Calculus of Variations and Integral Equation

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Lecture No. # 07

Welcome viewers to the NPTEL lecture series on the calculus of variations. This is the 7th lecture of this series. In the last lecture, we introduced fundamental concepts of the variations and related topics.

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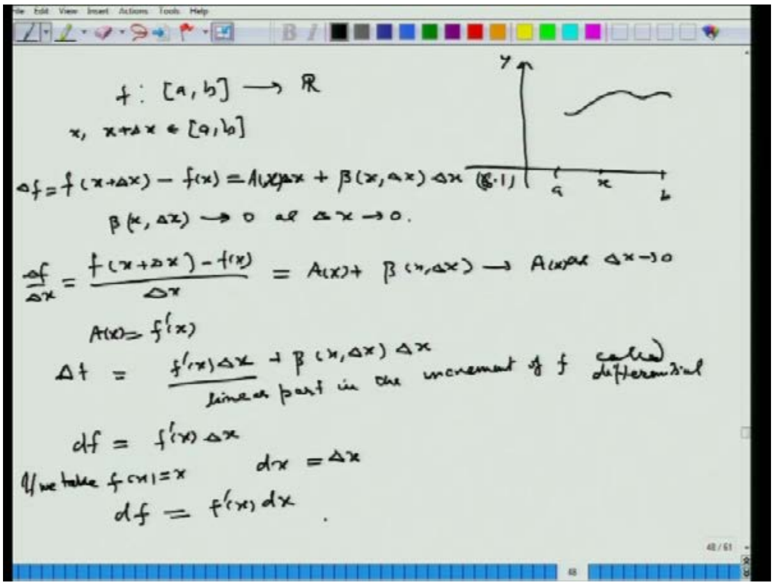

We started with the concept of variation of a function f defined from the interval a to b to R, and we defined the concept of differential of this function. Here the variation delta f is actually the difference of the values of f at x plus delta x and f(x). So, delta f, the variation in f is defined as f of x plus delta x minus f of x, which is actually equal to this A has to be a function of x. So, we will write the dependence of x on A like this, A of x A of x times delta x plus beta gamma delta x into delta x.

So, here this beta is assumed to have this property that it tends to 0, as delta x tends to 0. So, if f is differentiable, if and only if this 6.1 holds, that is the variation delta f has the property that it this difference is equal to A(x) times delta x pus beta (x, delta x) times delta x, this A will be function of x in general. So, that is, what is here 6.1 holds, if and only if f is differentiable. And we can see that this A actually has to be f prime x, so this A courses a function of x, which is the derivative of f at x, which can be seen just dividing 6.1 by delta x, and letting delta x tend to 0.

So, it this since, beta is tending to 0, as delta x tends to 0. So, this term goes to 0, and so, this tends the right hand side of this delta f by delta x tends to A of x, as delta x tends to 0. Therefore, this since this delta f by delta x limit, this is actually equal to by definition, the derivative of f at x, and therefore, A(x) must be f prime x.

So, we see that in the case of function, we have this result that variation delta f has to be then f prime (x) delta x plus beta (x, delta x) times delta x. This part, underline part is clearly linear in x linear in delta x, and therefore, it is called linear part in the increment of f, and this is what is called differential. And therefore, it is this differential is denoted by d f, and so, therefore, d f must be f prime x delta x. And if we take f(x) equal to x that the identity function, then you can see that f prime x is 1, and therefore, d x must be equal to delta x. So far independent variable differential is the same as the variation, and we have this d f differential of f equal to f prime x dx.

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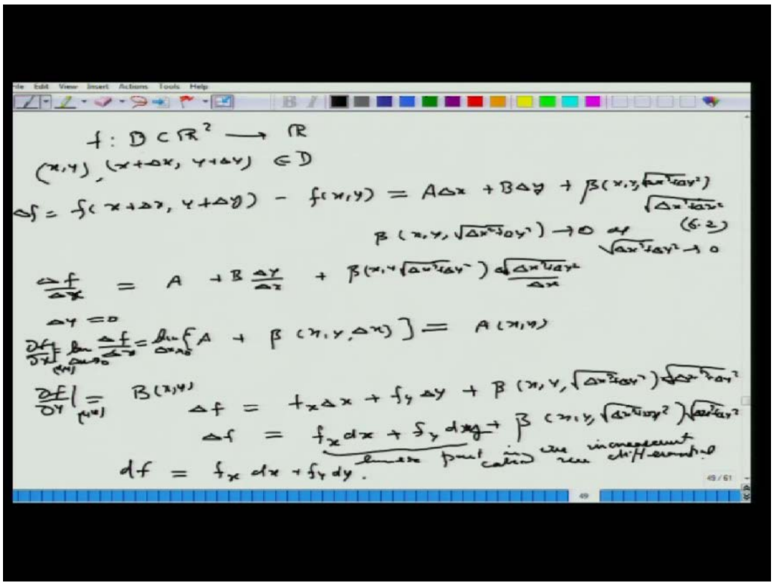

The this concept of differential can be extended to higher dimensions in the following manner, that it is f let us say in particular, we take function of two variables x, y. And assuming that this f is defined from domain in R 2 into R, and assume that x, y and x plus delta x, and y pus delta y are points in D. Then you can see that this variation d f is actually equal to f of x plus delta x, y plus delta y minus f (x, y). And we say that this f is differentiable, f is differentiable if and only, if this delta f the variation in f that is delta f is equal to this A.

So, A and B will be of course, functions of x and y here, and we can see that this difference has to be then equal to A delta x plus B delta y plus beta x, y, square root of delta x square plus delta y square times delta square root of delta x square plus delta y square. So, f is differentiable, if and only if 6.2 holds, and we can see that for the same process, we take delta y 0, and then divide by delta x, we see that this A has to be actually then equal to del f by del x at (x, y).

Similarly, B has to be... So, these are functions of x, y, because the derivatives R to be evaluated at (x, y), this at (x, y), similarly, this at (x, y). So, these A and B are actually, the partial derivatives of f with respective x and y respectively, at evaluated at (x, y). And so, this is the linear part in the these increments delta x and delta y. So, that is what we call linear part in the increment of the function f, and it is denoted as differential, as before in the one dimension case. And so, we denote d f the differential of f as f x partial derivatives of f with the respective x times d x plus f y d y. Since, this delta x and delta y are same as d x d y. So, we get this.

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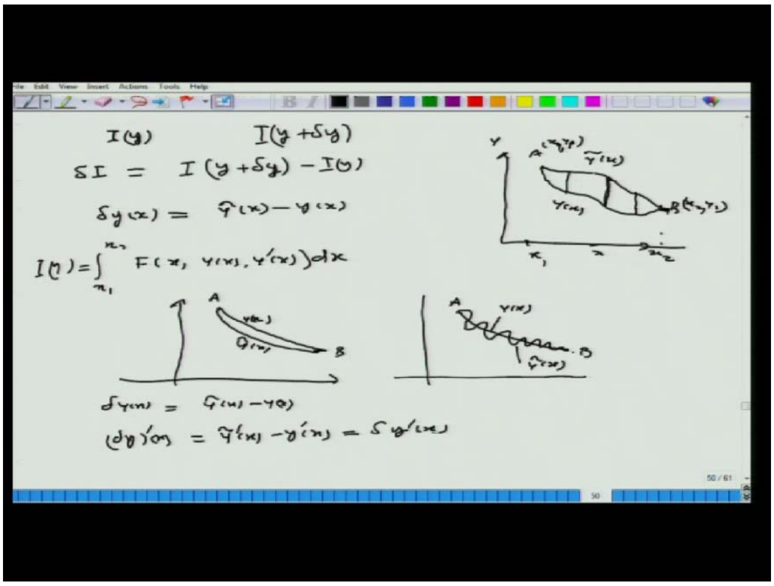

Now, we want to extend this concept of variation for the functional. So, we have this functional for example, this is our functional here. So, if we change, if we vary y, then what is the variation in I, this integral that is what we want to defined. So, here first we need to define the variation in y itself like, we had variation in x that was delta x, and which was same thing as the differential d x.

So, what is happening in this case, that is what we want to define, here let us say, y(x) is y(x) and y tide (x) are two admissible functions, that means, they pass through these two points, fix points A and B. And then the they have the property that the integral is this functional is defined. So, these y and y tilde (x) are admissible functions, and we want to see that what. So, the variation here in y is defined like this, delta y at x is y tilde(x) minus y(x).

So, this ordinate here, this ordinate at this upper one minus ordinate at this... So, this is the difference between those two ordinates. So, whatever is remaining inside these two is what defined as the variation delta y at x. So, if x changes over this whole interval, we have different delta y here like this, here and like this. So, it is a function of x, when x varies over the interval x 1 to x 2, and then here, we want to see that what is that A, I y plus delta y. So, here y will be replaced by y plus delta y, and what is this y prime, y plus delta y prime that is, what we will also come into picture.

And so, if we define delta y like this, y tilde (x) minus y(x), clearly then delta y prime is this y tilde prime minus y prime (x), and you can clearly see that this is same thing as delta y prime (x). So, variation in the derivative of the variation in this case is the variation of the derivative. So, that is how it is defined in this case.

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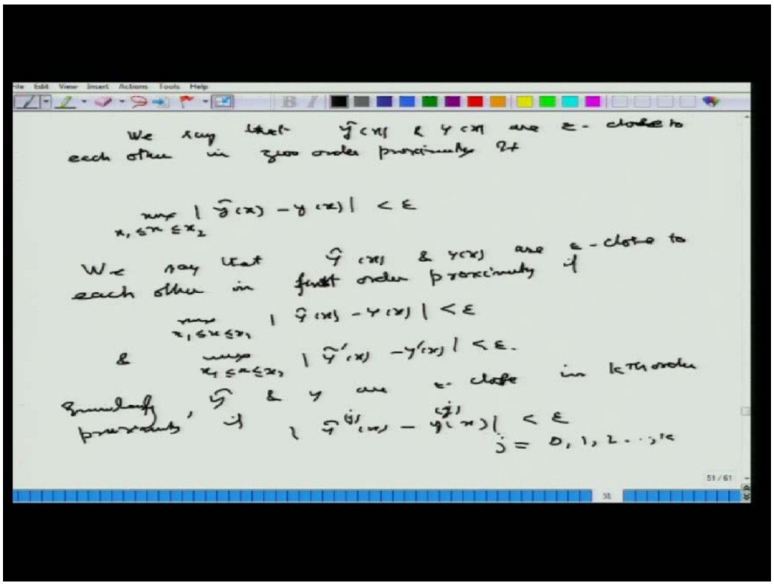

And we say that since, here the concept of nearness of these y and y tilde has to be made precise, and that is what we do in the following manner? We did we say that this two functions, these two functions y and y tilde are epsilon close, epsilon is the positive quantity, here epsilon close to each other in the zero order of proximity. So, this is the order of proximity, here we have been defining in the in the following manner; that we say that y tilde (x) and y (x) are epsilon close to each other in zero order proximity.

If maximum of this x 1 less than equal to x less than equal to x 2 of y tilde more absolute value of y tilde (x) minus y (x), that is the difference between ordinates, the absolute value of that is less than epsilon, for all x in the interval. So, we take maximum of the all these difference of ordinates, there should be less than epsilon.

Now, we see that these y tilde and y (x) are epsilon close to each other in first order proximity, if not only the difference between the ordinates of y, but the difference between the ordinates of y tilde are also epsilon close, for all values of x. So, that is what is stated here, the maximum of x 1 less than x less than equal to x 2 of the absolute value of the difference of ordinates at x is less than epsilon.

Similarly, difference between the ordinates of y prime, absolute value of that and maximum of these quantities over the interval x is less than epsilon. So, similarly, we can extend it to k k order proximity in the same manner, that all the derivatives y tilde (j) minus y (j), this is the jth derivative here. So, the difference between the ordinates of jth derivative are close epsilon close to each other. So, this is the… I mean the difference the between the ordinates of jth derivative is less than epsilon in the absolute value. So, here, where j is 1, 2 to n.

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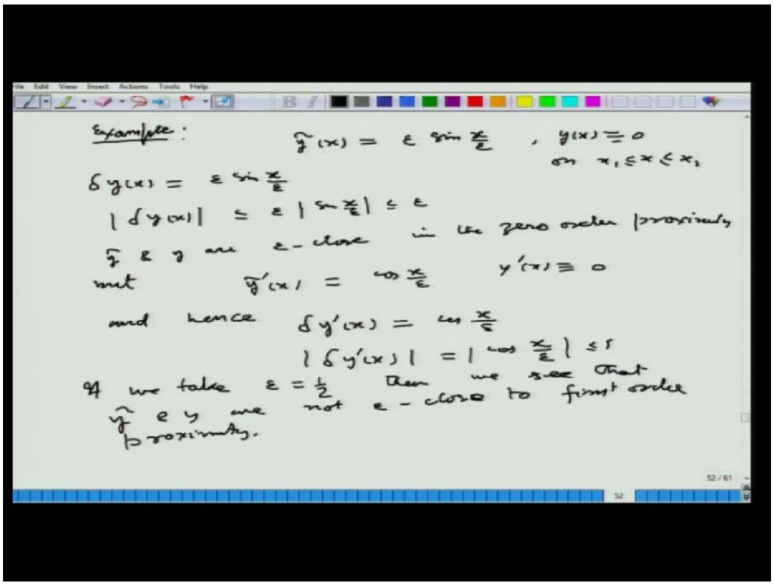

So, for example, if we consider this y tilde (x) equal to epsilon sin x by epsilon, and y (x) equal to identically 0 on x 1 less than x less than x2, then we see that this delta y at x is nothing but epsilon sin x by epsilon. And so, absolute value of this so absolute value of y, because y (x) is identically 0 here. So, this is less than equal to epsilon mod sin x by epsilon, and sin here is always bounded by 1. So, this is less than equal to epsilon.

So, here we see that here we f 2 l of the equivalent also less than equal to and all these… So, we see that these two curves are close epsilon close. So, y tilde and y are epsilon - close in the zero order proximity, but this y tilde prime (x) is cos x by epsilon. And so, and y prime (x) is identically 0 anyway, and so therefore and hence, this delta y prime (x) is cos of x by epsilon, and absolute value of this equal to absolute value of cos x by epsilon, which is only bounded by 1.

And so, if we take epsilon equal to half, then we see that y tilde and y are not epsilon - close to first order of proximity, their close to their anyway close to each other in zero order proximity for all epsilon positive. But they are not close to each other in first order proximity, if we take epsilon to be strictly less than 1, because if we take epsilon equal to half, we can see that this cannot be satisfied, this cannot be less than half in general here.

So, we can see that this is an example, where epsilon, where y and y tilde are close to each other only in zero order proximity.

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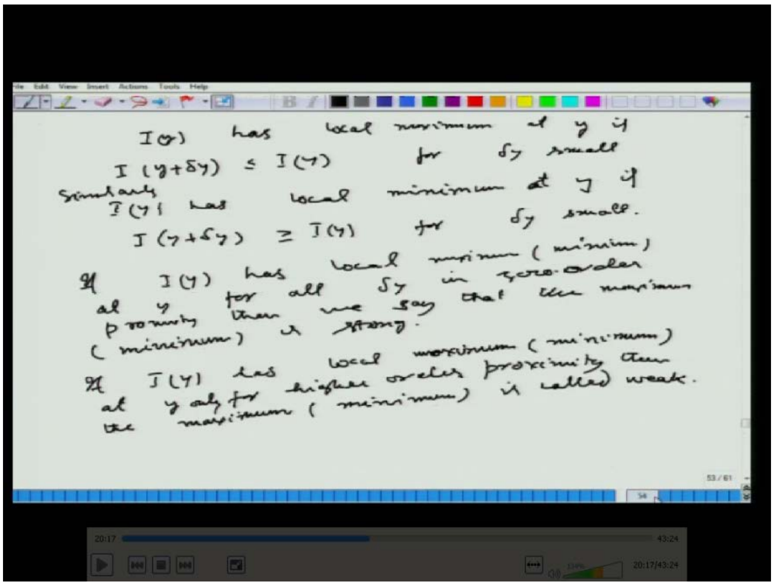

So, here we define that… So, if this I (y), we say that it is I (y) has local maximum

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Local maximum at y, if I y plus delta y is less than I (y) for delta y small. Similarly, I (y) similarly, I (y) has local minimum at y, if I y plus delta y is greater than equal to I (y) for delta y small.

Now, if if this I (y) has local maximum or minimum at y for all delta y in zero order zero order proximity, then we say that that the maximum or minimum is strong. If I (y) has local maximum or minimum at y for only for higher order proximity,

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Then the minimum or maximum sorry maximum or minimum maximum or minimum is called weak. Clearly, if the maximum is achieved at zero order proximity, it may strongly then it is also achieved in the higher order proximity, in the weak sense also.

And so but it there may be cases, where the closeness in the higher order proximity is required minimum may not exist at lower order proximities, then we say that such minimum maximum are weak. Now, here we define this variation in the same manner as we define for the functions, we want to extend it to the case, where we have the functional. So, first before going into that we need to define certain concepts of linearity and so on for functional.

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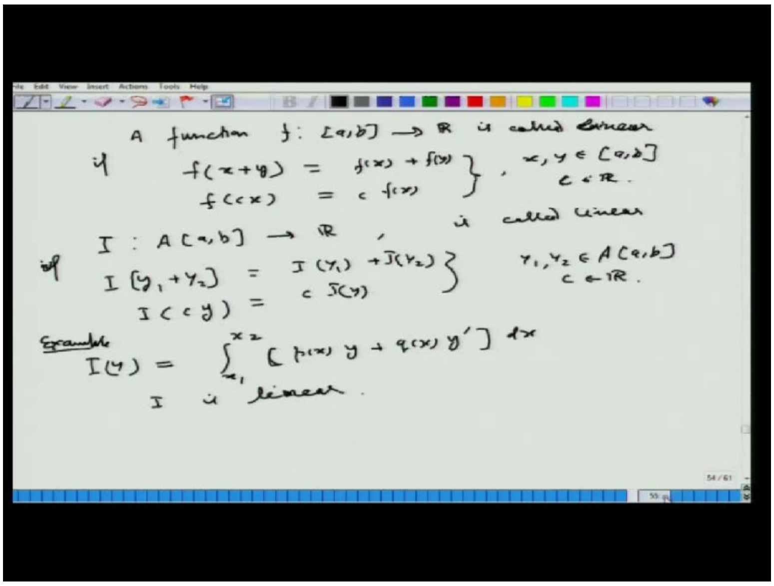

So, here we know that the function f form a to b to R is called linear, if l… If you have f of x plus y is f of x plus f of y, for x, y in a, b. And f of c x is c times f (x), for x, y for x in k, x, y in a, b and c is in R, c is a constant in R. So, similarly, if I here functional, where it is from admissible class define on certain interval a to b into R. So, here A is the admissible class of functions for which this I will make sense. So, then here, if you take two functions like this, y 1, y 2, then it should be like this I of y 1 plus I of y 2 and this called linear, if and I of this constant c times I (y).So, (( )) same manner, we define where y 1, y 2 are in this A the admissible class, and c is a number in this. So, linearity is defined in the same manner; for example, here this functional I of… example, here of the x 1 to x 2 here of p (x) y plus q (x) y prime d x. So, this I is linear here, I is linear this an example here of this.

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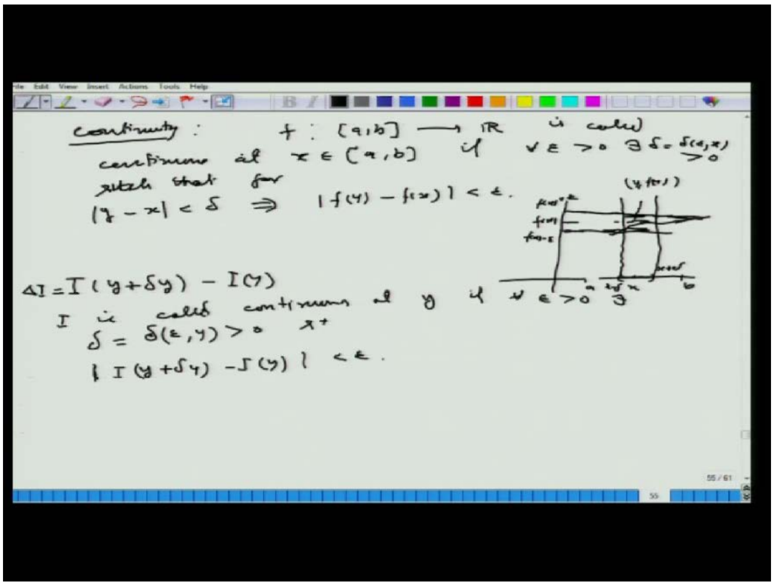

Next we define the continuity. So, to call the continuity, so f from a to b into R is called continuous. If continuous at x belonging to a, b, if for every epsilon greater than 0, there exist delta, which is a function of epsilon as well as x, and this is also positive, such that for such that for y minus x less than this delta implies f of y minus f of x is less than epsilon.

So, here it is like this A and B and function here, let us say, at this point what should happen at this point x, if you take a neighborhood around this all y's in this. So, the delta neighborhood delta, this is x minus delta to x plus delta, in this interval wherever that y is there this f (x), f (x) is here this is f (x). So, and this is epsilon strip, this is f (x) plus epsilon, this is f x minus epsilon. So, here these values f (y) must lie within this strip.

So, and this is, what is like this sorry.

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This x minus delta to x plus delta strip. So, this point x, f (x) or y f (y) must lie this is x f (x) and here…. So, within this rectangle, these value must lie y f (y) must lie here, y, f (y), this point must lie within this rectangle. So, that is what it means, if we have these values sufficiently, these values y sufficiently close to x then the values f (y) will also be lying sufficiently close here in this y strip. So, that is, what is the continuity of a function? So, here the continuity of functional, we will be defining in this following sense.

So, here if we consider this I (y). So, I (y) will be continuous at y, if we consider these neighboring variations. So, this is delta I. So, if we take this delta y sufficiently close to… So, this I is called continuous at y, if for every epsilon greater than 0, there exist delta, which will be function of this and y here, such that this absolute value of I y plus delta y minus I (y) will be less than epsilon. So, is extending the same manner as the function case. So, the continuity of this functional is defined in the same manner.

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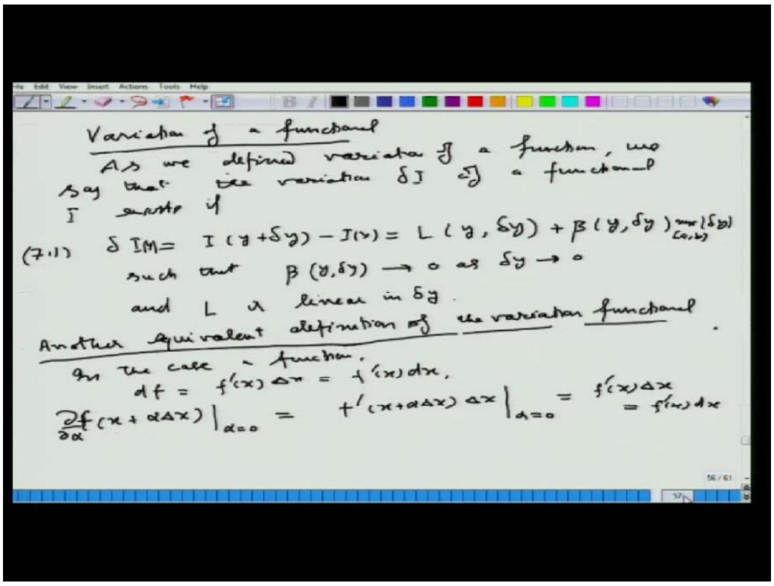

Now, we want to define the variation of a functional. So, we have already considered this I (y) and I y plus delta y, the difference between them, we say that, as we define define variation of a function, we say that the variation delta I of a functional I exist. If the following rules, if this delta I has the form, which is nothing but delta at delta there is a function, this I y plus delta y minus I (y), this has the form that there is a linear part here, it is function of y as well as delta y and sorry plus beta (y, delta y). And then the absolute value of the maximum times maximum of delta y is maximum is taken over the interval maximum over a, b, where I is defined, such that this beta (y, delta y) tends to and tends to 0 as delta y tends to 0, and this l is linear in delta y.

So, we say that this is a 7.1 we will call, we say that the variation of the functional I exist, if and only if 7.1 holds that is the delta I that the difference I y plus delta y minus I

(y) as the form that there is a linear part the increment that is L (y, delta y). This is linear in delta y plus, here the term beta (y ,delta y) times maximum of absolute value of delta y, this maximum is taken over the interval, where this I, functional I is defined. So, we say that this variation exists, if and only if 7.1 holds.

Now, here, there is another way of defining, equivalent way of defining the functional variation of a functional. Another equivalent definition...

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Of the variation.

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For example, in the case of in the case of function, we know that this d f, we have defined d f equal to f prime (x) delta x or delta x or same thing as f prime (x) d x. Here, we can see that this is also equal to… If we consider f of x plus alpha delta x and differentiated partially with the respect to this, and evaluated at alpha equal to 0, this is the same thing as f prime at x plus a alpha delta x. And then this argument differentiated with the respective alpha gives you delta x, this thing evaluated at alpha equal to 0. So, this gives you f prime (x) delta x is the same thing as f prime (x) dx from here.

So, there is this is a convenient way of defining, equivalent way of defining the differential of f, same way if we emitted this definition for the functional, we can consider in the following manner.

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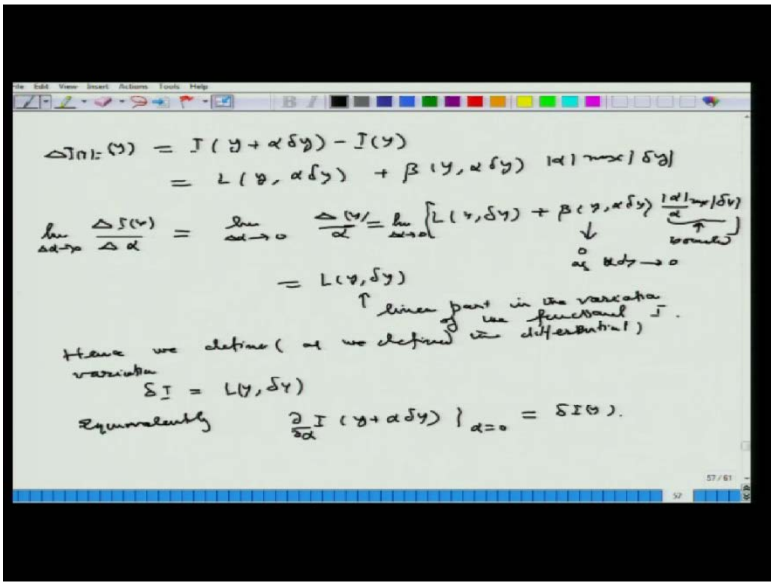

That we consider here, delta I (y) as I y plus alpha delta y minus I (y). So, this will be then equal to, if variation exist then from 7.1, this should be equal to L(y ,alpha delta y) plus beta (y, alpha delta y) and mod alpha maximum of delta y. And here, if we divide by this, So, if we divide by this, here delta alpha, and then take limit delta alpha tending to 0 is same thing as limit delta alpha tending to 0, this is delta of delta I (y) upon alpha, because delta alpha is alpha minus 0, which is same thing as alpha.

And so, you since this L is linear, we see that this will give you, y delta y plus, here we will have beta (y alpha delta y), and here will have alpha over is maximum on delta y and since, this quantity is bounded, this quantity is bounded and this quantity goes to 0 as delta y. Since, alpha tends to 0; therefore alpha delta y will tend to 0 as alpha delta y tends to 0. And so, this will be equal to this the whole thing limit of this delta alpha tending to 0 of the origin.

And so, this will be equal to L (y, delta y) which is the linear part in the variation, this is the linear part in the variation the variation of the functional. And so, this is what will be actually defined as… As... So, this hence, we define, as we define as we define the differential, as we define the differential, we define variation delta I as L (y, delta y), here this is you should change this notation here, this is delta I at y, this also will change to delta. So, that we use different notation for the variation, and it difference.

Similarly, here let me correct that. So, here also we change this to delta. So, this is the difference this capital delta will be used to define, the difference between the values at y of the functional I that y plus delta y, this delta is used for variation. So, this is a variation of the function y itself, and this is the difference and the values of the functional, and we say that if variation exist this variation, we will be defining as delta I.

So, here you say that the variation of the functional exist, if and only if the 7.1, which means that is the difference is equal to this linear part in the increment that is L (y, delta y) plus beta (y, delta y) times maximum of absolute value of delta y. And we see that provided this beta goes to 0 as delta y tends to 0, we say that the variation of I exist, and we, as we define the differential of a function, we define here the variation of this functional the linear part in the increment. So, that is what we have in the case of the functional.

Now, here we will consider in the same manner as we consider in the function case, we can have equivalent definition like this, that equivalently we define this I y plus alpha delta y and the partial derivative with respective to alpha. And evaluated to alpha equal to 0, this is what we have seen in this case, as the variation, and so, this is what will be called the delta I at y. So, this what equivalent definitions very convenient to use, this is what we will be using subsequently in our lectures.

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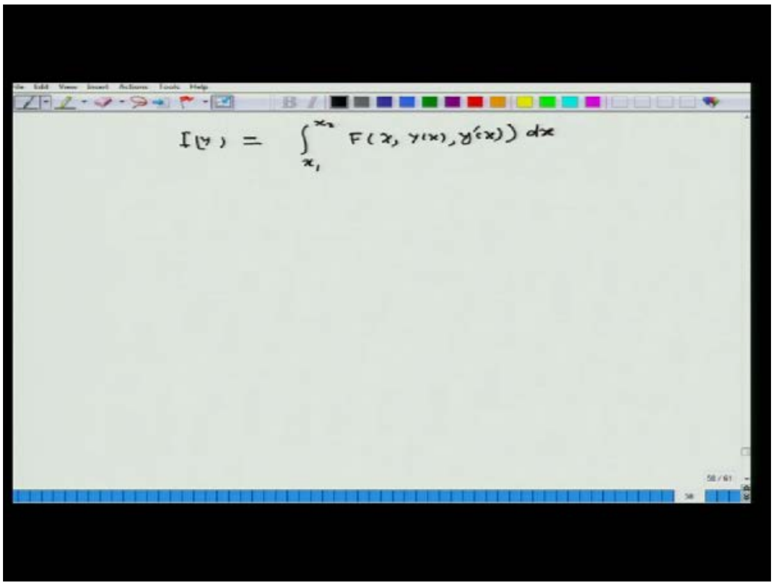

Now, let us consider here the case of the simple first which we consider this I of y, which is actually equal to x 1 to x 2 F x, y (x), y prime (x) dx. So, we would apply this equivalent definition here of the functional, the variation of the functional. And we will see that we will consider here, I of y plus alpha delta y, and then we will differentiated with respective alpha. And then we will equate that to 0, and see that the condition, which which to be satisfied by the function y is, what we will be getting as and the necessary condition for this y 2 optimize this functional. And that is what will be giving as the Euler's equation, which is to be satisfied by the function y in order, this functional to be optimized. So, that is what will be considered in the next lecture. Thank you very much for viewing this.