Calculus of Variations and Integral Equation

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Lecture No. # 06

Welcome viewers to the NPTEL lecture series on the Calculus of Variation, is the 6th lecture of the series. Now, we will start the discussion on the main topics, we will start with fundamental concepts of the calculus of variation, here we has we had mention in the first lecture, that certain problem initiated the study of the calculus variations.

I would mention three main problems here: the first one was already mentioned, that was the branches recount problem introduced by proposed by John Bernaly in 1696 and then subsequently there were many problem added to the discussion of the calculus of variation, like geodesic finding of geodesic; and then isoperimetric problems; and certain problems of mechanic and physics, they were formulated has optimization problems; that means, finding minimal or maximal values of certain functional. So, here I would mention here three main problems, which initiated the study of the calculus of variations.

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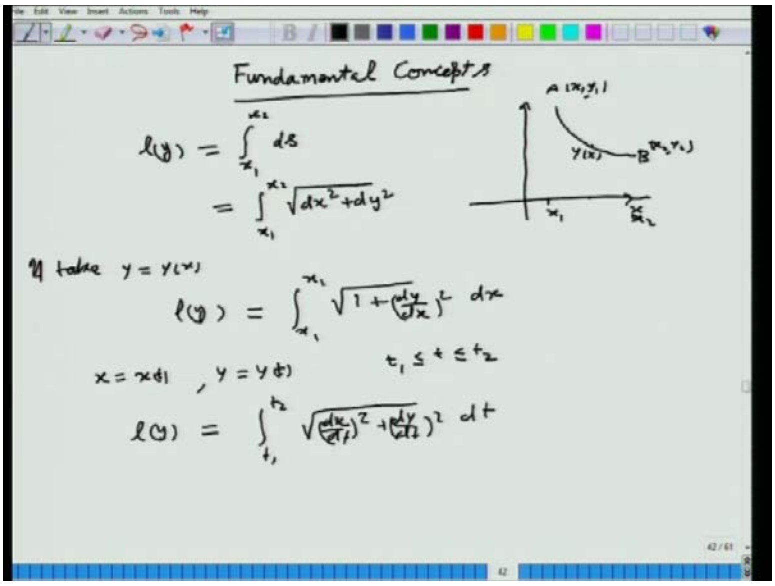

The first one is here, finding the length of function, like you have in the x y-plane and there or two points A and B; and a curve joining a smooth curved joining this you know, that the length of this cure is the functional that is square root of integral over let us say, this is the point x 1 an y 1; here x 1, x 2 and this the point (x 1, y 1) and (x 2, y 2); and so, we need to find this integral x 1 to x 2.

Integral x 1 to x 2 of this d s, where x is arc length and it is given by x 1 to x 2 square root d x square plus d y square, this is a plane curve. So, we have x and y variables only, in general, if we have three-dimensional curve, then d s is square root of d x square plus d y square d z square . So, here there are only the two variables. So, we have this.

Now, if we treat x as parameter and take y as a function of x, then if we take y as a function of the x, like this is y (x) for each x; here the value, this is (x y, y x) point on the curve; that the graph of the curve, then this l (y) is given by x 1 to x 2 square root. So, this becomes 1 here. So, x is a function of x itself; that is the identity function, and so, d x by d s gives you 1 plus d y by d x square; and this d x comes out here. Now, x has a function of t, y as function of t and t has range t 1 less t less than equal to t 2, then l (y) can be given as t 1 to t 2 square root, then we takes d x by d t square square plus d y by d t square and then this d t comes.

So, we can use any of these for has functional defining the length of the curve y. So, here we need to find a curve, such that the length is minimized; and we know the answer is the straight line joining these two points A and B. So, this is the first elementary problem of the calculus of variation.

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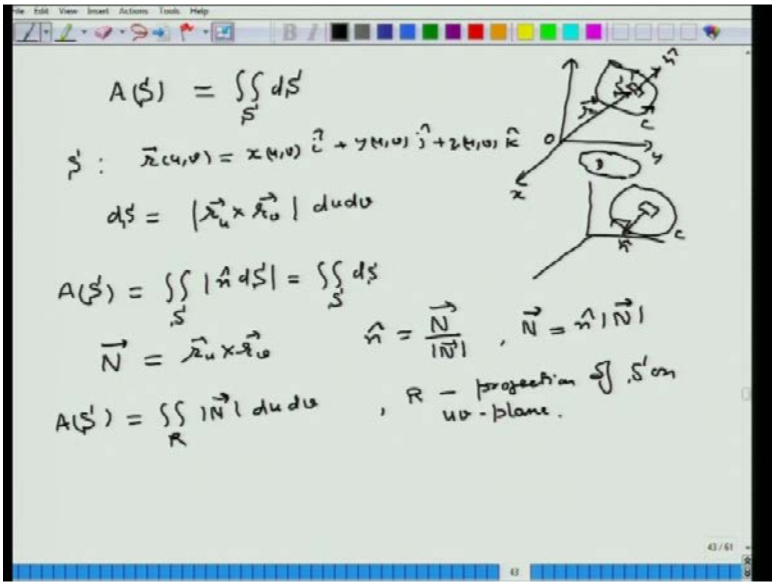

The next one is, we have seen that the surface area A (S) is given by, we know that; this is double integral over the surface S of the elements of the surface area d S here, you have three dimensional figure x, y, z and this is the surface bounded by the curve, let us say, c here and this is surface S. So, here element area is d S and we have seen that this is outward normal become actually, we need to fix, which one to define as outward normal, because there are two directions, one is the upper one and then the other one is the down side; that is also the normal to this surface.

So, which one to call outward here, we take the convention that; we take this curve c as positively oriented, when we take this anticlockwise; and if we consider this situation at the same surface is here; and we take the positive direction of c like this, then and the element d s is here, then the normal will be downward like this.

So, we have to see that here, the positive directions of c; that means, this is an oriented oriented surface the upper side here is taken like this, when the direction of this curve bounding the surface actually, we are considering the downside of the surface, and then the positive direction is the (( )) direction of the earlier one.

And so and outward normal in then will be the downside, downward normal to this one, which we had considered, just minus of this an kept here will be in this situation. So, we need to fix the orientation of the surface the beginning; and then see what, which side we are considered as the positive side of the surface.

So, here this we have seen that d s is actually, now here we parameterized S as like position vector of the point p here like this, this is position vector whether r, here and it will be function of two parameter (u, v). So, we take x (u, v) i plus y (u, v) j plus z (u, v) k; that we already explain, and then we know that this elementary d S is, nothing but r u cross r v absolute value that d u d v, and so, here now actually, we know that this we one can take as, so it n cap. So, this will be, nothing but A (S) will be over this S and absolute value of this n cap d S. So this, what, we have as d S, now this n cap d S till than B.

Now, if you defined this capital N as r u cross r v, then we know that; this normal to this surface, because this r u and r v constitute the tangent plane, and so, r u cross r v gives you a normal to the plane described by r u an r v vectors, and so, this n cap is the unit normal, which will be, nothing but capital N cap and add a arrow over this absolute value of is vector N. So, that is, what will be the unit vector in the direction of the in outward normal, and so, we can see that then this capital N will be given by N cap absolute value of capital N.

And so**,** Therefore, this A (S) can be seen that surface area will be here, will be given in projector one, let us say, this R is the projection of this on the u v-plane; rather here, we are taking x y. So, we will take that a particular case, here this is d, this is projection of the surface on x y- plane, and if we take projection on u v-plane that we denote by R. So R, and so, this will be nothing but absolute value of capital N d u d u, because absolute value of N cap is one. So, that is what we have, where R is the projection of S on u v–plane.

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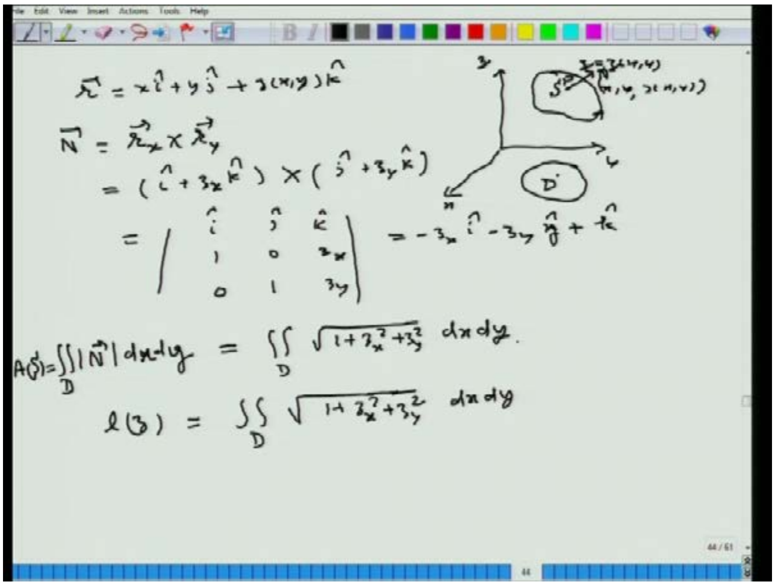

So, that is, what will be the formula for finding this surface area, in particular, if we take this case, where surface is projector onto x y-plane? So, this the surface, and it is projector on this x y-plane like this D, this is the surface S. And so, in this case, this is special case, our position vector r is given x i plus y i and plus let us say, this surface is given by z as function of (x, y); for each (x, y) here this gives you the points this p here, which is (x, y) and z (x, y) like this, which we have already described earlier.

So, this will be the position vector in this is special case, usually, if you take u v as parameter, then this x y will be functions of u and v, but now we are taking x and y as self as parameter, and then z is a function of x and y. And then in this case, we can see that N, which is, nothing but r x, now u is x. So, r x cross or r y and v is y here. So, this is, nothing but (i plus z x k) cross product (j plus z y k).

So, this is, nothing but i j k and 1 0 z x here and 0 1 z y. So, that is i times this 0 minus from minus z x i similarly, j give you minus z y k j and plus and then k components is 1 here k. So, this is what is actually, capital N here and this situation; and if we take the downside then we will have to take r y cross r x, which will be minus of N and will be pointed downwards.

In the case, when we take the other as the reverse directions of this as a positive orientation, positive direction of c. So, then that will be the orientation of the surface will be just reverse; that will be from downside of this surface as be here. So, here in this case, so we substitute in this formula; that A S is equal to absolute value of the this cap d u d v over this projection is now on R is now D, this 1 over D; and this is actually d x d y u and v are x y respectively. So, this is, nothing but and so absolute value of this gives you square root 1 plus z x square plus z y square and d x d y is what we had already seen this formula, so this the proof of that result given here.

So, here in this case, this surface area, where is taken x y taken as parameter, this surface area by this formula, and so, this is a function of... So, this here l of z, if you can take like this square root 1 plus z x square plus z y square d x d y. So, another more general functional here, so if you change z, then this number l is going to change, l z is going to change. So, this is a functional defined like this on high dimensional spaces.

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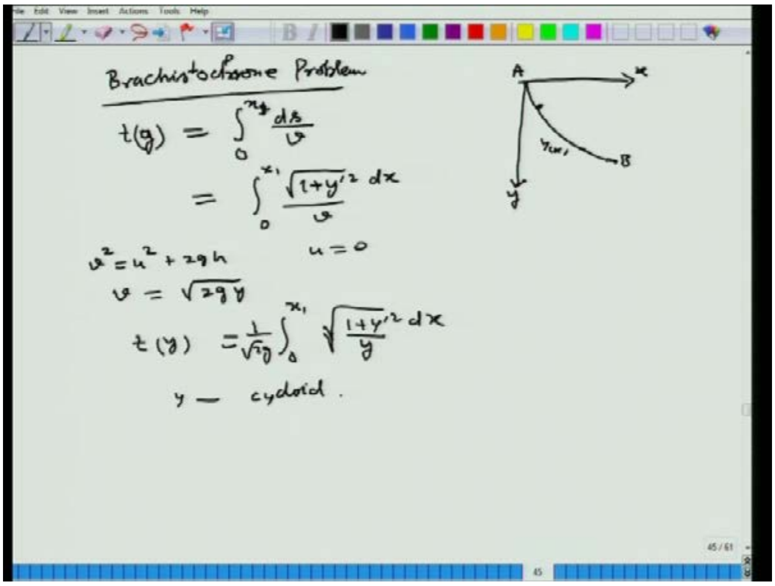

So, I would mention now, these other problems, main problems that first one was mentioned that; which was Brachistochrone problems.

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Which was described earlier, is a in the vertical plane you have to take. So, we take this one point A as the origin itself; and the other point B like this; and here we join this by smooth curve, this y. So, this x axis, this is y axis and we take this point as the shifted to originate itself, and so, this is the curve is y (x) and the point is sliding under gravity. So, we need to find the time, which we have already seen that this time t will be given by which will be a function of y and it is integral x 1 to x 2.

So, here anyway x 1 is 0. So, 0 to let us say, 0 to sum x 1 here, and this will be d s over v, and so, we will have this 0 to x 1 d s we know that is square root 1 plus y prime square d x and v here. Now this v is we know that v square equal to u square plus twice g h. So, u is 0, because it is a dropped from here, there is no force apply here, initially it just at rest.

So, you can be, you will rho, and so, that we can see that using this formula, we see that v, this v is actually root to g y h, height is actually taken of as y. So, substituting it here you see that t y comes out to be 0 to x 1 square root 1 plus y prime square d x over root 1, we can take this constant, how root 2 g and whole like this y. So, this is the functional with for the Brachistochrone problem and we, once we develop sufficient machinery to see that what are those functions, which optimized this, which give the least value, we will see that the the solution of this problem is actually, cycloid. So y terms have to be cycloid, y is actually cycloid, this what will be answered, once we have sufficient tools developed for solving these problems.

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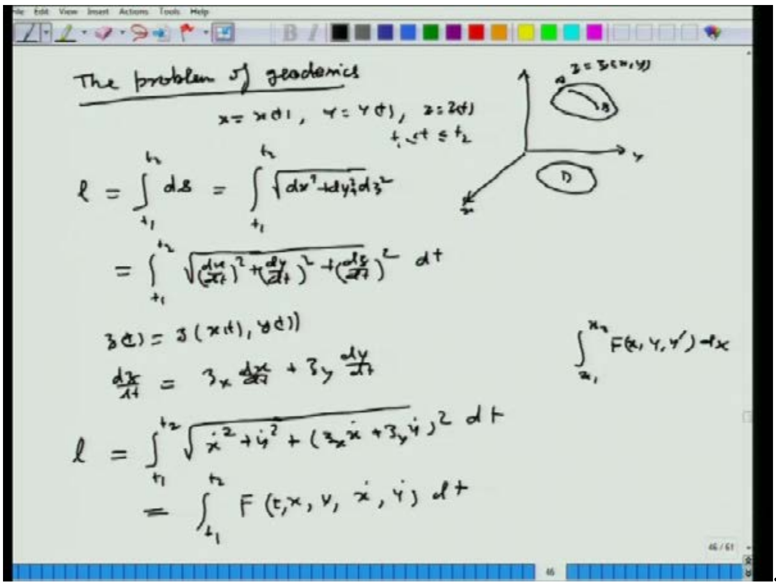

The other one is the problem of geodesics.

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This the generalization of these problem, here in this we have a plane surface here, the plane x y and there are two points and we want to find the minimum length, it joining these two points; and we see that in this case answered will be straight line, but in more the general problem like this, we have for example, if here this surface S is like this projection is D here, in this case and this z is given as function of this (x, y).

Now, here we take two points on the surface A, B; and then the curve joining this on this surface. So, this is the A here, this point B and this is curves join this two points, we want to find the a minimal the the curve, which as minimal length and these curve as to be on the surface. And this should have minimal length; that means, you take any neighboring curve it is length will be more than this one, such curves are called geodesics. So, in the in a plane thing, geodesic are straight line, here like on is spares, these are great circle and so on. We can have various others like cones on that also, there are nice geodesic, which we will be discussing in subsequent lectures.

So here, how do we formulate this as the problem of the calculus of variations? So, here we know that again this length formula will be given by... So, length of the curve, so here on this, we can parameterized this curve like, x of (t) y of (t) and z of (t) like this. So, let say and z as and t as the range between t 1 to t 2. So, that is, what will be parametrical this curve will be described by these three equations; and so then the l will be given by t 1 to t 2 here, d s and that is given t 1 to t 2 square root d x square plus d y square plus d z square.

And so, in terms of parameters t 1 to t 2, so d x by d t square plus d y by d t square plus d z by d t square d t; and then these x, y, z must satisfy this z (t) must be z of (x (t), y (t)) like this; and so, from this d z by d t must be z (x) partial derivative z (x) and then d x by d t z y d y by d t. So, like this. So, here there is a restriction that these points x, y, z must lie on the surface and that will be described by this equations.

And so, we can substitute here, and so, this l can be given as. So, this will be a function of z any way, and so, t 1 to t 2, here we will write short notations like x dot, dot means d x by d t x dot a square plus y dot a square plus z dot is square, this z x x dot plus z y y dot square root d t; and these z is again, so z (x), since z is the function of (x, y). So, z (x) and z (y) are also functions of (x, y).

So, this is like t 1 to t 2 and square root of, so some functional like this, of you have (x, y, x dot, y dot) d t, here t is not appearing explicitly, but in certain t may appear explicitly. So, this is more general case, which we had consider, we had consider only like this, x 1 to x 2 F (x, y, y dash) d x, here there are more dependable variable i, t is independent here. So, we can put t also here, although this is not explicitly function of t.

So, there are t is independent variable, x is a function of t; y is a function of t; and x dot y dot. So, there are more dependent variables here, this is only one independent variable, more dependent variables x and y. So, this is situation, which we will be the dealing with later. So, this is problem of finding geodesics on surfaces.

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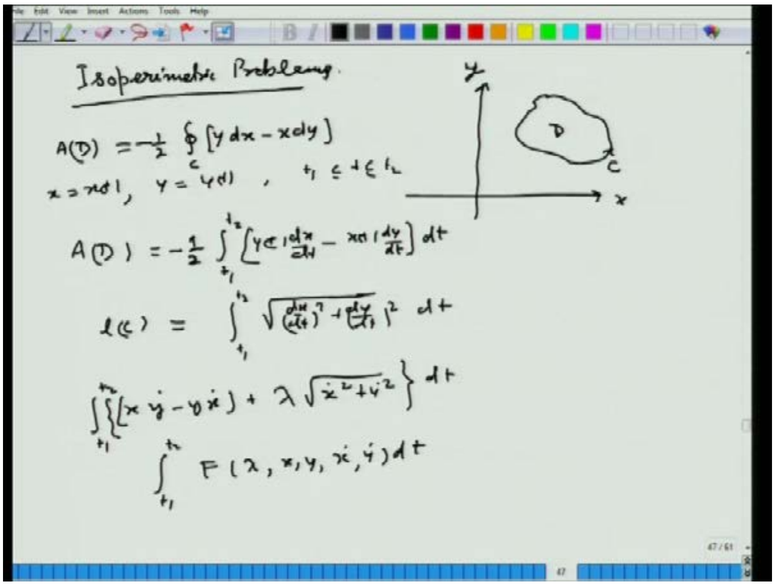

And then the third one is isoperimetric problem, which was mention earlier. So, this was considered ancient time by Greeks, where area as a certain surfaces, where to be found, such that the bound their boundaries are of fixed length. So, that is, what is known as isoperimetric problem? Iso means same, parameter parameter should be same.

So, here you have situation like this, certain curve is given of fixed length; and this is the area here, so this is the domain D; let say, x y-plane and this C is having fix length. So, we need to optimize this area of D, such that the length of C does not change. So, that is what is isoperimetric problem, we know the answer is disk here D has to, so this C has to be circle; and D has to be the inside area of; that means, so disk; and so, that is answer, which we will be getting, when we have sufficient tools to study these problem.

So here, how would you posses, we have seen that; this area can be given Area of D by using that greens theorem result, we can see that it is given by half of integral over C y d x minus x d y like this. So, here and this length is a fixed here C. So, we again take this as a; C as a parametric thing. So, with this, we will if we take x as function of t, y as function of t, then this Area of D is given by 1 by 2 t 1 to t 2 here, this range is y (t) d x by d t minus x (t) d y by d t] is x d y. So, this is sorry this is x d y or minus of this, we can put it here, so minus of this. So, this is actually x d y minus y d x. So, we can put minus sign here like this.

So, here we will have this one and the length of this C curve is given; that is t 1 to t 2 and square root d x by d t whole square plus d y by d t whole square. So, this is what is to be done here; and so, we use here, Lagrange's method of multipliers, and so, we can take the functional like this, t 1 to t 2, here y is dot minus x y dot minus of this. So, let us, put that minus sign inside; so, x y dot minus y x dot plus lambda square root x dot square plus y dot square and d t. So, this will be the functional, which will consider here lambda will be Lagrange's multiplier, it is undetermined, it have to be determined, we need to find this x and y as well as lambda here. So, this can be taken as a functional F here, and again you have only t is not appearing explicitly, F as a function of lambda also is their x, y, x dot, y dot d t. So this kind of functional optimizing.

So, these are three main problems actually created lot of interest in the theory of the calculus of variations; and so here, now what are the tools, which we use in order to see that to answer these questions, what are those curve be optimized, which optimized these functional. So, here we need to find certain conditions, which are known as necessary conditions. It is an Euler's equation, actual known as Euler's conditions, which is to be satisfied by that; that is what we will be deriving in these lectures. And so, let us first understand how we are going to consider the variation of these functions here.

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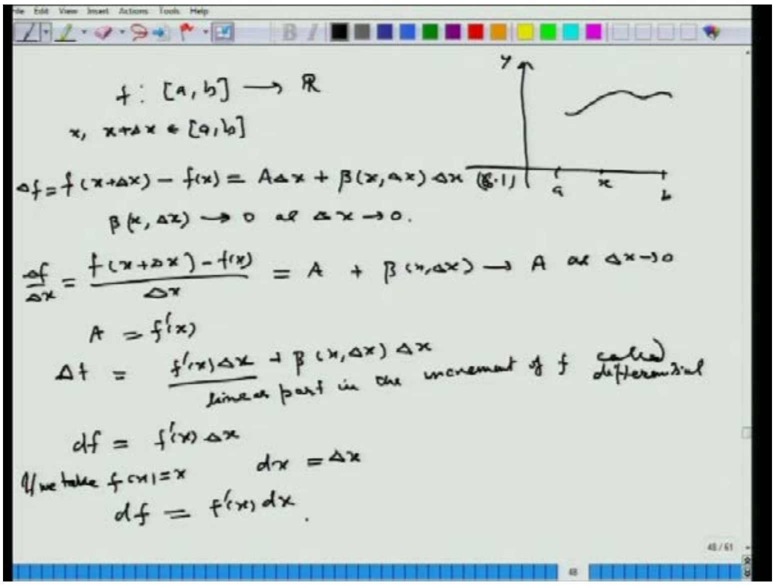

Now first, let us see that; how we do it in case of functions. So here, we have a function f on that interval let us say a, b to R; and so, this case here a, b; and function is like this and around x, we can see that; such that x plus delta x is also in that interval; so, x and x plus delta x being in this a plus a, b in this interval. So, then f of x plus delta x minus f of (x) should be if this function is differentiable, then we says that this has to be like this A delta x plus sum beta is the function of x, delta x delta times delta x.

So here; so, we considered here a function f from a, b to R; and this is situation here, this is a, b and x is an interior point here, and x plus delta is also in that interval. So, delta x is increment in x, if f is differentiable, then we, In fact, f is differentiable if and only the following holes that f of x plus delta x minus f of x is 8 times delta x, A is a constant here, not a function of x or delta x and plus this term beta (x, delta x) delta x.

Here, this beta (x, delta x) this stands to 0; has delta x tends to 0. So, we can see that if we divided by delta x. So, we know that f of x plus delta x minus f of (x) divided by delta x; it gives you A plus and beta (x, delta x) and this goes to A has delta x tends to 0. So, clearly then A has to be f prime x. So, this is the linear part in the increment here, see f (x) is, this delta x is increment in x and this is delta f given by f of x plus delta x minus f (x). So, this increment in f is actually, A times delta x plus beta (x, delta x) delta x; this function beta is assume to satisfied this condition.

So, f is differentiable, if and only if this holds. This is let say (6.1). So, 6.1 holds, if and only if f is differentiable; and we can see that we divide by this delta x here. So, this delta f by delta x is equal to this; and this tends to A as delta tends 0. Therefore, we know that this is used for derivative a prime; so, follows that A has prime x; and so, here this part is called linear part in the increment, because it is the linear in delta x.

So, and that... So, this A or rather; so, therefore, this delta f is actually a prime (x) delta x plus beta (x, delta x) delta x, here this is the linear part in the increment in the increment of f, and so, this is what called differential, and so, this called differential and we denoted d f. So, d f is actually f prime (x) delta x, this is differential of f and this clearly differential coefficient; it is coefficient of the differential delta x here, because now if we take f x equal to x, you known that d x equal to delta x; and so, we can replace this, if we take this, then d x is equal to delta x for this independent variable, this increment is same thing as differential, and so, d f equal to f prime (x) d x. So, that is, what we frequently use that; we just say that dividing by d x, we get actually, which is Layman's language of saying that we divided by d x and we get a prime x.

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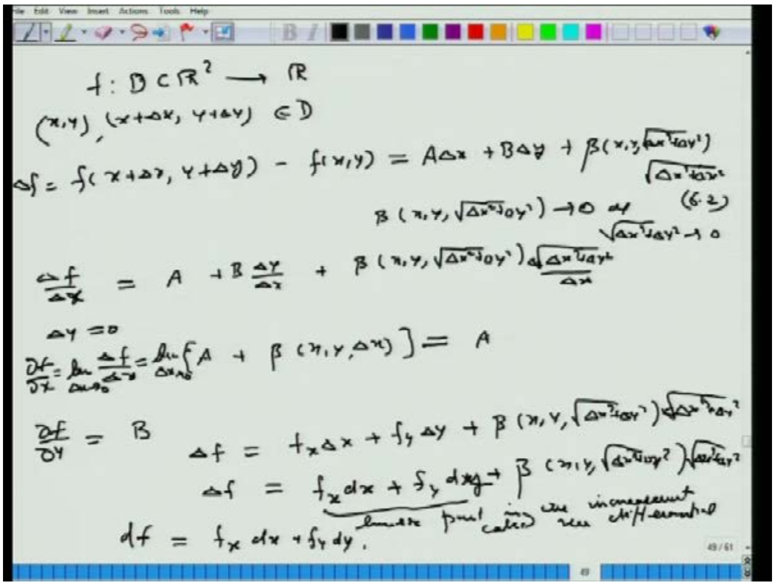

So, but in the case of two variable, what happens that if you have f as a function of R 2 or rather a domain in R 2 R and then, we can take any point (x, y) in (x plus delta x), (y plus delta y) these points in D, then f of x plus delta x, so delta f here again, y plus delta y minus f of (x, y) is given by in the same manner, it will be A delta x plus b delta y plus beta x, y, square root of delta x square plus delta y square times square root of delta x square plus beta y square; such that this beta x, y, square root delta x square plus delta y square goes to 0, as square root is delta x square plus delta y square goes to 0.

So, we say that f is differentiable, if the following happens that this let say, this is (6.2). So, (6.2) holds if and only if f is an differentiable; and we can again see that; if we divide by delta x and fix delta y, do not, we do not change delta y, only delta x is changing an delta x tends to 0. Then if we considered this like this delta x, then we have A plus B delta y delta x plus beta x, y, square root delta x square plus delta y square, and here times square root delta x square plus delta y square delta x.

So, this goes to see, we take let us say, we take delta y to be 0. So, take delta y to be 0, then we see that delta f over delta x is A plus beta x, y, delta x, and then like this and you have this goes to 0, this goes to A as delta x tends to 0, which is the particular case, because delta y, we have taken 0 here. So, this condition will be satisfied; and so, this is what is actually. So, A now this is, nothing but del f by del x, it partial A has to be then this goes to in the limit limit of delta x tending to 0. So, let me put it here, in limit delta x tending to 0 of this. So, this is actually equal to A.

Similarly, we have delta f by delta y will turn out to be B. So, we can see that here, this delta f is actually f x partial derivatives delta x plus f y delta y plus beta x, y, square root x square plus delta y square square root delta x square plus delta y square; and we have this as delta x is same thing as d x plus f y d y plus beta x, y, square root delta x square plus delta y square square root delta x square plus delta y square.

And so, this is the linear part of increment, linear part in the increment because this is linear in this delta x, which is, nothing but d x delta y, these are independent variables and therefore, delta x will be same thing d x, which we have seen and delta y will be d y here, and so, this d f will be given. So, that linear part is called the differential, called the differential and so, d f is given by f x d x plus f y d y. So, that is the situation for functions.

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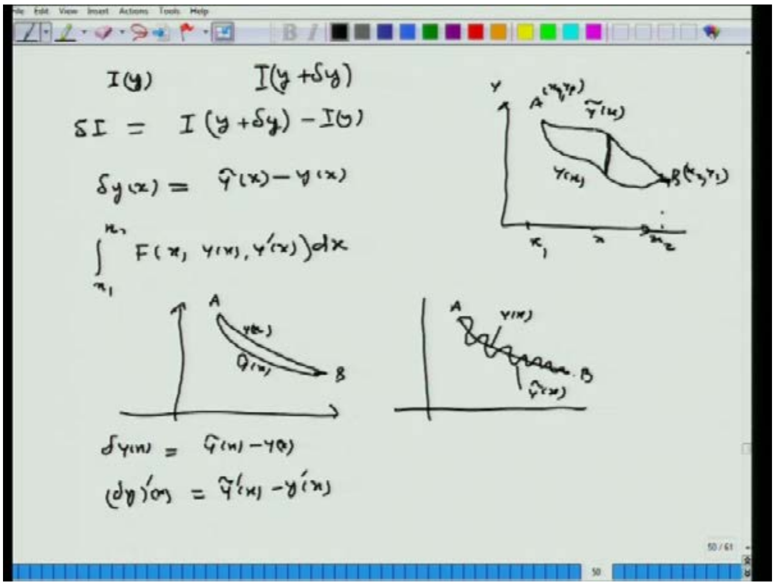

Now, let us see in the case of functional. So, we have I (y) here, now we give increment to like we give increment to x, we give increment we give increment to y, and what happens to the increment to I. So, this is the functional I here. So, if we consider this y plus delta y. So, what is that delta y we mean and what is the functional this, and then delta I will be given by I y plus delta y like this minus I (y). So, how do we defined this that is what is to been seen, what is this delta y here.

So, what we do here, let us say, we have and there are these two points A and B; and this curve y (x) and this is neighboring curve y 1 (x). So, these y (x) and y 1 (x) these are joining these two points A and B. So, this is the point (x 0, y 0) or (x 1, y 1) and this point (x 2, y 2) and these are two functions y (x) and let say, is not y 1 here; let say, this is a y tilde the neighboring curve here.

So, this delta y as a function of (x) is a difference this y tilde (x) minus y (x). So, at each point here this x 1, this x 2 here, and so at this point x difference here, y tilde (x) is here, and y (x) is here. So, this is the, this difference of these ordinates is what is delta y (x). So, as x moves from here to here, we get this various value of delta y. So, this is the increment in y; and now, what happens this I, which will be a functions of (y plus delta y) now here, I of (y plus delta y) and minus I (y). So, that is what will be denoted as the increment in the function or variation variation in the function.

In the same manner, this situation here; so, how do we defined the here, they actually thinks here, like we had x going to x plus delta x the differential delta x, now here we are seen that delta y is there, we are seeing that this y tilde (x) is near to this in what sense that is what is to be made presides here, because this functional because this functional x 1 to x 2 here, you have F x, y (x) and y prime x. So, it involves y prime (x) also y (x) as well as y prime (x).

So, what is the variation in y prime (x) that is to be seen for example, if we take this situation that this point A and B, now these two curves like this, if you consider or the other one curve is like this; and the other curve is going like this. So, all though this is also close to the let us say, this is the original of y (x) this is and this is y tilde (x). So, y (x) is straight one, this one is y (x) and the curved one is y tilde (x). So, we see that though this delta y (x) is y tilde (x) minus y (x) here; and in this case, also we will have delta y (x) will be same thing here; and so we see that here the variation will be too much in the derivate. So, here all though, this if we defined this (delta y) prime (x) will be y prime (x) minus y tilde prime (x) minus y prime (x).

So, here in this case both this will be (( )). So, if this less then epsilon let say, then this will have similar to the epsilon distance, but here this thing will be difference will be too much in the case, when we take derivates here. So, we need to define the closeness in the sense of order of proximity. So, that is what we will be constant now.

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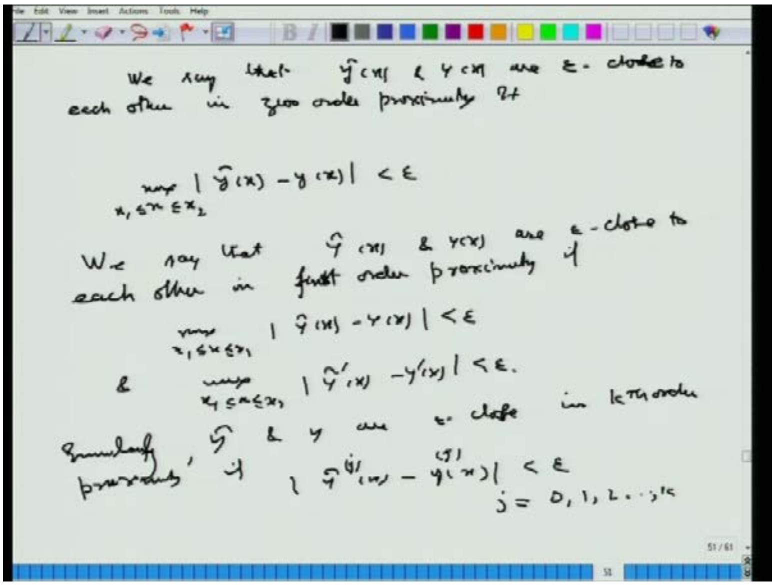

So, we defined here the concept of proximity of order k, we say that we say that y tilde (x) close to each other in the zero order proximity, if for every epsilon greater than 0 there exist delta greater than 0; such that if x minus x dash. So, here this will write. So, this y (x), absolute value of this y tilde (x) minus y (x) less than epsilon, so y tilde here. So, this is called epsilon order of proximity before (( )). So, y (x) is close to epsilon is to epsilon, close to each other. So, each order 0 order, if this, if we have the following for all x here.

So, this rather, we can see maximum, where x less than x 1 less than x 2 is less then epsilon, now we generalize this, we say that y tilde (x) and y (x) are epsilon close to each other; in first one is order proximity, first order proximity, if maximum of x 1 less x less than x 2 is y tilde (x) minus y (x) less than x 1; also an maximum x 1 less than x 2, y the prime (x) minus y prime (x) is also epsilon. So, this can be generalized and k order proximity similarly, y tilde and y are epsilon close in k th order proximity, if this y tilde j th derivative (x) minus y j derivative (x) is less than epsilon, for j equal to 0, 1, 2 up to k.

Zero order, zero means then the function itself. So, that is what. So, there are not only close like this, see these are only zero order close; not the first order, whereas this is first order close first order proximity of this curves y then y tilde. So, like this, we will be considering the continue various concept continuity and differentiability, as we considered this for functions, we will be defining the variation of functional in the next lecture.

Thank you very much for being this.