have all components non-negative; and then we have to find the best solution out of all these and this is what we will go; and therefore, you see that here it was so it was so convenient to have a quality constraints, because I have a nice characterization of feasibility here; of course, feasibility for Ax equal to b, then I have to modify that characterization or may be design the algorithm is such a way that I only locate the feasible solutions for which x is non-negative that means all the components are non-negative.

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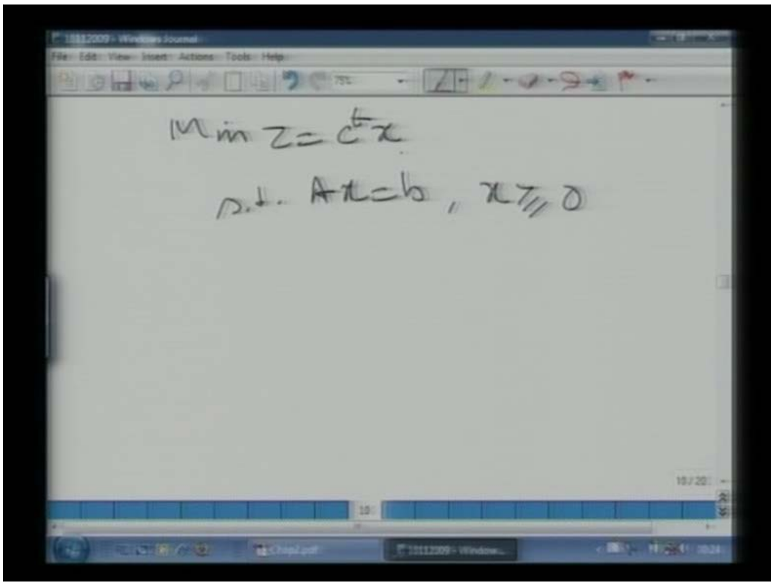

So, this is the task that lies ahead of us and what is we will let me state the problem again minimize z equal to c transpose x subject to Ax equal to b and x non-negative. So, here in case the problem does not have a feasible solution the algorithm will detect that and tell me that the problem is in feasible; and therefore, there is no question of looking for the best, because I stop there once I know that the problem is in feasible; if the problem is feasible then, of course, the attempt would be to locate the one, which gives me the best value in terms of either a max or a minimum.

I will conclude this lecture by just recalling all that we have discussed today. Essentially, I try to introduced the axioms of linearity, and I told you how by using these axioms of linearity, we can formulate certain optimization problems as linear programming problems; and then I tried to show you what task algorithm will have to accomplish in order to the able to solve a linear programming problems, I started discussing a little bit of the required mathematics behind it. And in the next lecture I will continue with the formulation of the algorithm, and show you the various features of the algorithm which will solve linear programming problems.