Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Module No. # 01
Lecture No. # 08

Welcome viewers to the NPTEL lecture series on the Calculus of Variations; this is the 8th lecture of the series. In the last lecture, we had got introduced to the concept of variation of a functional.

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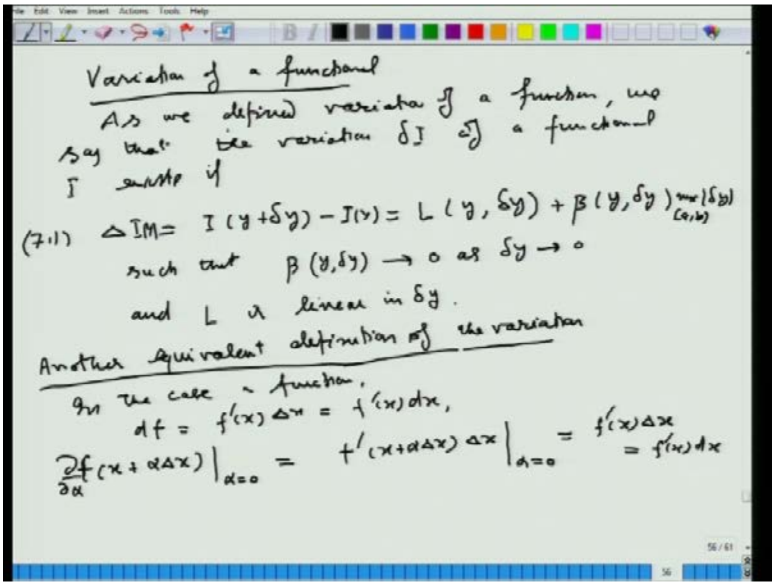

We first recalled the definition of the differential of a function, which was defined like this, we had a function f x from a to b to r and we considered it is differential are defined as d f equal to f prime x d x where for independent variable x d x equal to delta x. So, for independent variable the differential is equal to the increment itself.

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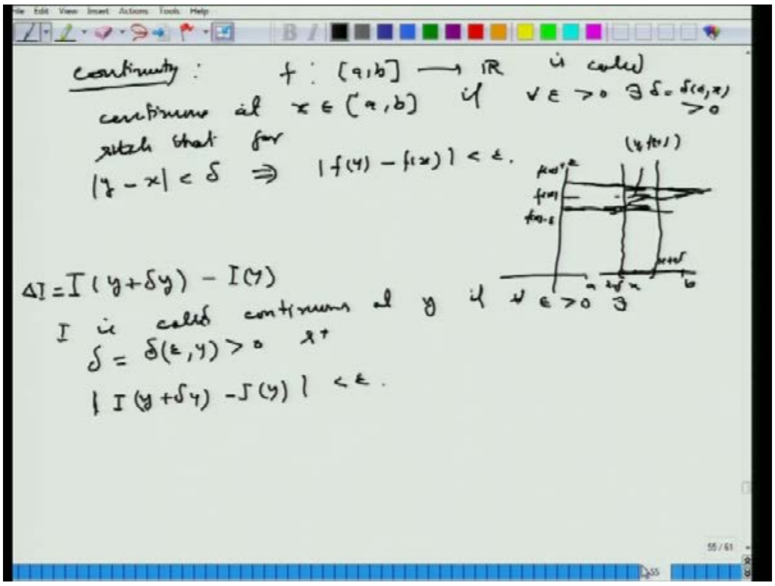

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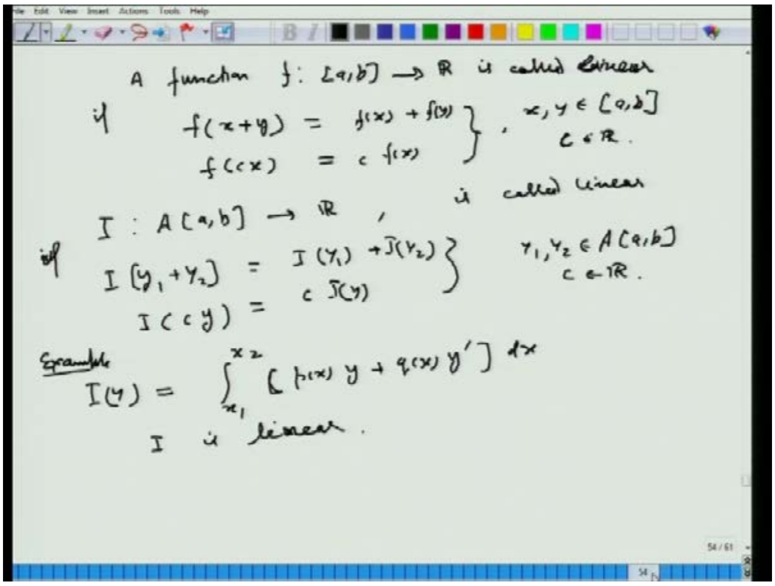

So, for a function f we saw that its increment delta f is actually equal to f x plus delta x minus f x, and that if it f is differentiable then, it can be written as as we have seen, in the case of a function we have.

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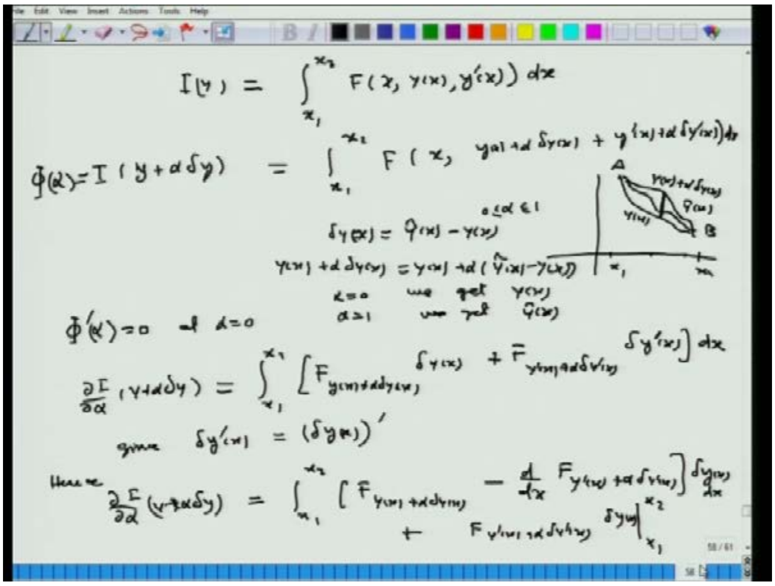

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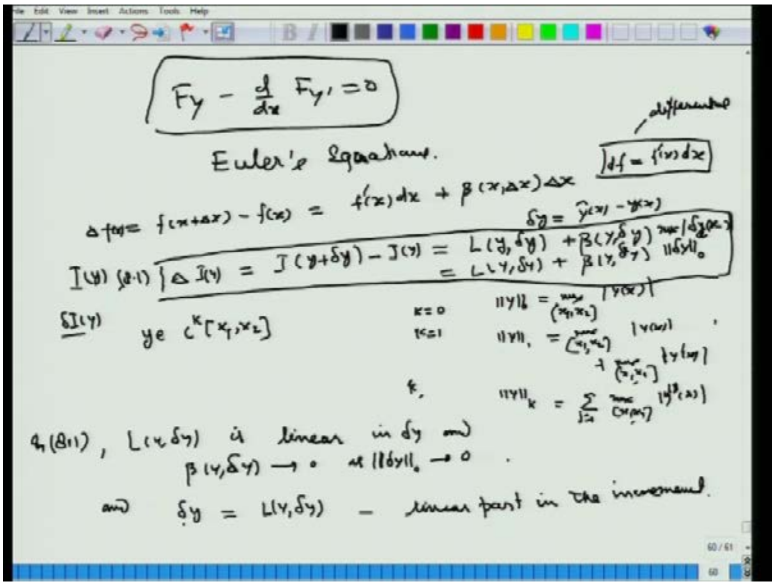

Here, delta f equal to f of x plus delta x minus f of x, and if f is differentiable, then we saw that this is actually equal to f prime x delta x or d x is same thing d x plus some beta x delta x times delta x. Similarly, for the functional I y, we saw at this the increment delta I y, which is a same thing as the difference I at y plus this delta small delta y denotes the variation in y which is, so delta y we know that delta y equal to y tilde x minus y x, this is the difference between the ordinates of 2 curves y x and y tilde x.

So, for this minus I of y and we say that, like we defined f is differentiable if and only if, the following holds then similarly, for the functional I we say that, I the variation delta I exists small delta I, small delta I exists if and only if, if this increment capital delta I which is by definition I of y plus delta y minus I Y, if this difference is equal to here some linear part in the increment, which is function of y as well as this variation delta y and plus some beta y delta y and here, the absolute value of this maximum of mod delta y, here this maximum is that, maximum norm in c k.

See for any c k, x 1 to x 2 if y belongs to this, we define this norm here, for k equal to 0 we define norm of y which is maximum of mod y x maximum over x 1, x 2 interval and if k equal to 1, we define this as norm this as 0 norm we can say and then y 1 norm, maximum over x 1 to x 2 of mod y x plus mod y prime x, maximum separately we take maximum of this plus maximum of x 1 to x 2 of y prime x (Refer Slide Time: 04:45).

So, similarly for any general k we take y k, for general k we take the c k norm as summation maximum over x 1 to x 2 of this y j norm j equal to maximum of Y j th derivative of x like this. So, here we take the zeroth norm that is this one maximum here. So, we say that this increment delta I if this difference is of the form if it can be expressed as l y delta y plus beta, delta delta comma delta y and this is the zeroth norm define this of the increment delta y this can also be written like this l y delta y plus beta y delta y and this is zeroeth here maximum of this is like this and zeroth norm like this this (Refer Slide Time: 05:47).

This maximum is over that interval x 1 to x 2, so here and this l in this linear in, so L y delta y is linear, let us say this is 8.1 in 8.1, we have L y delta y is linear in delta and this beta y delta y… sorry this is small delta, because like in independent variable here d x and delta x are same similarly, here for this independent variable y, here we will have this increment and the variation same (Refer Slide Time: 6:47).

So, this delta y this goes to 0 as this maximum delta y 0 norm goes to 0 (Refer Slide Time: 8:00). So, we say that this I the functional I, has variation this delta I y provided 8.1 holds and this linear part and this delta y is defined as l y delta y which is the linear part in the increment.

So, that is the definition, here this linear part like here this linear part is called differential, we say that this d f equal to f prime x d x, so that is what is called differential. Same way we define the variation delta y here which is the linear part in this increment given by 8.1, if 8.1 holds for the functional I, so this was equivalently then defined, so by the fact that this d f is also del f of x plus alpha delta x over del alpha at alpha equal to 0, same way this we have seen alpha equal to zero same way this we have seen that the, this delta I y, here is this is d f at x of this, and then this is defined as del I y plus alpha delta y del alpha alpha equal to 0.

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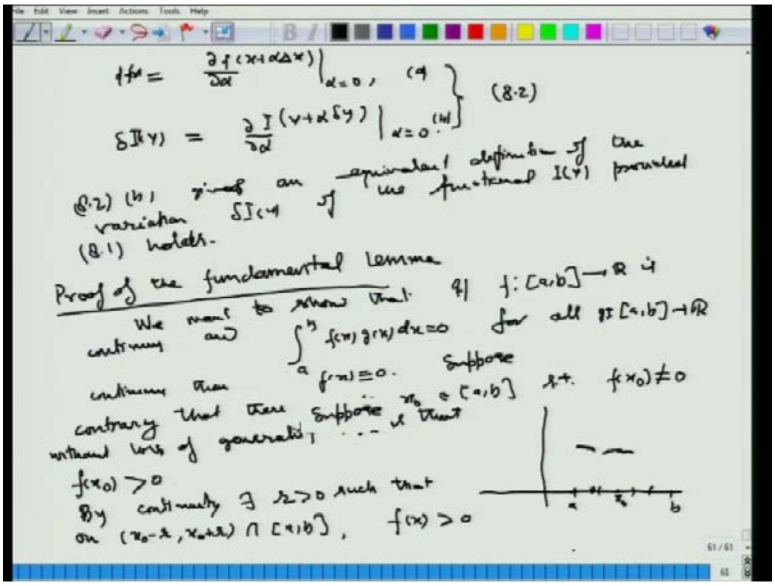

Here, this is an equivalent definition of this, this is 8.2, so here this second part of this 8.2 let say this a and this is b. So, 8.2 b gives and equivalent definition of the variation delta I y of the functional I y provided 8.1 holds.

We are saying that variation exist if and only if 8.2 hold, so under that assumption we see that the equivalent definition of the variation delta I, can be given by this 8.2 b and so this is what we use for getting necessary condition, which is eluders equation for any function any function optimizing the functional I.

So, that is what we have got in the last lecture, here we have got. So, for this functional we saw that, so this functional we considered that is the first step here the simplest one where we I y is given by x 1 to integral x 1 to x 2, f of x y y prime x d x; here, y is assume to satisfy certain additional conditions than continuity, because this integral involves y prime and so this should be integral should be defined in the sense of remand.

So, y is assumed to to be in the admissible class where y prime will have a smoothness property. So, here this function, functional I is unchanged to function phi alpha using this using this substitution here, I y plus alpha delta y. So, then it becomes x 1 to x 2 f of x y plus alpha delta y x plus y prime x plus alpha delta y prime x, because here delta y x is taken like this y prime x minus y x, and here only alpha is changing this y and y prime y tilde are fixed here, only we are changing alpha therefore, this functional it then becomes only function of alpha, and we use the usual rule of finding the values of alpha for which this phi will become maximum. So, we see that if y has to if y has to optimize, because y is obtained when alpha is taken 0, so we see that in the necessary condition for y to optimize this is the phi prime alpha must be 0 at alpha equal to 0.

So, that is what we impose, here we differentiate this functional with respect to alpha partially and we get the following x 1 to x 2 integral of f y x plus alpha delta y x and then this argument is differentiated with respect alpha, which leaves delta y x plus, then f is differentiated with this third argument that is y prime plus alpha delta y prime x, and then this is differentiated with respect to alpha which leaves delta y prime x d x.

Now, here with this second term in the integrand we use integration by parts, here we shift this derivative onto this f y prime plus alpha delta y prime which gives us this minus d by d x of f of y prime x plus alpha delta y prime x. And here, this term is not disturbed and so we take out this which is feed from here delta y; and we take that common from both the terms here, and we get the boundary terms like this evaluated at x 1 to x 2.

Now, since at the moment we are taking only fixed boundary conditions that the points a and b are fixed and therefore, since y and y tilde pass through these 2 points, so we get delta y at x 1 and delta y at x 2 equal to 0; because delta y since delta y at x is y tilde x minus y x and therefore, this at x 1 as well as x 2 will be 0, because both y and y tilde are passing through the points a and b, and so we get this del over del x partial derivative of this I y plus alpha delta y the following, because the boundary term vanishes (Refer Slide Time: 15:34).

And here we use the fundamental, so we put alpha equal to 0 here, which then gives us this 7.2 which was described in the last lecture and now, this integrant has to be 0, which follows from since this variation is arbitrary here, because all this y and y tilde or you can take any arbitrary y and y tilde and therefore, this is true for any variation here and therefore, we uses fundamental lemma of the calculus of variation like this; that if you have two functions f and g continuous, if we have this function f fixed function which is continuous from a to b to r and this integral a to b f x g x d x equal to 0 for all continuous functions on this interval then f x must be identically 0.

So, this is to be proved which will establish today in this lecture, so we use this to conclude that this integrant is 0 and that is what is called Euler's Equation, so Euler's Equation is the necessary condition for y to optimize the functional I, so here we establish this fundamental lemma, so this proof of the fundamental lemma. So, we want to show that, to show that if f from a to b to R is continuous and integral a to b f x, g x, d x equal to 0 for all g from a to b on R continuous, then f x must be identically 0, so suppose contrary that f x is not identically 0, so suppose contrary that there exists x their exits x zero in the interval a to b such that, f x is 0 is not equal to 0; now here if situation can occur that it could be either a or b. So, is like this if x 0 is a or b, so if x 0 a then f x 0 could be here or it could be below here, so we can multiply by minus, we can replace f by minus f, and assume that it is on the upper side of this x x.

So, without loss of generality, suppose that f of x 0 is positive since it is not 0, so either it is negative or it is positive. So, assuming that if it is negative we can multiply by minus and then consider minus f x instead of f x, and then conclude for minus f x and whatever holds for minus f x with than conclude from that for f x. So, we can assume without loss of generate that f of x 0 is positive here, so if it is at the end points then we take only, if it is the left end point we take the interval; some interval never would round a on the right of this and if it is the right end point, then we take neighborhood around this point around b in the inside interval a, b; if it is an interior point like this than we can take never around this x 0, so there exists, so by continuity there exists r greater than 0 such that such that here, x 0 minus r to x 0 plus r intersection this a, b such that on f x is positive.

Like since it is positive, so by continuity it, continuous to remain above here, so there is the neighborhood around this such that, this is will remain here. Similarly, if it is here then some were f of x is 0 here and so by continuity remains above the x axis similarly, if it is the right end point then it remains above like this by continuity, so we will have this thing.

Now, we construct here, so by continuity there exists r greater than 0 such that on this interval neighborhood around x 0, so x 0 minus r to x 0 plus r intersection a, b, so if this is x 0 is interior point then this can be taken in in inside. So, and this intersection will be this interval x 0 minus r to x 0 plus r itself, but if it if x 0 is the end point, then only the right side of that will come in the intersection or if x 0 is the right end point that is b then you will have only the left side of this interval coming in the intersection.

So, that way we have a neighborhood around here this either this end point or if x 0 is interior point then like this and if x 0 is the right end point b then we have this interval around this which we can denote.

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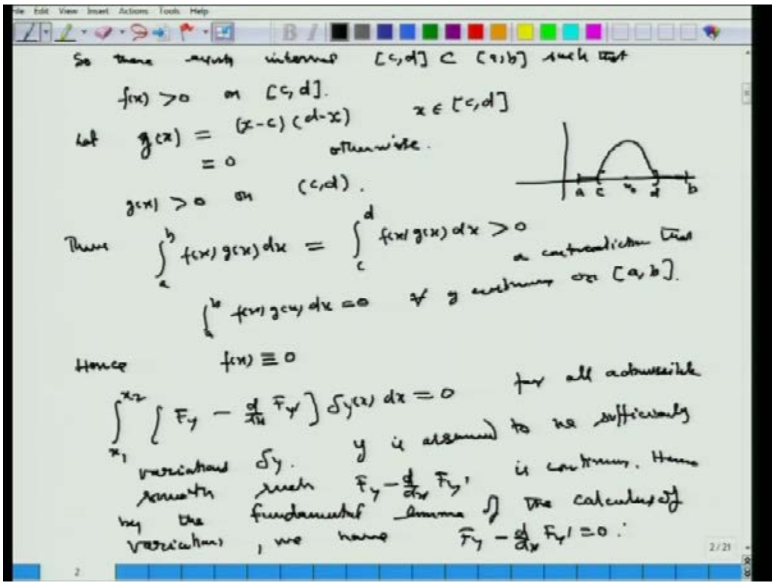

So, there exist interval let us say c, d which is inside a, b such that, this f x is positive on c, d; now we construct eta like this, so we construct eta let eta x or g x b x minus c into d minus x. Here, if x belongs to c, d and g x equal to 0 otherwise, then we can see that here on this interval, if it is so this is right if a 0 is right end point then c d can be taken like this, if it is interior point the c d interval can be taken like this or c d interval can be taken right like this if x 0 is right end point.

So, this interval here eta will be taken positive like this and vanishing at the end points at as well as at the other parts, so eta is like this or the g x rather, so here you have this interval a to b (Refer Slide Time: 25:28). So, let us see supposing that x 0 is here interior and you have this interval c d like this, so g x is something like parabola like this, this is parabola here.

And here it is 0 end point that c also it is 0 and here this is 0 and on c and d it is strictly positive, and so g x is positive on open interval c, d, thus integral a to b f x g x d x is since, the g x 0 outside, so it reduces to c to d f x g x d x and here both the things are strictly positive, so we get strictly positive a contradiction diction that integral a to b f x g x d x 0 for all g, all continues g, continuous on a to b, really g is continuous here, but the integral a to b, f x g x d x is strictly positive it is not 0. So, there cannot be such a case that, so therefore, hence f x must be identically 0, so now, we have got this, now here we have seen that x 1 to x 2 f y minus d by d x of f y prime delta y x d x equal to 0, and this is for all admissible variations delta y, here we are assuring that, y is assumed to be sufficiently smooth such that, is f y minus d by d x of f y prime is continuous. Hence by the fundamental fundamental theorem lemma of the calculus of variations; we have this f y minus d by d x of f y prime equal to 0, which is Euler's equation, is known as Euler's equation.

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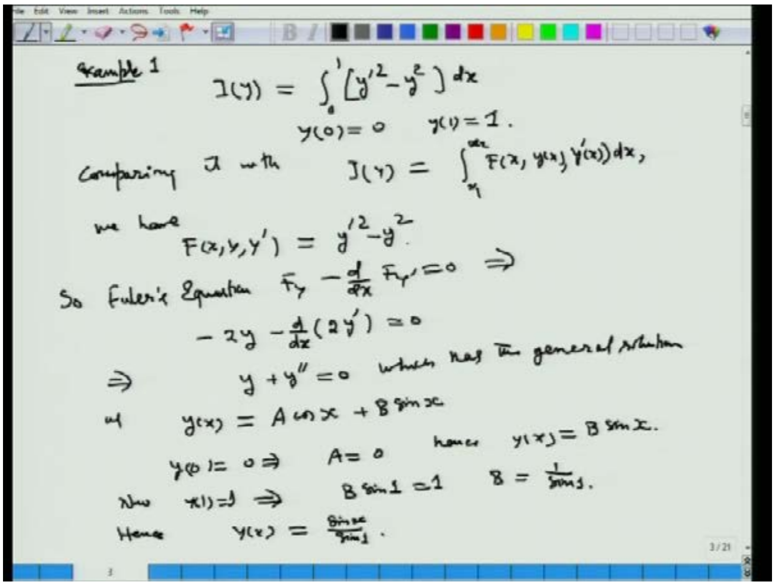

So, let us see some applications of Euler Equation, here we consider a few examples, the first 1 is, we consider few examples, the first 1 is let us consider, I y integral 0 to 1 y prime square minus y square d x y 0 equal to 0 and y 1 equal to 1.

So, here we have the integrand F, so comparing it with the general form, so comparing it with the general form I y if x 0, x 1, x 2, F x, y x, y prime x d x, we have F x, y, y prime equal to y prime square minus y square. So, Euler Equation f y minus d by d x of f y prime equal to 0 implies in this case that minus 2 y f y is minus 2 y here, minus d by d x of this f y prime is 2 y prime equal to 0. So, this implies that y plus y double prime equal to 0 and which has solution, general solution rather as y x equal to A cos x plus B sin x. Now, we use the boundary conditions y 0 equal to 0 implies A equal to 0, so then hence y x is B sin x, now y 1 equal to 1 implies that B sin 1 equal to 1 and therefore, B equal to 1 over sin 1, and thus hence y x is actually sin x over sin 1, so that's how we obtain. So, this y x is the function which will optimize this functional, weather it will minimize or maximize that will be subsequently seen, we will have sufficient conditions to determine whether an external actually minimizes or maximizes.

Now, the second example is I y equal to 0 to 1 y prime square plus x y d x y 0 equal to 0 and y 1 equal to 1.

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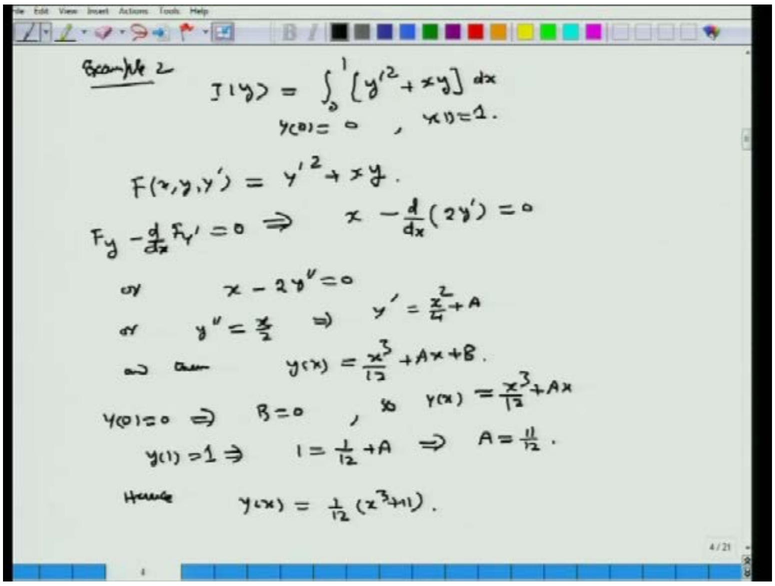

So, in this example now, comparing it with the standard form you see that integrand is so this is F x, y, y prime is function is y prime square plus x y. And so Euler's Equation F y minus d y d x f y prime equal to 0 implies F y is x here minus d by d x of 2 y prime equal to 0 or x minus 2 y double prime equal to 0 or y double prime equal to x by 2, so this implies y prime equal to x square by f 4 plus A and then y of x equal to x cube by 12 plus A x plus B.

So, now this A and B are to be determined using the given conditions, so y 0 equal to 0 implies that b equal to 0 and, so so y is x cube by 12 plus A x, now y 1 equal to 1 implies that 1 equal to 1 by 12 plus A also implies that, A equal to 11 by 12 hence, y x equal to 1 by 12 x cube plus 11. So, that is the external given here, now the next example we consider here I y is x 0 to x 1 rather x 1 to x 2 we are using, x 1 to x 2 integral y square d x and conditions are y at x 1 equal to A and y at x 2 equal to B.

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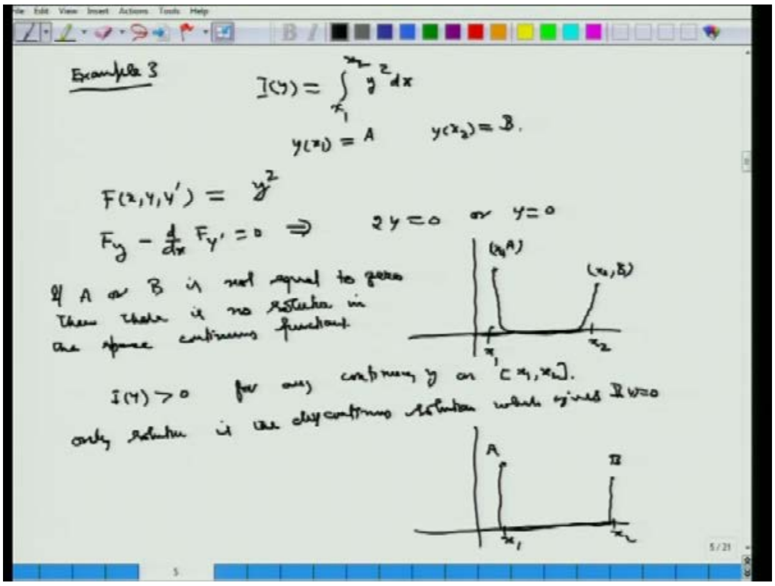

Now, here F is x, y, y prime is simply y square. So, the variables x y had y, only variable y is present and x and y prime are absent in this. So, we have Euler's Equation F y minus d by d x of F y prime to 0 implies that 2 y equal to 0 or y equal to 0. So that means, x axis is the solution of this Euler's Equation and if, so here section is like this only this x axis is the here this is x 1, this is x 2. Now, if A and B are not 0, if A or B is not equal to 0 then there is no solution in the space of, in the space of continuous functions.

So, if our admissible class is the space of continuous functions on the interval x 1 to x 2 then there is no solutions here, because this if let us say both x 1 and x 2 are not 0, if both A and B are not 0, so this is x 1 A, A and this is that the somewhere here x 2 b and you see that here it cannot pass through these two points, because y is identically 0 here, so y equal to 0 for all x, so this is the candidate for optimizing the functional I and even if we try to take approximation here like this, but it will never be 0 it can be reduced I can be reduced as much as we want, but 0 cannot be made here, for any y here this will be I y will be positive for any continuous y, y on x 1, x 2 and only solution is the discontinuous solution, is the discontinuous solution which gives you the, which gives I y equal to 0 that is following, so this is x 1 is x 2 and here is A and here is B, so this simply drops like this and goes like this and, so that is a discontinuous solution here (Refer Slide Time: 40:45). So, in this case there is no solution in this case of continuous functions.

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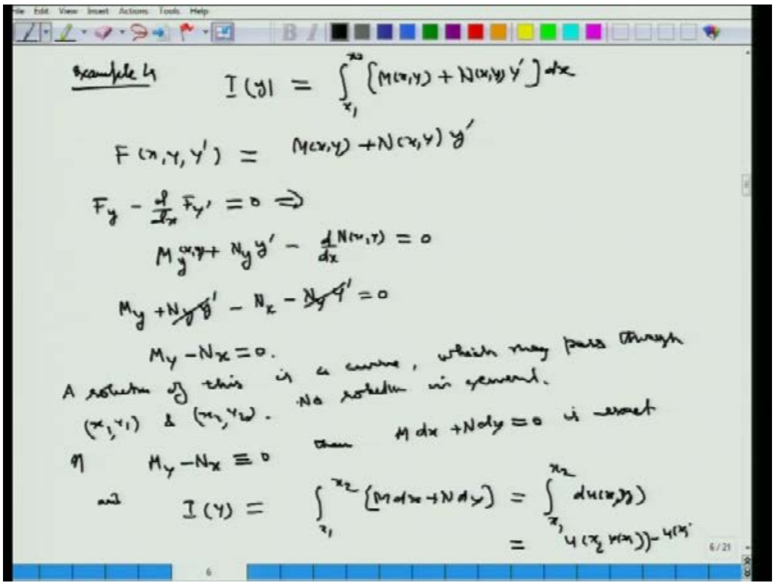

Now, the next example is, if you consider I y equal to x 1 to x 2 here some function M x y plus N x y, y prime d x, so here F x, y, y prime is actually equal to M x y plus N x y, y prime. So, Euler's equation F y minus d by b x of F y prime equal to 0 implies M y plus N y y prime minus d by d x of N x y here all these are functions of x, y equal to 0.

So, that this M y plus N y y prime minus, now this is total derivative of N x y, so first gives you N x minus N y y prime equal to 0, so this cancels here, so we get M y minus N x equal to 0. Now, here this gives a curve and if it does not pass through the points x 1 to x 2. So, solution of this this is an algebraic equation in x and y, and so this curve which may not pass through x 1, y 1 and x 2, y 2; if the pass also then there is a only 1 curve and this is not a problem of optimization in this case, and if so no solution in general if M y minus N x is identically 0, then this reduces to an identity then we see that that M d x plus N d y equal to 0 is exact and then I y is x 1 to x 2 M d x plus N d y and this. So, there exist then some function of x y since it is exact, so therefore, this will be some function u existing such that, M d x plus N d y is equal to the total derivative of u of this x y here, and so this is at the value u at x 1, x 2 at y at x 2 minus u at x 1 y at x 1 and so this is a fixed quantity, a fixed quantity. So, therefore, in this case also it is not a problem of optimization, it is not a problem of the calculus of variations.

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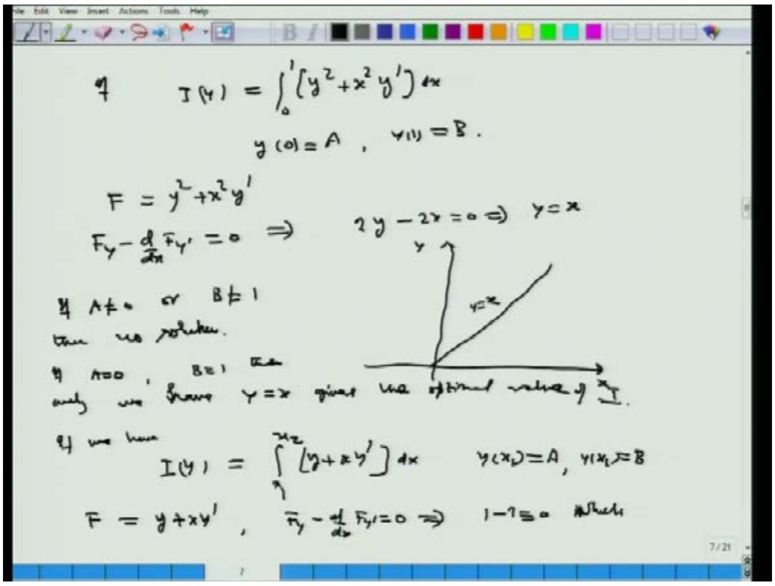

For example, if you take, if we have the lecture in this case I y 0 to 1 y square plus x square y prime d x and y 0 equal to A and y 1 equal to B, then in this case we have F as y square plus x square y prime and F y minus d by d x of F y prime equal to 0 implies that 2 y minus 2 x equal to 0, so implies y equal to x. So, this line y equal to x is the y equal to x, is the candidate for this optimizing this functional and if A, if A is not equal to 0 or if B is not equal to 1 then no solution (Refer Slide Time: 47:49). So, and if A equal to 0 and B equal to 1 then only we have y equal to x s extremely s s the candidate, x gives the optimal value of I.

Now, if we have I equal to I of y equal to x 0 to, x 1 to x 2 y plus x y prime d x and y at x 1 equal to A and y at x 2 equal to B. Now, in this case we have F s y plus x y, so F y minus d by d x of F y prime equal to 0 implies that 1 minus 1 which is 0 identically, so we get the identity here hence which is an identity.

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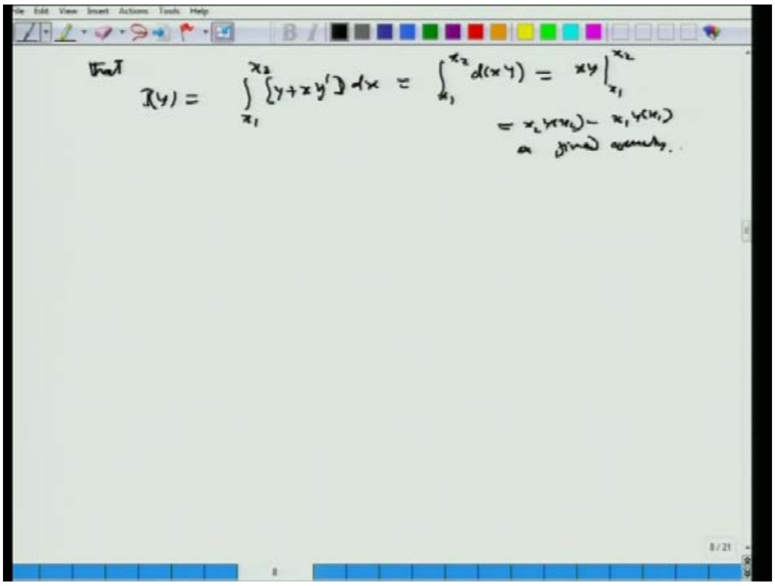

Now, we see that we observed that, that this I y equal to x 1 to x 2 we can write it this y plus x y prime d x can be written as x 1 to x 2 d of x y here and so this is actually x y evaluated, x y evaluated at x 1 to x 2, so this is x 2, y at x 2 minus x 1, y at x 1 and which is in an which is fixed quantity, a fixed quantity here.

So, this is not a problem of optimization, so like this there are many problems which although appear as optimizing problem, but as problem of calculus of variations, but they are not, so we will actually take the genuine cases of such problems in the next lecture, thank you very much for viewing.