**Calculus of Variations and Integral Equations**

**Prof. Dhirendra Bahuguna**
**Department of Mathematics and Statics**
**Indian Institute of Technology, Kanpur**

**Module No. # 01**
**Lecture No. # 05**

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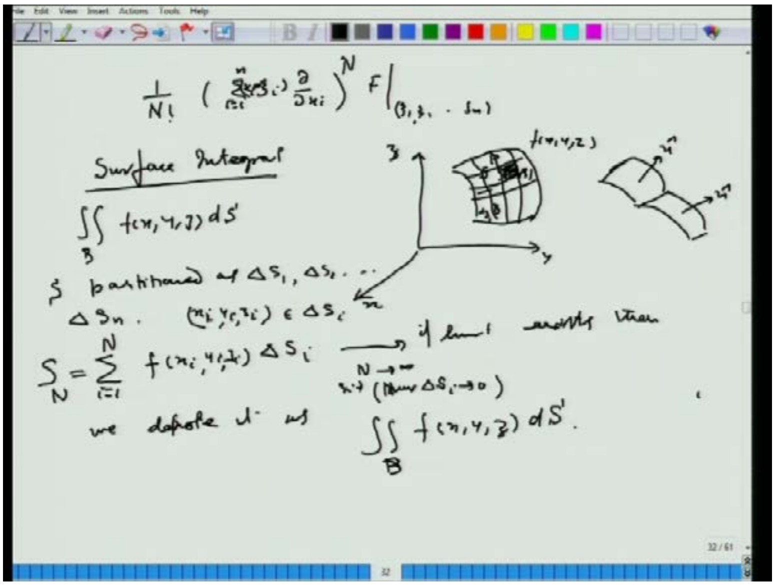


Welcome viewers. The NPTEL lecture series on the calculus of variations; this is the fifth lecture of the series. In the last lecture, we discussed the concept of surface integral where surface b is in three dimension space x, y, z and a function is defined on the surface, that is f x y z for each point p which is whose coordinates are x, y, z. This point is on the surface, and the values of this function defined on all points on the surface, and it on its boundary b is the surface, and delta b is its boundary; this function is assumed to be continuous on the surface and on its boundary also.

So, then the surface integral double, integral over b f x, y, z d S, this is capital S and not the arc length is small s; here d S is the element area on the surface, the typical surface element is like this, shaded darkened curved square or curved rectangle. This surface is partitioned into these element areas d S delta S 1, delta S 2 and so on up to delta S n, and x i, y i, z i, is a point in this delta S i, then the integral is defined as the limit of the sum summation i equal to 1 to capital N, when N is the number of these element surface areas; this number of elements partitioning the whole surface, and then we take this summation f of xi, y i, z i, where x i, y i, z i is the point in that element area delta S i, times the area of that element that is delta S i; and if this limit exists, we define it as the integral, double integral over b f x, y, z d S.

Here, this integral will be defined. It will not been depending on the way we partition it, for any partition when we pass to the limit, it should give you the same value. So, under these sufficient conditions like, f is continuous and surface, this surface is piecewise smooth, and its boundary is piecewise continuously differentiable, then this limit does not depend in the manner we partition it; and so the integral will be defined.

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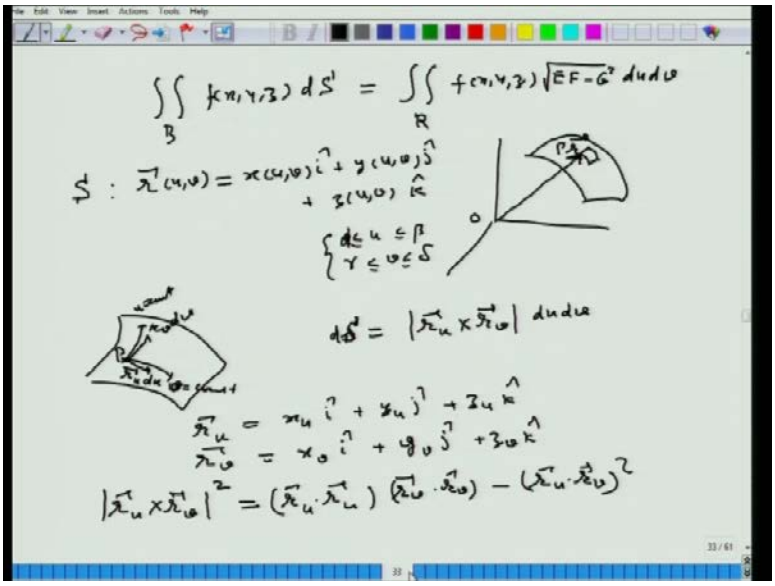

Here, how we calculate this surface area. We as we know, we have some experience of calculating on the flats planes; the functions which are defined in certain domain like this. So, here x i in the x y plane, we have this flat surface area d and a function z is defined on this, and we have seen that this surface area is given by this formula. This also is a particular case of the more general one which we are going to consider.

So, here we have for any point p on this surface, like this. Here, this surface is parameterized by two parameters u and v. So, position vector of a point p that is the director vector o, p - starting from o and ending at p. So, here here, it will have three components x, y, z. So, each component is a function of u, v; those parameters and so, we get x u v, y u v, z u v like that, these are the components. So, the vector r will be given by x u, v i plus y u comma v j plus z u comma v k. And these u, v are within the range of these bounded intervals alpha, beta and gamma delta.

So, here then this element area d S; we have seen that this comes out to be the cross, absolute value of the cross product r u cross r v d u d v. Because r u d d u is the tangent on the curve v equal to constant, and r v d v is the tangent element on the curve u equal to constant. So, this curved surface element like this darkened one, will be actually given by, which is d S equal to absolute value of the cross product r u cross r v; and since d u d v is positive it comes out. So, that is the element area d S and. So, here r u is the partial derivative with respect to the parameter u, and r v is the partial derivative of r position vector a with respect to v; so each component gets differentiated partially with respect to those variables, and we get this and so, r u cross r v square.

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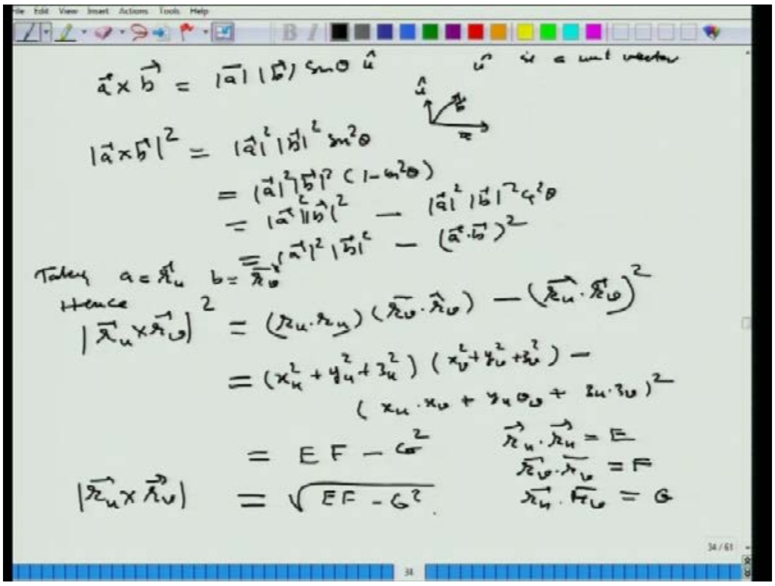

Here we use the formula, which we proved that for any 2 vectors a and b in r 3, we see that a cross b is given by the absolute value of a cross b square is given by, absolute value of a square times absolute value of b square minus a dot b; this is the dot product of 2 vectors that is a 1, b 1, a 2, b 2, a 3, b 3 where a i's and b i's are the components of a and b respectively, the square of that.

So, we get this r u cross r v absolute value square, like this and so, this r u dot r u is E, that is how we define that r u dot r u is E, and r v dot r v; this is F and r u dot r v, this is G.

So, this absolute value of r u cross r v square comes out to be E F minus g square. So, therefore, absolute value of r u cross r v is square root of E F minus g square; we take here positive square root, and that is what we have here, and so, our formula here; this integral comes out to be in terms of, now these are x y u, x y z are functions of u, v here. And this we have calculated that d S equal to square root E F minus g square d u d v. So, that is nothing but this integration over the flat surface d u d v, which we already know how to calculate.

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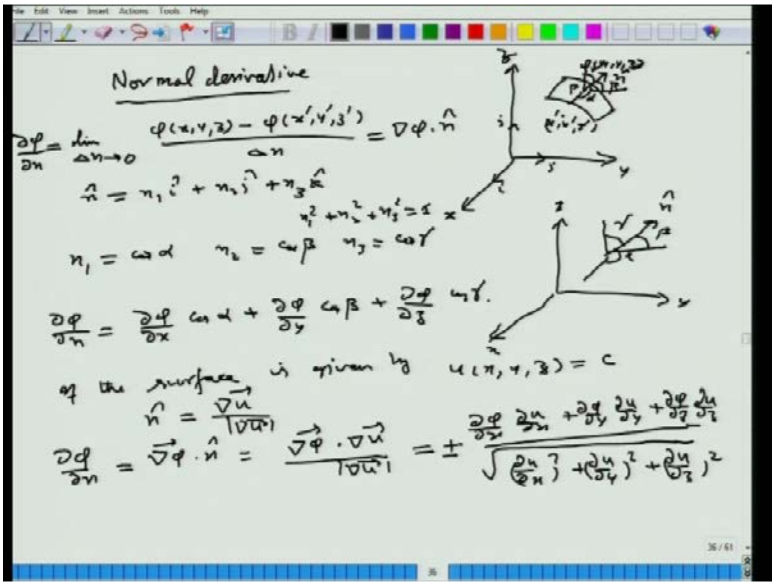


So, that is the surface integral formula in terms of a parametric representation u v, and then also we have Normal general Normal derivative in three dimension. As we have defined the Normal derivative of any function here. So, we have some surface here like this, and some function phi is defined on this. So, phi x, y, z which is defined at each point on this surface. So, Normal derivative of this phi is. So, Normal derivative of phi which was defined like, we defined it into two dimension, we define it limit of delta and ending to 0 of phi x, y, z minus phi x prime, y prime, z prime. Here is a part of the surface; the close surface like this, where here this is x dash, y dash, z dash inside the domain and the point x, y, z is on the surface.

So, here it is assumed that phi is defined at those interior points x dash, y dash, z dash; so, this limit over delta N, and we as before we have seen it in two dimensions it turns out to be gradient of phi dot N cap. So, where N cap is the unit Normal this N cap, N cap is the unit Normal which is N 1 i plus N 2 j plus N 3 k; i, j, k are unit vectors in x y z directions respectively. And we see that this unit vector therefore, N 1 square plus N 2 square plus N 3 square; this should be equal to 1. These are direction cosines, they are also given like this; if this Normal makes this angle alpha, beta, and gamma. So, here is like, I will blow it up in this way; is with x axis, is x, y and z. So, this z the angle is gamma with this x axis alpha, and this beta.

So, N 1 is actually cos alpha, N 2 is cos beta, N 3 is cos gamma. Where this N cap is the unit vector; here, this is Normal to the surface and it makes angles alpha, beta, gamma with the coordinate axis, and so, these are the direction cosines given by cos alpha, cos beta, cos gamma. So, we can see that this can be calculated, and so this is also the another notation of this is del phi over del n. So, we can see that del phi over del N is del phi over del x; gradient vector is del phi over del x, and then a first component is N 1. So, that is cos alpha plus del phi over del y cos beta plus del phi over del z cos gamma.

So, this another convenient formula for this. If in terms of direction races, we have a certain numbers here, small a, small b, small C then we divide by it is a square root summation a a square plus b square plus C square. So, that is also can be seen. Here, if let us say this surface; if this surface is given by z equal to some function z, x, y; then we know that the normal. So, in that or in terms of u x, y, z let us say, if you give u x, y, z equal to some constant c; if you can solve it for z explicitly, we can express this z as a function of x y also. So, then we know that the Normal will be gradient of u over. So this, in this case then N cap comes out to be gradient of u over absolute value of the gradient of u, and then we get the directional derivative. So, del phi over del N is we have gradient of phi dot N cap. So, that will be gradient of phi dot gradient of u over gradient of u.

So, this in component form, this is del phi by del x del u by del x plus del phi by del y del u by del y plus del phi by del z del u by del z over square with plus minus sign, square root here del u by del x square plus del u by del y square plus del u by del z square.

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So, plus sign or minus sign, we have to see that which side we are going to take so that, cosine of the angle should be positive that, and so, that we have to choose either plus sign or minus sign accordingly. If we choose plus sign if del u by del phi by del N, this positive; then we choose plus sign and if this is negative then we choose minus sign.

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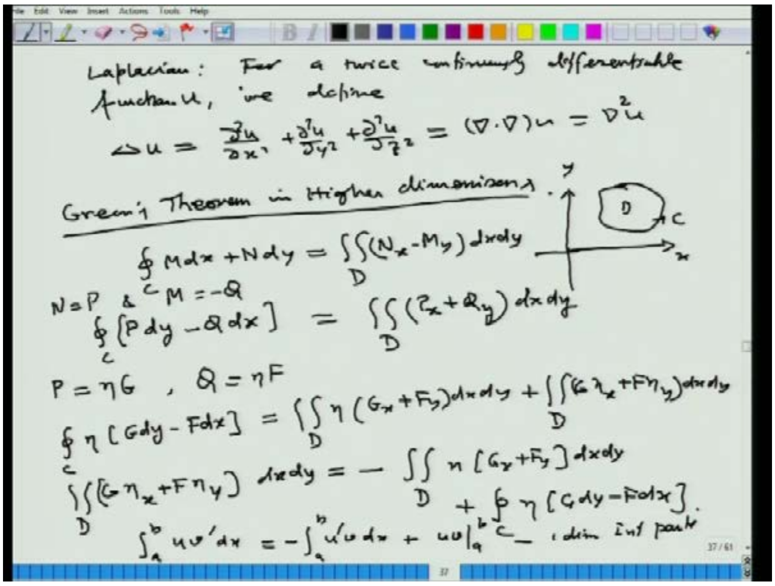

Now, here the Laplacian will be defined Laplace operator or Laplacian is actually defined as, the Laplacian of for twice continuously, differentiable function v can be define, we define this Laplacian that of u continuous function u, we define Laplacian of u as del 2 u by del x 2 plus del 2 u by del y 2 plus del 2 u by del z 2; which is nothing but gradient of gradient del dot del like this, of u or in short it is written like del del square u; this another notation for the Laplacian. Now, we go to green's theorem in higher dimension. Firstly, we have seen that green's theorem.

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In higher dimension now we will see. So, recall in two dimension, we had seen that this; if you have this case like in x y, if you have some certain domain like this. And here, this boundary of this is let us say c, and its positive direction is taken like this anticlockwise, and here we have seen that if M and N are smooth functions here; u C that this M d x plus N d y. This was equal to over d N x minus M y d x d y. This we had established in the earlier lecture; that were M and r continuously differentiable in d, and the continuous on a piecewise continuous on the boundary, and C is piecewise smooth, then we have this result which is green's theorem in two dimension. Now, we can generalize this in higher dimension. So, first let us see certain applications in two dimension itself, so we have, we can write this in a slightly different form like this. So, if we take M equal to like this; so, we can write this as N equal to p, let say N equal to p; and M equal to minus Q. Then we get p d y minus Q d x, and this line integral will be then equal to o d, we get p x plus Q y d x d y. So, just if moving this minus sign from here, and then that minus sign will come here.

Now, we can... So, this is a I mean just same form with different choices of M and N. So, now, we take p equal to some eta G, and Q equal to eta F; here this eta G N and F they are continuously differentiable functions, and its on D and continuous piecewise continuous on the boundary of D, that is on C and C is of course, assume to be piecewise smooth; that means, that each point the tangent is well defined tangent, Normal or defined except at finitely many points. So, there... So, putting this P and Q here, we get the following which is on this side we get eta. So, put this P equal to eta G; So, we get eta G d y minus F d x equal to here. So, we will have two terms here, this double integral, because there are 2 functions here. So, x derivative will give you eta G x plus eta x G.

So, like that we have and collecting the since here, you get eta G x plus F y d x d y; here this these are all partial derivative, as sub scripted means the partial derivative plus double integral over D eta. So, G eta x plus F eta y d x d y. And so, we take one term on the other side. So, we will get like this, here. So, we have. So, we can write this over D, like this G eta x plus F eta y d x d y, we take this on the other side to get minus D eta G x plus F y d x d y plus this boundary integral eta G d y minus F d x. So, this is the general form of integration by parts; so, greens theorem gives us the general way of the same formula of which we had in one-dimensional recall, that integration by parts formula was a to b, u v prime d x is minus u prime v d x plus a to b and u v evaluated at a to b.

So, this is one-dimensional. So, this is one-dimensional integration by parts; and so, this is two-dimensional integration by parts. So, this is generalization of the earlier integration by parts formula which we discussed. Here, we can see that, here the x derivatives there on eta, y derivatives there on eta, and G, N, F are not having any derivatives here. So, if you shift these derivatives on G, we get a minus sign here like we got here. So, we got minus sign and eta becomes free of derivatives and this x derivative get shifted on to G; similarly, y derivative get shifted onto F like this, and this eta becomes free of any derivative we gain minus sign here. So, this is the generalization of integration by parts formula of one-dimensional. So, this is two-dimensional integration by parts formula.

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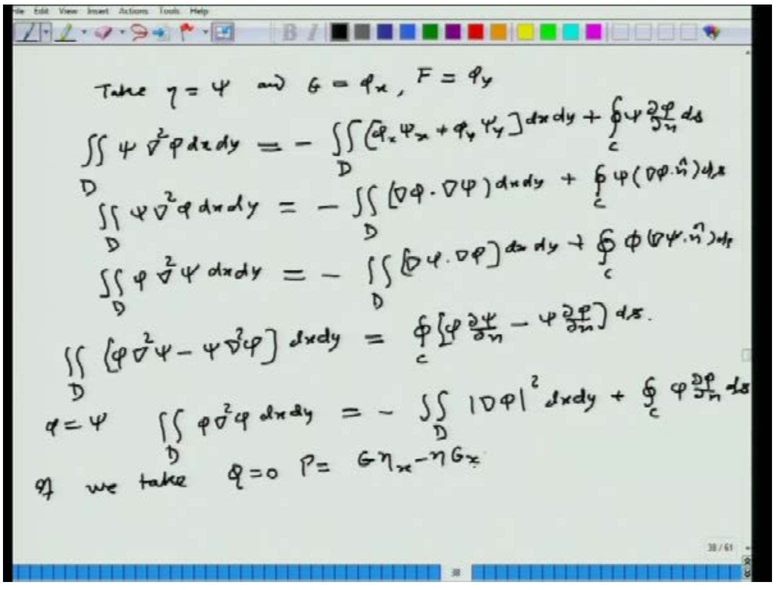

Similarly, we can have higher dimensional integration by parts formula. So, we can repeat this process like this, if we take. So, take now eta equal to psi, and G equal to phi x, and F equal to phi y. So, here in this we will take eta equal to psi and g equal to phi x and F equal to phi y. So, we get this thing here, and here these x derivative will give you G x x like that; So we get finally, psi del S square phi d x d y equal to minus, this is phi x psi x plus phi y psi y d x d y plus the boundary integral psi del phi by del n d s. Here, this s is the arc length. So, here we will see that, here if you put eta equal to eta equal to psi here, and g equal to phi x and F equal to phi y, we get from this; here we get g x will give you phi x x and this will give you phi y y; so that is the Laplacian two-dimensional Laplacian here, each of eta equal to psi, eta equal to psi and you get Laplacian there and then in this term here, we get G equal to phi x. So, this is eta x mean psi x and F is phi y and eta is eta y psi y, so we get this term here with minus sign.

This can also be written slightly differently like this, and this is the dot product of the gradient of phi dot gradient of psi. And here this is; obviously, this is we know that this is gradient of phi dot N cap d S. So, this is the in a vector form we have this, and if we interchange the phi and psi we get phi here, and del square psi d x d y this remains the same, because it is a symmetric. So, we get gradient of psi dot gradient of phi d x d y plus over c.

So, psi here sorry this will be phi now, and this will be psi dot N cap d S. So, subtracting this we get the following greens identity, that phi this we can subtract from this. So, phi del square psi minus psi del square phi d x d y; these two terms cancel each other, and so, we get this boundary integral phi, we either write like this or we can write del S phi del n form also. So, this is phi del psi by del n minus psi del phi by del n. So, this is greens; one of the greens identities expressed like this. And if we write phi equal to psi here, we can see that; then we will have here del square this dot product will give you square of these facing. So, if we write phi equal to psi, in this if you write phi equal to psi. So, we will have this phi del square phi d x d y will be minus D mod del phi square plus phi del phi by del n d x.

So, there are various forms of identities obtain like this, which can be by proper choice of M and N or in F and G, we can see various forms of greens identities. We can go to higher order integration by parts also if we take. So, if you take this Q equal to 0, and p equal to G phi x minus sorry G eta x eta x minus eta G x. Then we get. So, we substitute here P equal to P equal to G eta x minus eta G x and Q equal to 0 here.

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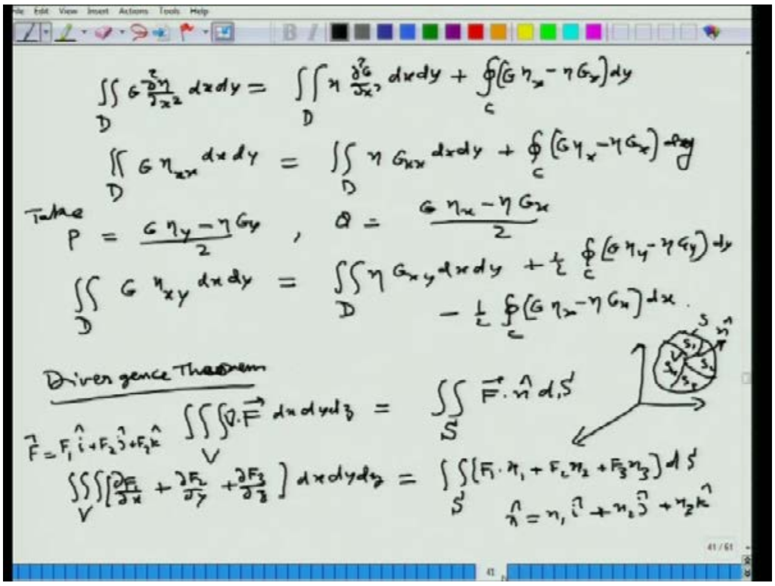

So, you will get the following thing; integration over D we get G del 2 eta by del x 2 now, d x d y and this is over D eta del 2 G over del x 2 d x d y plus the boundary integral C over C G eta x minus eta G x d y. Because we have chosen Q equal to 0 here, Q equal to 0. So, we get only see we have taken Q equal to 0 here. So, this term will not be there this. We will have only p d y and p is chosen like this, G eta x minus eta G x. So, we get at the same thing here, and we get... So, this is what is now higher order, I mean higher order derived integration by parts here. Because now, 2 derivatives are being shifted on G; here eta x x we can write it like this also in short form G eta x x d x d y; this is over D eta G x x d x d y and plus G eat x minus eta G x d y.

So, here this x x first we shift one derivatives, we give 1 minus sign; and then again we shift x derivative, we get those x x derivatives on G and eta eta becomes free. So, this is shifting of derivatives; here, because of 2 derivatives are being shifted, we get plus sign here, and on the boundary we get this term. So that is the integration by parts formula for second order derivatives. Similarly, we can take now, P equal to something and Q equal to 0 So, than we will have P equal to G eta y minus eta G y by 2 and Q equal to G eta x minus eta G x by 2. So, then we get for mixed integral, mix derivatives thus eta x y d x d y. So, this is now, these this these derivatives are shifted onto this. So, eta becomes free and G x y d x d y over D, and you get those boundary terms 1 by 2 C G eta y minus eta g y d y plus 1 by 2; sorry minus minus sign here, minus 1 by 2 G eta x minus eta G x.

So, this is for the next derivative. Now, we go to three dimension case and we will have the following result. So, here we use the divergence theorem which is…

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What it says, here this for given any bounded volume v here, and f. So, here divergence means del dot, here del dot F d x d y d z is actually equal to the surface integral or the surface of this v, F dot N cap d S, here we have the following picture. So, certain bounded volume V is enclosed by the surface S. So, this surfaces is S at any point the outward normal is given by N cap as usual N, the points are there in the domain in V. So, the this is what is the divergence theorem. In component form we get this; so, F S components F 1, F 2, F 3; F 1 i plus F 2 j plus F 3 k. So, then we get here F 1 del F 1 over del x plus del F 2 over del y plus del F 3 over del z; that is the divergence of F, because del operator is a del x i plus del y j plus del j k. So, dot product gives you this. d x d y d z; here, F these component phi are assuming to be continuously differentiable in V, and continuous on the surface S - And S is piecewise smooth that is the on those patches, whereas S is smooth and is well defined that is continuous function, at every point except on certain boundary of those patches; like it could be a like this.

So, in these patches let say, S 1, S 2, S 3 and S 4; this N is a continuously defined on those patches S 1, S 2. So, here this side surface integral side, we get F 1 N 1 plus F 2 N 2 plus sorry these are scalars components. N 2 plus F 3 N 3 d S. Here, N cap as usual have components N 1 i plus N 2 j plus N 3 k; this is unit normal.

So, those are defined those cosines direction cosines of the normal here. So, how do we see this theorem, here we will component wise we show that, this is integral of this del F 1 del x is F 1 N 1 d S and so on. So, we show let us do it for the 1 the last term.

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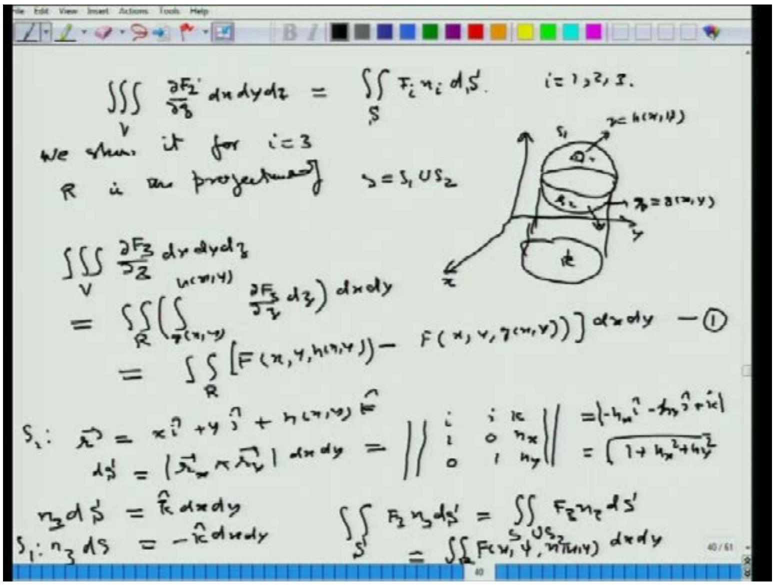

So, last term we will show that this, over V del F 3 over del z d x d y d z equal to this over S; this is F 3 n 3 d S. So, like that we have. So, are what we are to show that like this, that F i, this i equal to 1, 2, 3. So, we show it for i equal to 3. So, what we will have, this way again will do it for a simple case like this that here, we will take the surface S parameterized by the x y here coordinates, and So, this the surface S is projected on to this R here, this is the projection of this, and let say surface is like this. It take the simple case that surface has 2 parts - the upper part, the lower part we write it as z equal to g x y, and the upper part z equal to h x y.

So, assuming that the surface can be represented like this, and this R is the projection of this surface; here this is the projection of this. And so, R is the projection of projections of, of both the parts let us say this is S 1, and this is part is S 2 So, S is S 1 union S 2. So, then we can see that this triple integral over V del F 3 over del z d x d y d z, here will be like this over R, and then here z component will be g x y to h x y of del F 3 over del z d z and d x d y. And so, it will be over R here, F 3. So, z differentiation and integration will cancel each other. So, we will get the boundary terms x y and h x y minus F evaluated at x, y, g x, y, d x d y.

So, here we can see that, this is what we will see that this is actually equal to the right hand side of that. So, here we see that if we write this parametric representation like this x y. So, then the for the upper surface, we see that R comes out for any point P here, position vector R will be given by x i plus y j plus z component is there x on S 2; h x y k and so, this element area here, will be so, d S here; will be r x cross r y, absolute value d x d y, and we can see that this is nothing but so, r x... So, absolute value of their determinant like i, j, k and here, you have 1 r x is 1 and this is 0 and this is h x here and 0, 1 h y here, absolute value of that. So, we get here minus h x i.

So, let us absolute value of that minus h x i minus g h y j and plus k. So, we here this square root of 1 plus h x square plus h y square, that is what we will get here. And so, here we see that this is the, here normal this cos gamma will be positive, and on the lower part we will see that; here we get. So, this side is nothing but S of F 3 dot N 3 d S, we have seen that, here let us first seen that what is this actually element area. See here, so N 3 d S; this is the vector form of this element area will be actually equal to, N 3 is k d S will come out to be simply d x d y. So, you will have this N 3 d S will be like this, because they component N 3 component is just 1 here. So, N 3... So, N 3 component is the k here. So, we get this.

Now, on the lower part we get N 3 d S on S 1; we get this as minus, because here you will have in the negative direction. So, will, but minus k d x d y. So, we can see that, here this is you get minus sign on this. So, we get finally, the same thing here F x y. So, this dot product n 3 will be 1 here, n 3 will be… n 3 is a k component. So, it surplus value will be plus 1 here, and here you because of this you get minus. So, this minus sign will be adjusted and so, and so, this is one.

So, we get from this integral surface is F 3 n 3 d S S, F x y on the lower S as the union S 1, S 2, so on S 2 we get positive one. So, that is h x y. So, let us first like this; this is S 1 union S 2. So, F 3 N 3 d S.

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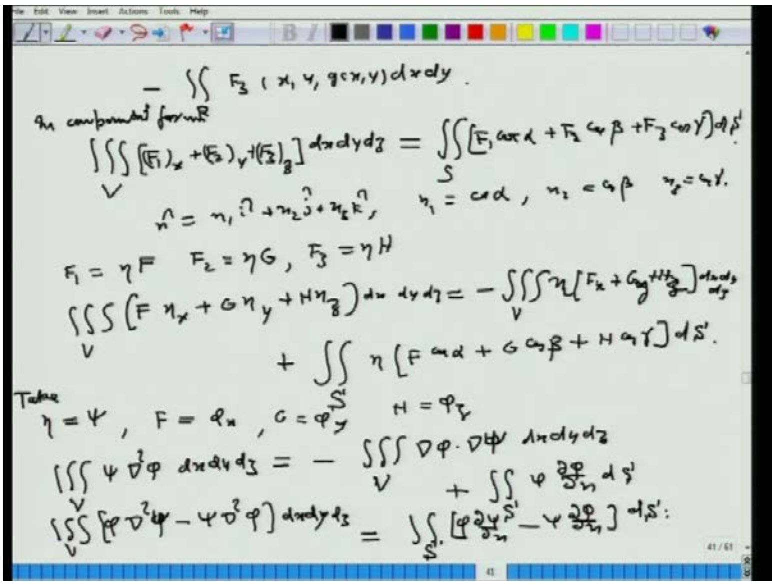

So, first on S 2 that is with positive sign F x y h x y and d x d y, because dot product here will have this 1 and so, d x d y and over this region R minus minus over this region R again, and F 3 x y g x y d x d y. So, we see that it matches with the quantity here for each i similarly, we can do for other components, and so we get overall by adding all the terms we get the divergence theorem.

Now, So, here in the component form, it can be written like this. As you have seen already them. So, in component form, we have this that over triple integral V, we get F x plus F y plus F z d x d y d z; we get over this S F 1 that is or we can write it in cosine form cos alpha plus F 2 cause beta plus F 3 cos gamma d S. Where N 1 is cos alpha as usual, N 2 is cos beta N 3 is cos gamma. N has components, N 1 i plus N 2 j plus N 3 k.

So, we have this; now, if we choose as before we choose F 1 sorry, this was F 1 F 1 x, F 2 the components F 3 y. So, F 1, F 2, F 3; let me write it neatly. This is F 1 x plus F 2 y plus F 3 z.

Let the divergence of F here. And so, we get these from the divergence theorem; now, if we choose F 1 equal to let us say eta F F 2 equal to eta G plus F 3 sorry comma F 3 equal to eta h; where again eta F, G, N, H are assume to be sufficiently smooth; and so, we kept these over V, here F eta x plus G eta y plus H eta z d x d y d z, and we get one more term here, we take it on the other side with minus sign now; and so, we will have over V eta F x plus G x plus G y sorry plus H z d x d y d z of that boundary term is now surface here. So, we get surface integral over the surface S eta, and then you have F cos alpha plus G cos beta plus H cos gamma d S.

So, you can see that, this is now three-dimensional integration by parts; here, derivatives on eta. So, here eta x, eta y, eta z; now, we shift these derivatives on F, G, N, H. We get this minus sign here, and so, we get F x plus G y plus h x here, and eta becomes free of derivatives and here, this is the boundary term; here, boundary is the surface. So, we get this.

And if we put, eta equal to take eta equal to psi, and F equal to phi x, G equal to phi y, and H equal to phi z; we get this high order integration by parts. So, psi del S square phi d x, d y, d z minus, here this is gradient of phi - dot gradient of psi; this will have 3 components here phi x, phi y, phi z gradient of psi d x, d y, d z and plus this boundary term over S psi del phi by del N, which is now three-dimensional direction normal derivative d S.

If we interchange the phi and psi, and then subtract we get the similar formula which we had earlier. So, this over V of phi del square psi minus psi del square phi d x, d y, d z will be, this term will cancel and we will have only the boundary term phi del psi by del m minus psi del phi by del n d S. So, the more general form of greens theorem which is actually consequence of the divergence theorem. So, with these preliminaries whatever things are required, we have discussed here in all these lectures, next time on in we will actually start the our lectures on the calculus of variation, we introduce certain concepts again; and then we will start the Euler's conditions and various concepts related to the calculus of variations. Thank you very much for viewing the lecture.