Calculus of Variations and Integral Equation

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Statics Calculus of Variations and Integral Equations

Hello, welcome viewers to the NPTEL lecture series on the calculus of variations. This is the ninth lecture of the series; in the last lecture, we discuss several examples as applications of the Euler's theorem and in the various example series of applications of the theorem.

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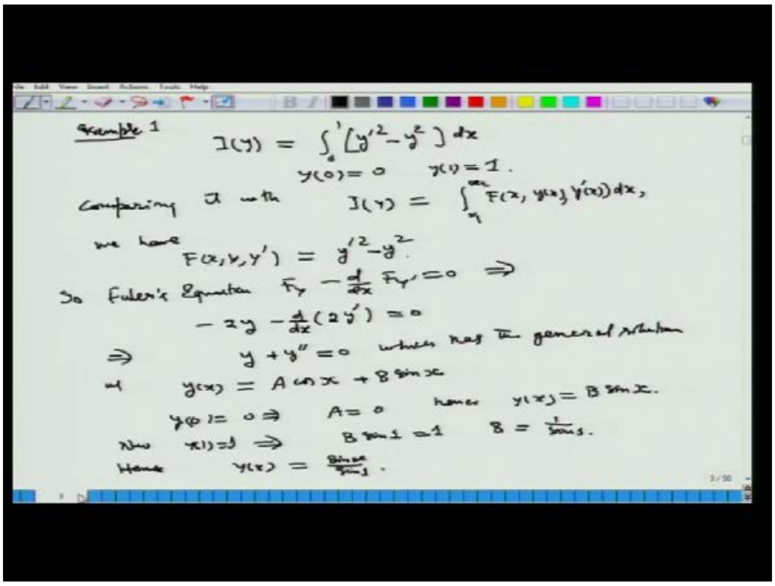

First example, we considered I(y) the functional I(y) as integral 0 to 1 y prime square minus y square d x with the adjoining conditions y(0) equal to 0 and y(1) equal to one comparing it with general form of the functional I(y) equal to x integral x 1 to x 2 F(x y y prime)dx, we have F(x) y prime square minus y square. And in this

Euler's equation implies that y plus y double prime equal to 0. This differential equation, this is second ordered differential equation and so y is to be assumed that admissible functions will have twice continuous differentiability of this y is to be assumed here and so y(x) is the solutions of this differential equation y plus y double prime equal to 0 given by A cos x plus B sin x and the given conditions then imply A equal to 0 and B equal to 1 over sin 1 and So, we get as externals y of x equal to sin x divided by sin 1.

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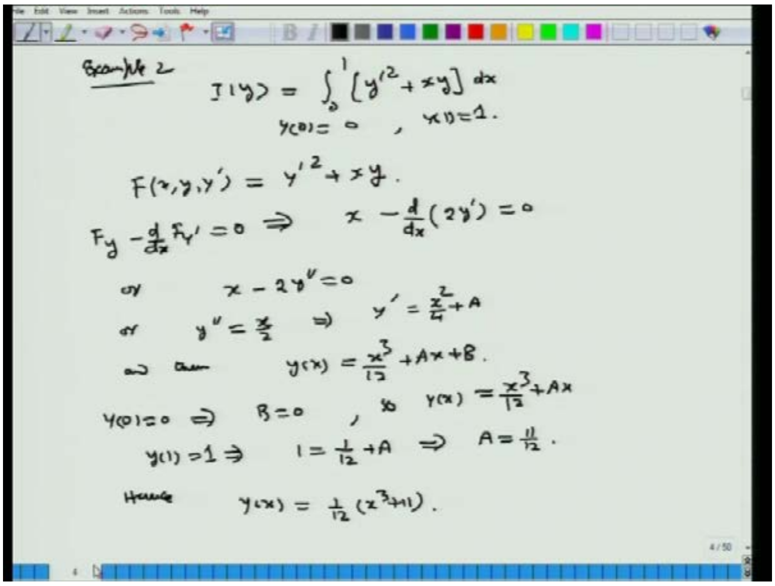

In the next example, we had considered I(y) equal to integral 0 to 1 y prime square plus xy d x. So, here f is y prime square plus xy and so, Euler's equation F of y minus d by d x F y prime equal to 0 implies that x minus 2 y double prime equal to 0.So, we get y equal to x cube by twelve plus A x plus B and these A and B constants, We will have to be determined by the given conditions. So, we get y(0) equal to 0 implies B equal to 0 and y(1)equal to 1 implies that A equal to 11 divided by 12. So, we finally get y(x) equal to 1 by 12 times x cube plus11.

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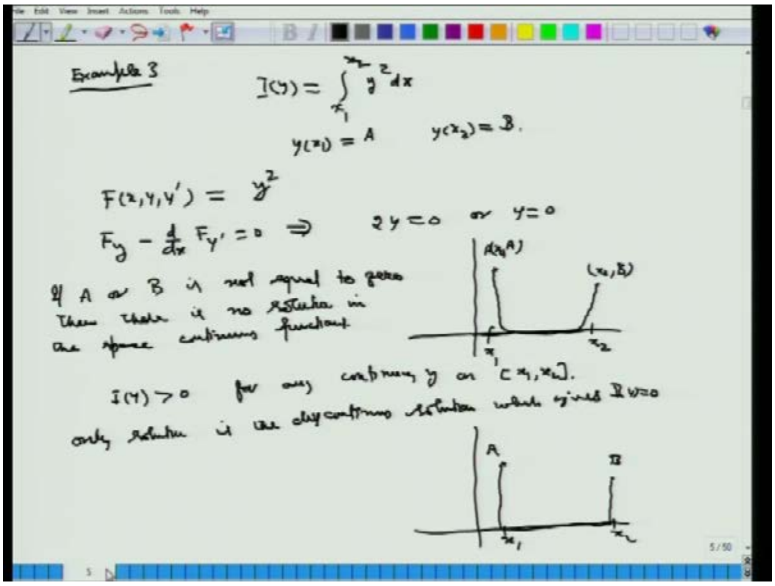

So, that is the external and in the next example we had considered I of y equal to integral x 1 to x 2 of y square into d x and here the given conditions are y of x 1 equal to A and y of x 2 equal to B. Where A and B will have to be chosen suitably and here F is y square and. So, Euler's equation F into y minus d divided by d x into F into y prime equal to 0 implies that y equal to 0.

So, if A and B are not on the x axis; that means, A and B are not equal to 0 respectively then we have this (x 1, A) and (x 2, B) not on the x axis and then there is no solution of these problem in this space of continuous functions. Only solution to this problem is discontinuous function given by this figure.

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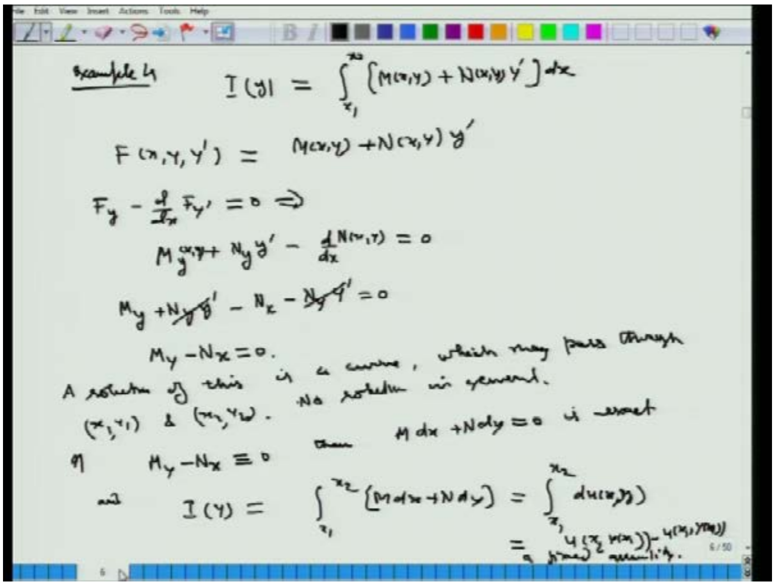

So, we do not have externals in general in this space of admissible curves. Here in the next example we considered I of y equal to integral x 1 to x 2 M plus N into y prime M and N are function of x and y .Here F is M plus N into y prime and. So, Euler's equation implies that M y minus N x equal to 0 and. So, here this will have to be solve for x y or y as a function of x and. So, the solution of this equation is a curve and in general it may not pass through this given point (x 1, y 1) one and (x 2, y 2).

So, there will not be a solution of these problem in general and if it as then there is only curve which is passing through those two point and in the second case when M y minus N x equal to 0.If it is an identity then we see that integrand can be written as M d x plus N d y and it is an exact differential and so by the property of exact differential there exit function u of x comma y such that M the total derivative of u is equal to M d x plus N d y.

And. So, here this integral finally, becomes the values of u at these end points. So, u of x 2 comma y of x 2 minus u of x 1 comma y of x 1 and so this is fixed value and so this not problem of the calculus of variation.

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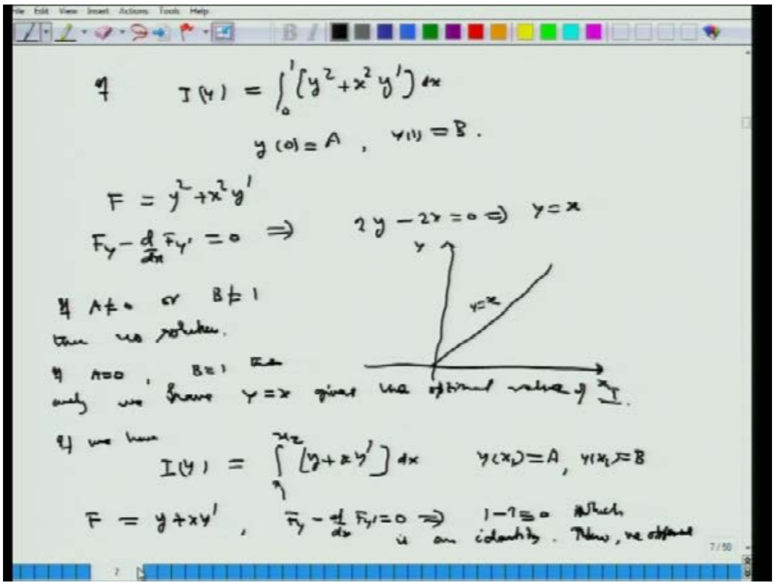

Next we consider I of y equal to integral 0 to 1 y square plus x square into y prime of d x. Here y of 0 equal to A and y of 1 equal to B then F is equal to y square plus x square into y prime. So, Euler's equation implies that y is equal to x. So, if this y of 0 equal to A and y of 1 equal to B are not on the diagonal y equal to x then there will be not any solution here. So, it is only for the case when A equal to 0 and B equal to 1 we have this y equal to x as an extrenals of this problem.

In the next example, we consider I of y equal to integral x 1 to x 2 y plus x into y prime d x and again with those same condition y of x 1 equal to A and y of x 2 equal to B.F is equal to y plus x into y prime and we see that here we get M y minus N x reduce to the identity here 1 minus 1 equal to 0.

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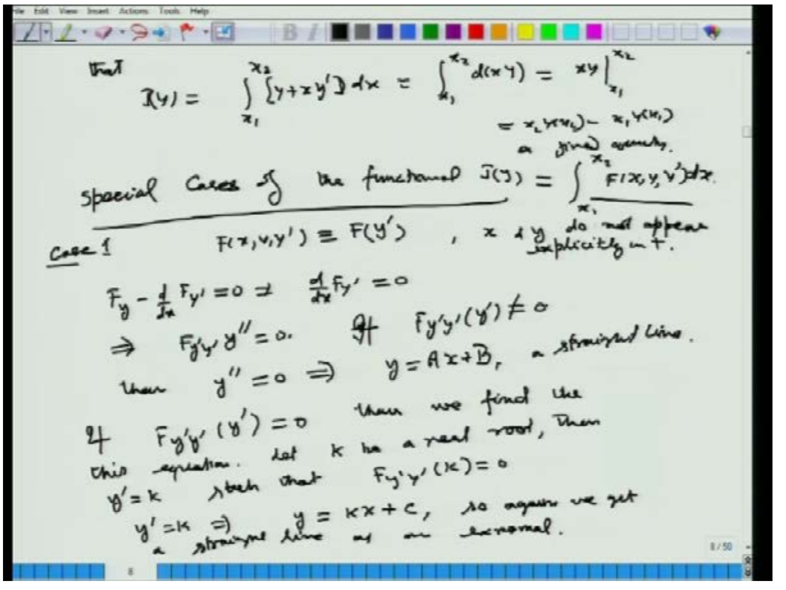

So, here we see that this is an exact differential which can be seen that it is actually d of x into y is equal to x into y evaluated at x 1 to x 2. So, that is x 2 into y of x 2 minus x 1 into y of x 1 which is a fixed quantity and so this not a problem of the calculus of variation.

So, that is where we had arrived at the… In the last lecture now we consider here special cases of this functional

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I of y is equal to integral x 1 to x 2 F x comma y comma y prime into d x.

So, when some of the variables are not present then we get some special cases. The first special case we consider here that F is function of x comma y comma y prime. So, let say this is the only function of y prime.

So, here x and y do not appear explicitly in F and in this case we have F y minus d divided by d x of F y prime equal to 0 and that d divided by d x of F y prime equal to 0.

Now, this is only function of y prime and this implies that F y prime y prime y double prime equal to 0.So, if this F y prime y prime thi

s also function of y prime if this is not equal 0 then y double prime must be equal to 0 which implies that y equal to A into x plus B.

So, we get a straight line straight line as externals and. So, here we get family of straight lines for different values of A and B we will get different lines and then these A and B are to be determine by the given condition. So, here the extremely is a straight line if this F y prime y prime is not equal to 0.If this F y prime y prime of y prime is equal to 0 then we find the roots of this equation . So, let say k be a real root then we have that is then y prime equal to k such that F y prime y prime of k equal to 0. So, for any root of this equation like this F y prime y prime of k equal to 0.So, y prime equal to k implies that y equal to k into x plus some constant C here.

So, here this C is to be determining by the given condition here and k is determine from this equation again so, we get a straight line as an external. So, in all this cases where this F y prime y prime is not equal to 0 we get an externals as straight line and if F y prime y prime is equal 0 then we find the roots of this equation and let us say if y prime is equal to k. If k is a root of this I mean y prime is equal to k and we get y is equal to k into x plus C. Again, we get a straight line as an external of this Euler's equation as a solution of this.

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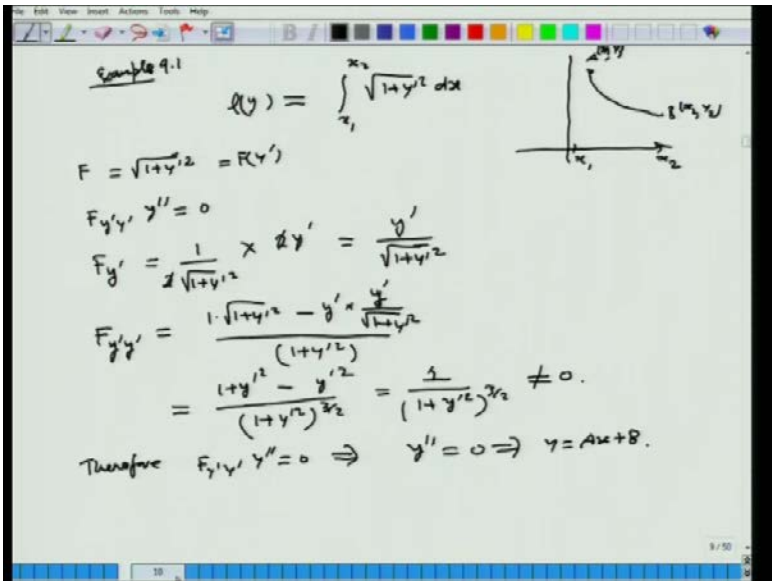

Now, here let us see various examples of this case the first one is let us say this is 9.1. So, here this l or length of y is given by x 1 to x 2 square root 1 plus y prime square d x. Here as before we have these two points A and B and, this curve joining these two points and this is x 1 this is x 2 this point is (x 1,y 1) and this is(x 2, y 2) and so this gives you the length of this curve. Now, here we have since this is a function of F here is equal to square root 1 plus y prime square and this is a function F of y prime only. So, we should have F y prime y prime y double prime is equal to 0. So, that is in the Euler's equation.

So, in this case here F y prime is equal to 1 divided by 2 into square root 1 plus y prime square and then into 2 into y prime. So, we will have y prime divided by square root 1 plus y prime square then again we differentiate partially here it of course, it is only y prime is available. So, it is ordinary derivative of that with respect to y prime.

So, we get here 1 into square root of 1 plus y prime square minus y prime into y prime divided by square root 1 plus y prime square and then whole divided by 1 plus y prime square.

You will have this as 1 plus y prime square minus y prime square whole divided by 1 plus y prime square to the power three-twos is equal to one divided by 1 plus y prime square to the power three-twos. So, this is not equal to 0 and hence therefore; this F y prime y prime y double prime equal to equal to 0 implies that y double prime equal to 0 which implies that y equal to A into x plus B. So, you get externals as straight line as expected.

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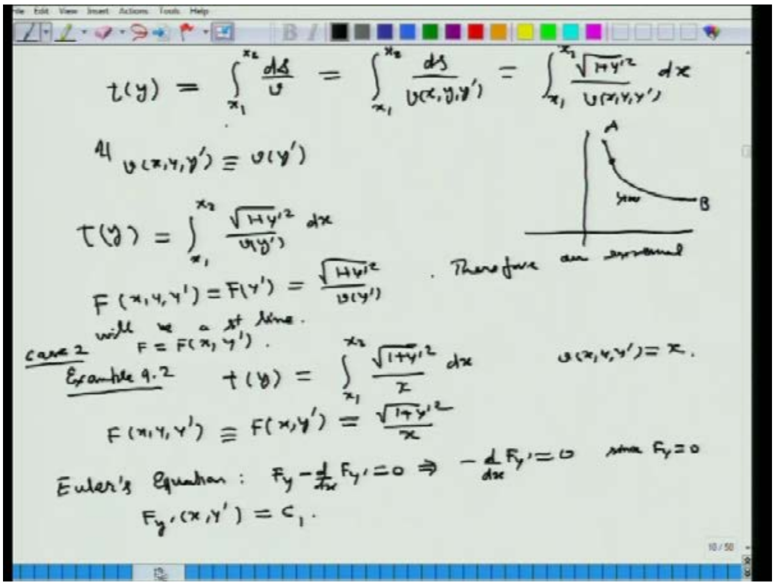

So, the another case of this we have seen that if you take t as time taken by a particle moving along y then we know that this is given by x 1 to x 2 d s divided by v where this v is the velocity of the particle along the moving along the curve. So, in general this let me write the general dependence. So, here usually this v would be a function of (x, y, y prime).And so this would be like x 1 to x 2 square root 1 plus y prime square divided by v x comma y comma y prime into d x.

So, that is the general form of the functional which gives you the time along a given curve here. So, you have these 2 points A and B and particle is moving along this curve y is equal to x and here v is the velocity which is a function of these three variables in general.

So, if v of x comma y comma y prime is function of y prime only like in the case where it is moving in a plane where then gravitational force is not playing any role then its velocity will be just v of y prime only. So, that can be taken as the function of y prime. So, in this particular case we have t of y is equal to x 1 to x 2 square root 1 plus y prime square divided by v of y prime. So, here again this F is just function of prime x y prime is a function of y prime only which is nothing, but square root 1 plus y prime square divided by v of y prime.

So, therefore; the externals will be a straight line. So, let us call it 9.2. So, in this case if we consider this t of y as x 1 to x 2 square root 1 plus y prime square and then velocity as a function of x that is here x itself for particular case. So, v of x comma y comma y prime is equal to x here.

So, in this case here F is function of x comma y comma y prime only which is equal to square root 1 plus y prime square divided by x. So, let us say this is case two where F is function of x comma y prime only.

So, this is an example of this case. So, here how do we do this? So, here and Euler’s equation is which is F y minus d divided by d x into F y prime equal to 0 implies that minus d divided by d x into F y prime as before because this since F y is equal to 0. And so we get this F y prime equal to which is function of x and y prime as the first integral. So, this Euler’s equation gets integrated easily here we get F y prime is the function of x comma y prime equal to c 1 and then once more we integrated it to get the externals.

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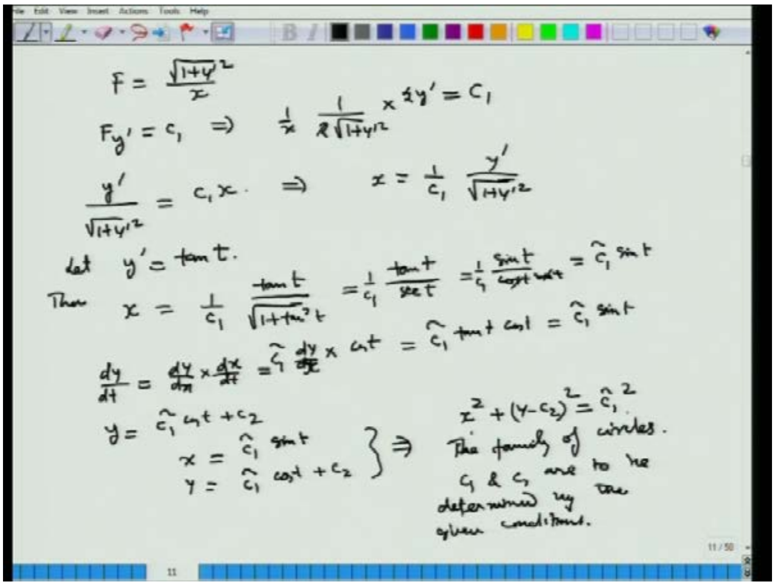

So, let us apply this case two this example. So, here we have F equal to square root 1 plus y prime square divided by x here. So, F y prime equal to c 1 implies that 1 divided by x into one divided by two root one plus y prime square into 2 into y is equal to c 1.

So, we get 2 cancels. So, y prime over square root 1 plus y prime square equal to c 1 into x. So, here we will try to get parametric form of the equation. So, this implies that x equal to 1 divided by c 1 into y prime divided by square root 1 plus y prime square.

So, we substitute y prime equals to something and then we will try to get the solution here. So, if we take y prime equal to tan t where t is a parameter here then we see that x equal to x equal to one divided by c 1 into tan t divided by square root 1 plus tan square t. So, that is equal to tan t divided by sec t. So, this is equal to sin t divided by cos t into sec t. So, this cancels. So, you get 1 divided by c 1 here. So, c 1 tilde sin t. So, that will be the x solution here and. So, now, to get y solution here we know that this d y divided by d t equal to d y divided by d t whole divided by d x divided by d t.

So, that this is tan t and here this d y divided by d t will be then... So, d x by d t x is this. So, that will be c 1 tilde into cos t. So, that will it be x equal to c 1 sin t and d y by. So, one. So, this will be actually. So, here the d y divided by d t is equal to d y divide by d x into d x divided by d t. Here d x divided by d t is equal to c 1 tilde cos t. And so d y divided by d x sorry which is c 1 tilde and d y divided by d x is tan t into cos t. So, that is c 1 tilde sin t and therefore; y equal to c 1 tilde cos t plus c 2. So, we get this as solution here and so we have x equal to c 1 tilde into sin t and y equal to c 1 tilde into cos t plus c 2. So, eliminating t from here gives you x square plus y minus c 2 squares equal to c 1 tilde square. So, we get circles as the externals family of circles.2 parameter families of circles here and then c 1 and c 2 are to be determined by the given conditions.

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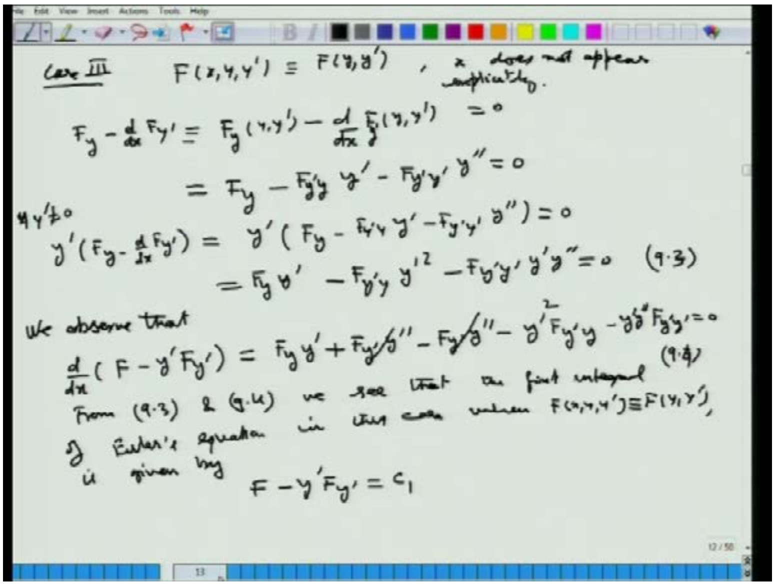

Now the next case where we get this F which is in general x comma y comma y prime which is equal to function of y comma y prime only. x variable is missing here x does not appear explicitly.

So, here F y minus d divided by d x into F y prime will be equal to in this case F y minus d divided by d x of this d divided by d x which is a function of y comma y prime and so this is equal to F y minus F y y prime minus F y this is F y. So, let me rewrite it again. So, this F y which is also a function of y comma y prime minus d divided by d x into F comma y prime and so this is equal to F y we will not write the dependence over y and y prime here explicitly and. So, this is d divided by d x of pressing… Which is opening this here so, we get F y and then since this is a total derivative so, here variables will have to be then differentiated with respect x. So, F y partial derivative of with respect y and then y differentiated with respect x. So, we get y prime minus here this is F y prime.

So, y prime y and then you have y prime minus F y prime y prime y double prime. So, if we multiply this thing by y prime since this is actually equal to 0 this is equal to 0. So, if y prime is not equal to 0 then we see that y prime this here will be then y prime of F y minus F y prime y into y prime minus F y prime y prime y double prime this is equal to 0 So, padding in the expanded form. So, you have y prime into F y minus F y prime y into y prime square minus F y prime y prime into y prime y double prime equal to 0.

So, let us write this also in the same order F y y prime. So, multiplying by y prime we get this. Now we observe that this d divided by d x of F minus y prime into F y prime is equal to since F is a function of y y prime. So, F y y prime plus F y prime y double prime minus F y prime double prime and here the second term minus first so, we get F y prime and y double prime. So, here we have differentiated y prime and this taken as factor here out and then minus y prime and derivative of F y prime with respect to x totally.

So, here that will be F y prime y and y prime. So, we get y prime square here minus y prime and then F y prime y prime and y double prime. So, that is what we will write it here equal to 0. So, we see that these two are the same because this cancels here and this is same thing as let say this is 9.3 and this is 9.4 other 9.2,9.3. So, from 9.3 and 9.4 we see that this first integral the first integral of Euler's equation in this case when F of(x, y prime) is actually F of y comma y prime. Function of y and y prime only is given by F minus y prime F y prime equal to c 1.

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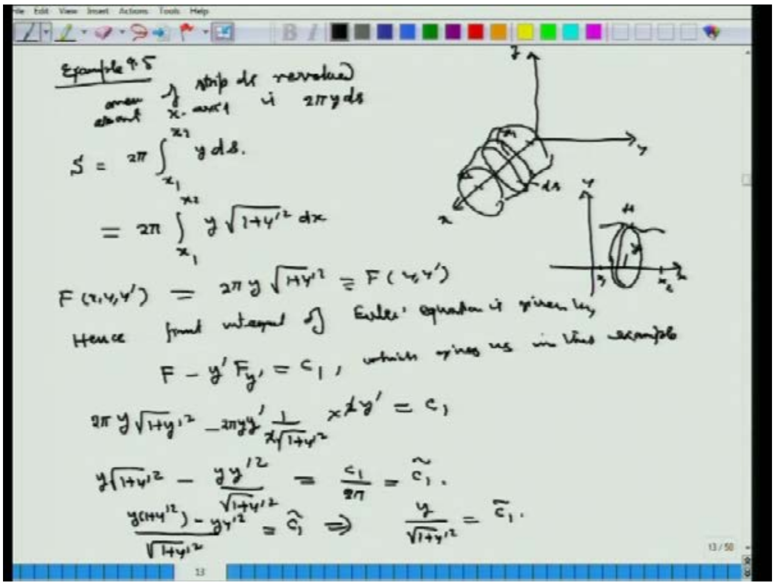

So, first integral is readily available here and then this can be integrated once more to get the externals. So, let us see this is this case in certain examples. So, here say example 9.5.So, here we consider the case where this x axis y axis and z axis and let say there is a curve given here this is from x 1 to x 2 and then this curve is rotated about x axis. So, we get like this surface generated like this. Now here if we consider d s element here and then we see that this d s is strip is rotated like this.

So, area of this d s step will be given by 2 into pi into y into d s where y is the vertical length here. So, area of the strip d s revolve about x axis is 2 into pi into y into d s. So, total surface area will be given by 2 into pi into integral x 1 to x 2 y into d s.

So, this can be seen like you have x into y. So, this curve was like this is d s element here and this is y distance here. So, this gets rotate like this d s element. So, therefore, this is 2 into pi into x 1 to x 2 this is here x 1and this is x 2 y d s is given by y into square root of 1 plus y prime square d x.

So, that it is the surface area of the object obtained by revolving this curve about the x axis. So, here F of x comma y comma y prime in general is equal to 2 into pi into y into square root of 1 plus y prime square equal to function F of y comma y prime only. So, we will see that hence the first integral of Euler's equation is given by F minus y prime into F of y prime equal to c 1which gives us in this example.

So, 2 into pi we will observe in c 1 itself and we will have y square root 1 plus y prime square minus y prime and then this. So, two pi F y prime means 1 divided by 2 roots 1 plus y prime square and then 2 y prime equal to c 1.

So, this 2 will cancel with this and 2 into pi taken on the other side. So, we will have y square root 1 plus y prime square minus y prime square divided by square root 1 plus y prime square equal to c 1 divided by 2 pi which is equal to c 1 tilde.

So, here 2 into pi into y should also be there which against

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into y prime. y prime this factor is coming from here. So, taking the LCM here we get this term will cancel. So, will have y into 1 plus y prime square minus y prime square equal to c 1 tilde which implies that y divided by square root 1 plus y prime square equal to c 1. So, here again we write in the form we take parametric representation.

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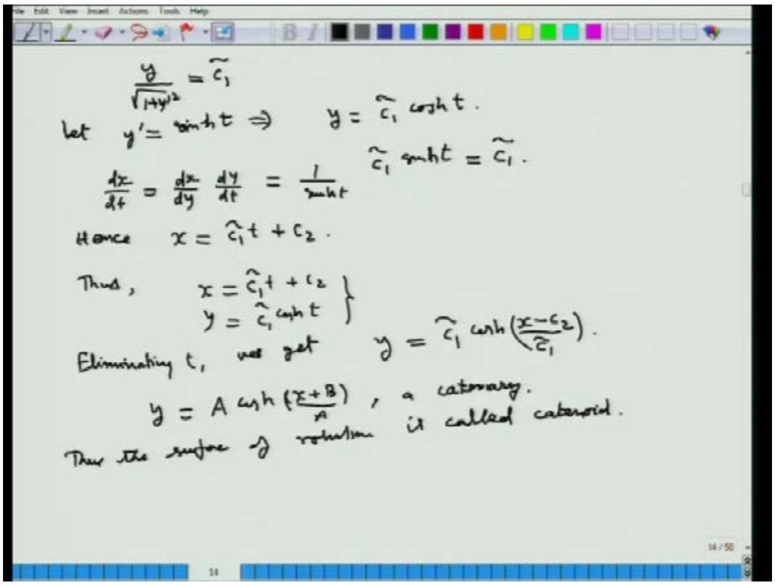

So, we have y divided by square root 1 plus y prime square equal to c 1 tilde and if we take let y prime equal to sin hyperbolic t we see that this implies that y equal to c 1 tilde you get cos hyperbolic t.

So, y comes out in this parametric form and now to get a d x divided by d t is equal to d x divided by d y into d y divided by d t like this. So, this d x divided by d t comes out to be. So, this is 1 divided by sin hyperbolic t and d y divided by d t is because y is this you get c 1 tilde sin hyperbolic t and so this you get only c 1 tilde.

So, hence x comes out to be c 1 tilde into t plus c 2 thus in the parametric form we have x equal to c 1 tilde into t plus c 2 and y equal to c 1 tilde into cos hyperbolic t.

Now this t can be eliminate here we get y as a function of x this c 1 tilde into cos hyperbolic x minus c 2 divided by c 1 tilde. So, adding in a standard form we can write that y equal to A into cos hyperbolic x plus some B divided by A like that.

.So, that is the form of solution which tells us that it is catenary. Which is and the surface thus the surface of revolution is called catenoid.

So, in this case the externals are caternary here those are the curves which will minimize the surface area now we come to the problem of brachistochrone which was introduced in the first lecture.

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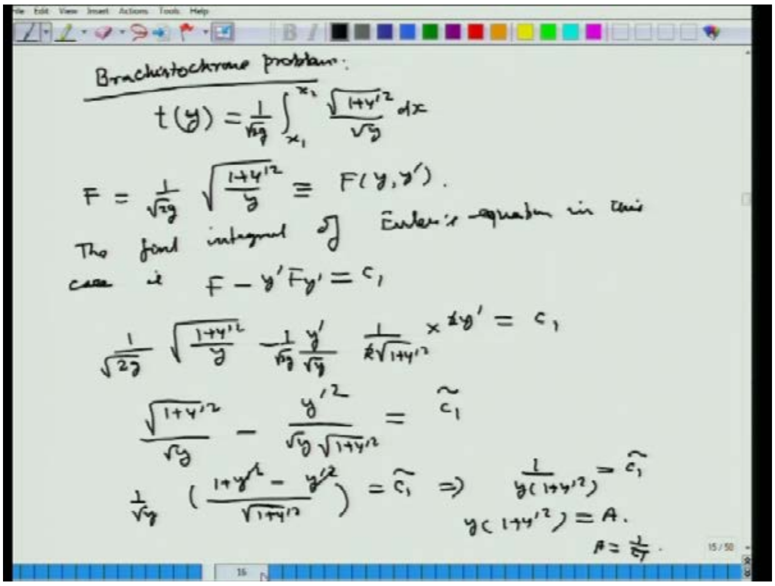

So, here recall that t of y is equal to x 1 to x 2 square root 1 plus y prime square we had got root 2 into g here and then root 2 into y into d x. So, here F is 1 divided by square root 2 into g into square root 1 plus y prime square divided by y.

So, this again a function F of y comma y prime. So, the first integral of Euler's equation in this case is F minus y prime into F y prime equal to c 1. So, here we will have 1 divided by square root 2 into g into square root 1 plus y prime square divided by y minus y prime divided by square root y we can take out here and you have 1 divided by 2 into square root 1 plus y prime square and into x square into y prime equal to c 1.1 divided by square root 2 into g here also.

So, taking this 1 divided by square root 2 into g on the other side merging it with c 1 and simplifying this we get the following in this case. So, let us try it square root of 1 plus y prime square divided by square root of y minus you get these two cancels here. y prime square divided by square root y into square root 1 plus y prime square is c 1 tilde here. And we can see that in this case you will have 1 divided by square root y is taken out and then you have 1 plus y prime square minus y prime square divided by square root 1 plus y prime square equal to c 1 tilde.

So, this cancelled here. So, we get 1 divided by squaring it we get y into 1 plus y prime square equal to c 1 tilde or inverting it we get y into 1 plus y prime square equal to let us say constant A. where A is 1 divided by c 1 tilde.

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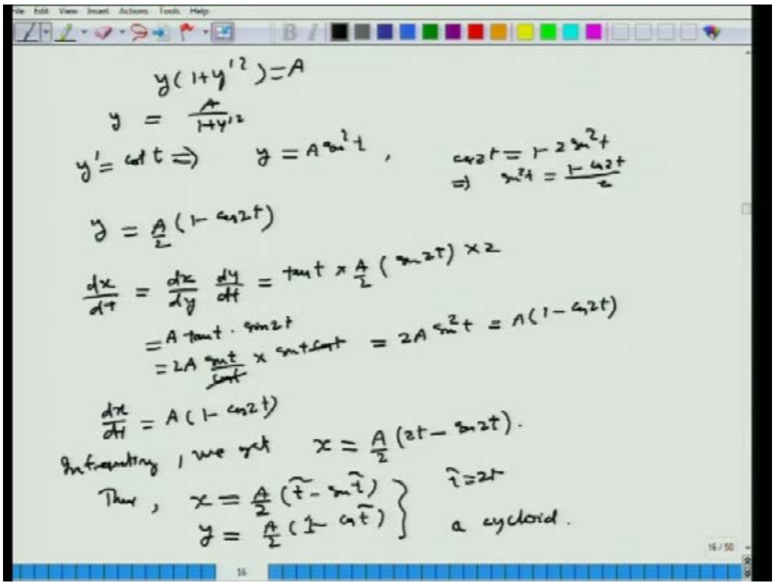

So, we get in this case y into 1 plus y prime square equal to A and so y is equal to A divided by 1 plus y prime square. So, if you take y prime equal to cot t here has a parametric representation we see that y equal to A. So, this gives you cosine x square. So, you get sin x square t here and then writing it as using the formula cos 2 into t equal to 1 minus 2 sin x square t. So, this implies that sin square t is 1 minus cos 2 into t by 2.

So, we get y as A divided by 2 into 1 minus cos 2 into t. Then we need to get x in terms of t. So, d x divided by d t is sending as d x divided by d y and then d y divided by d t here. So, d x divided by d y is inverse of that. So, tan t and d y divided by d t into A divided by 2 and then you have different session of this will give you plus sin and here you have sin 2 t into 2.So, this gives you A into tan t into sin 2 into t.

So, this is 2 into A into sin t and you will have this divided by cos t into sin t into cos t. cos t get cancelled. So,2 into A into sin x square t.So, then writing this as again A into 2 into sin x square t has 1 minus because this sin x square t is written here 2 into sin x square t as 1 minus cos 2 into t.

So, we get d x divided by d t as A times 1 minus cos 2 into t integrating this we get x equal to A divided by 2 and into 2 into t and then you get minus sin 2 into t divided by 2. And so that 2 we have taken out.

So, you get the parametric form like this thus x is equal to A divided by 2 into t we write as t tilde minus sin t tilde and were t tilde is 2 into t and y equal to A divided by 2 into 1 minus cos t tilde and we know that this is the solution of this curve is called cycloid.

So, externals are cycloid in this case which is as expected. So, mention in the first lecture. So, next we will be considering the functionals which will have more variables and more dependent variables and their higher order derivatives and then finally, will be constrained the functional which will have more independent variables and functions of those independent variables that will be constrained in the next lecture thank you very much for viewing this.