Calculus of Variations and Integral Equations

Prof: Dhirendra Bahuguna

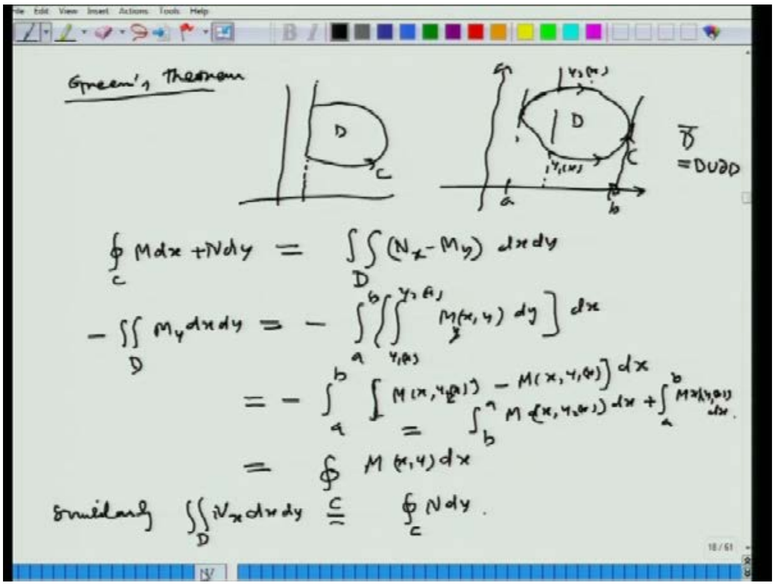
Department of Mathematics and Statics

Indian Institute of Technology, Kanpur

Lecture #04

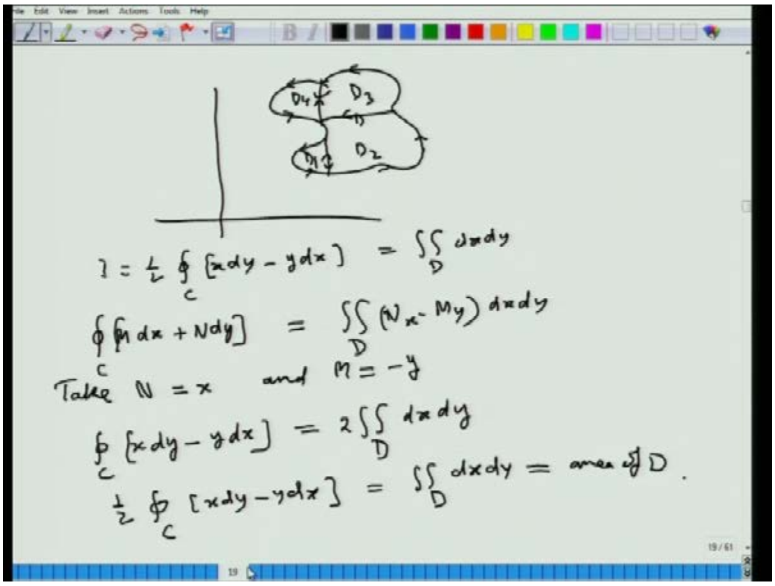
Welcome viewers to the lecture series of the calculus of variations, is the fourth lecture of the series. Let us recall what we did in the last lecture.

(Refer Slide Time: 00:28)



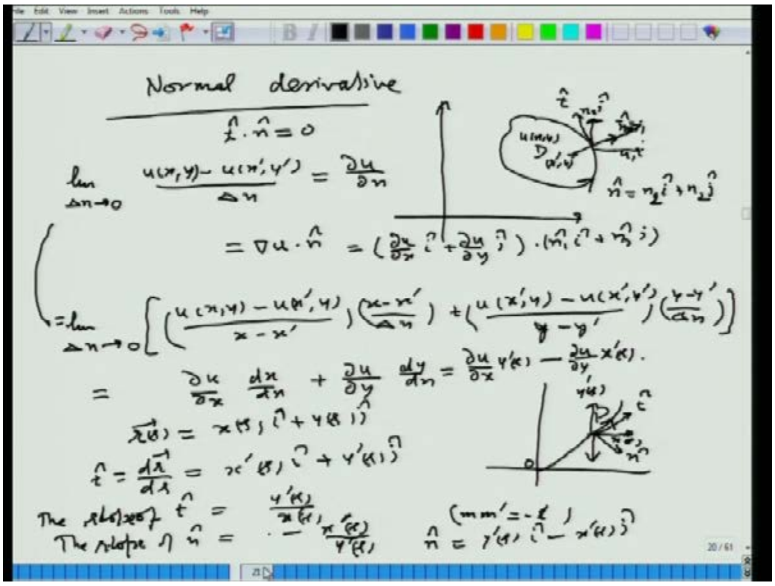
We discussed the concept of integration over the domain of x, y plane which give us the famous theorem - Green's theorem which is stated that integration over the domain D in the x, y plane can be written as the line integral over the boundary of its domain. So, that is what is stated here, the integration over this closed curve of M dx plus N dy. So, this is the line integral here which is the integration of this functional integrant over the boundary c of this domain D which is equal to the integration of the N x minus M y dx dy on the domain D. So, this is what was established in the last lecture.

(Refer Slide Time: 01:24)



And then, we extended this, firstly, we did it for very simple domain and then we extended it to more general domains like complicated domains like this which was divided into simpler domains, and then, added the integrals over those individual sub domains.

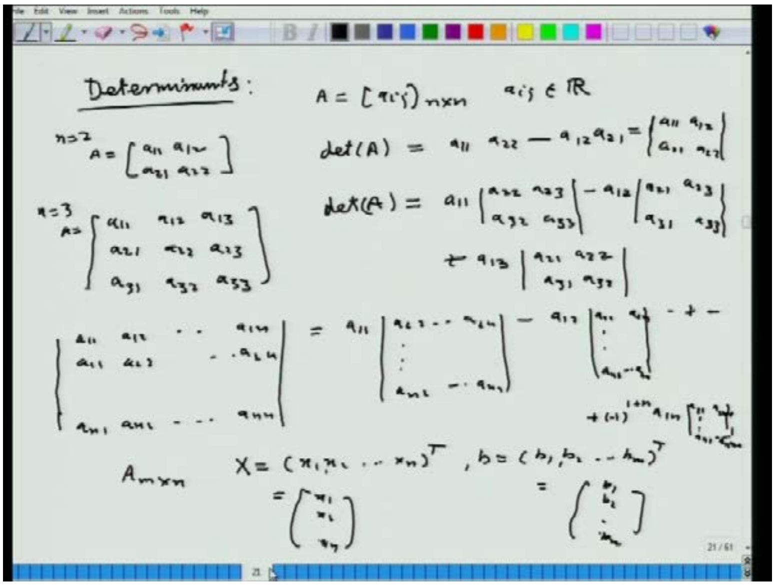
(Refer Slide Time: 01:42)



Then, we considered normal derivatives which the particular case of directional derivative. Directional derivative in any vector is given by gradient of the function which is being differentiated along a given vector. So, gradient of that function dot product with the unit vector in the direction in which we are differentiating it.

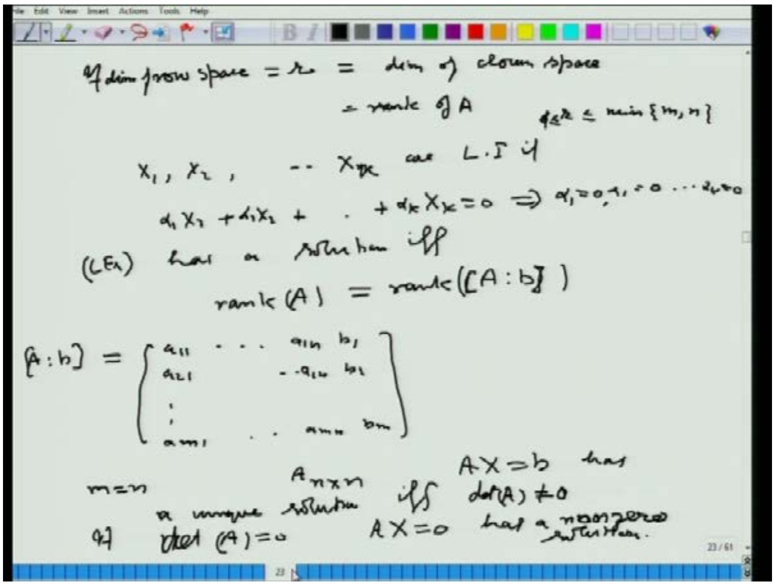
So, in particular, if we take the direction normal direction, so that is what is called normal derivative which was defined here in this manner. Since the function is defined only inside D. So, we took minus of what is usually taken that u(x dash,y dash) minus u(x,y) whole delta and limit delta n tending to 0. So, we took minus of that since the function is defined only inside the domain D. So, that is what will be defined as the directional normal derivative del u by del m, which we have seen that it is a gradient of u dot product with the unit vector in the direction of outward normal. Which we had seen we added and subtracted this u x prime y, and then individually these limits were taken here and then we saw that, it is nothing but the gradient of u dot n cap, n cap is the unit vector outward unit vector in the normal direction.

(Refer Slide Time: 03:14)



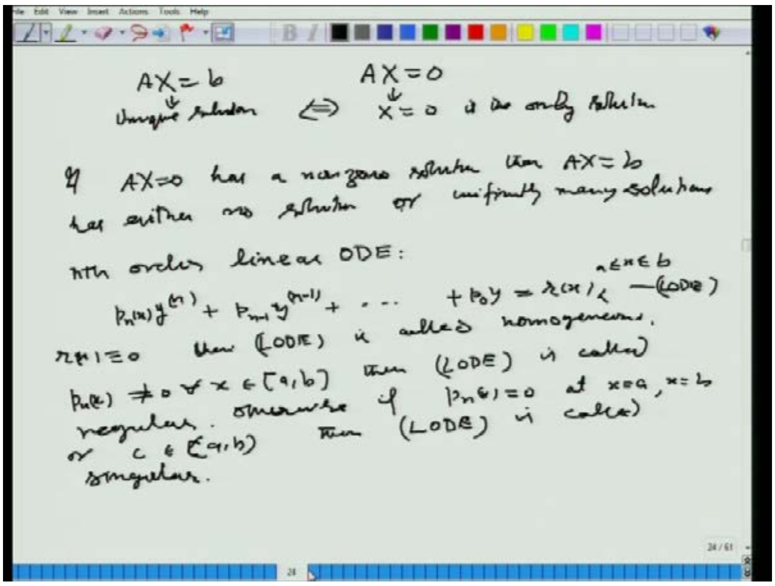
Then, we consider determinants and various concepts related to determinants and associated system of linear equations and when we saw that the system has n by n system has a solution if and only if the rank of the matrix is same thing as the rank of the augmented matrix.

(Refer Slide Time: 03:22)



In case when we have a square system that is n equations and n unknowns, then it is equivalent to saying that determinant of A, because now we can consider determinant. So, determinant AX equal to b this system has a unique solution if and only if determinant of A is non-zero. And if determent of A is 0 than A is singular, and then these homogeneous system has non-zero solutions and therefore, the non homogeneous system will have either no solution or infinitely many solutions.

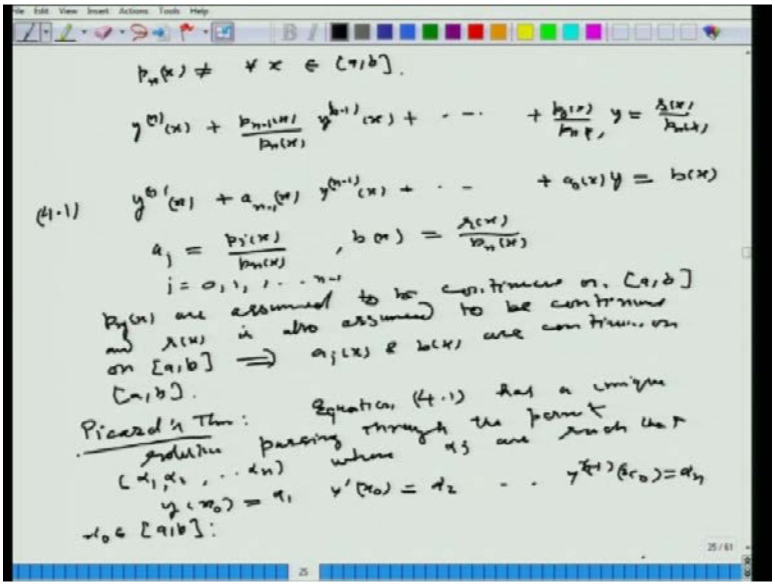
(Refer Slide Time: 04:11)



Then in the last point of the lecture, we stopped at consideration of n th order ordinary differential equations which was taken in this form p n x y n th derivative plus p n minus 1, it is also for… All these p n's are continuous functions of x and they are the coefficients of the various derivatives like p n is the coefficient of highest order derivative, the highest order derivative here is n. So, that is why this is n th order ordinary differential equation. These is linear in all these derivatives and on the right hand side we have R x function and if R x is identically 0 we call homogeneous.

And if this p n x is never 0 on the interval (a,b) then we call this equation a regular equation, and if p x vanishes either at n points or in the interior of the interval open interval (a,b) then we call this question singular equation. Examples of singular equations are Bessel equations, Lysander equations and various other those equations which cannot be solved in the usual way, then one goes for power series method and (()) in general to solve those problems. Here we will be consist only the regular problems and here…

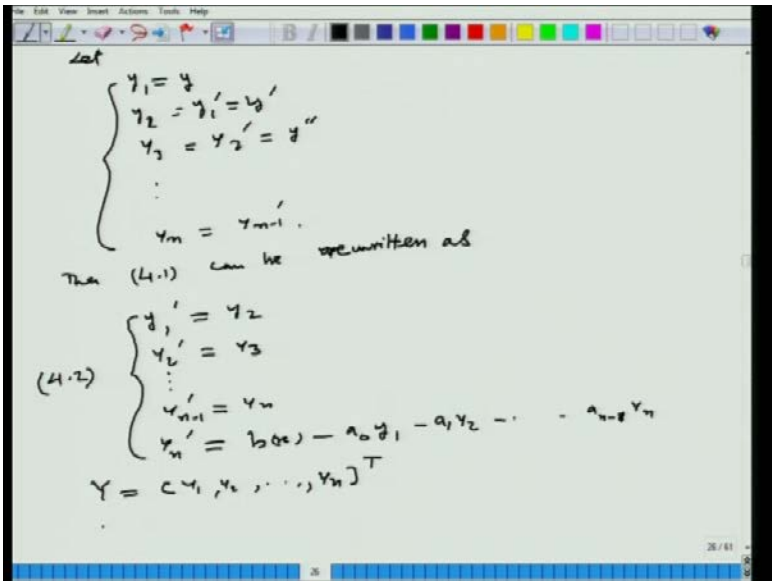
(Refer Slide Time: 05:41)



So, we assume that this p n x is not 0 for all x in (a,b) and therefore, this equation can be written as we can divide by p n x throughout and then get y n x plus p n minus 1 x divided by p n x y n minus 1 x and so on plus p 0 x divided by p n x, y equal to r x over p n x. So, this can be re-written in this form y n x plus a and minus 1 x, y n minus 1 x and so on plus a 0 x y equal to here b x, where these a j is p j x over p n x, and b x is the j equal to 0, 1, 2 up to n minus 1, and b x equal to r x over p n x.

Now, since all these p ns are p n is p j x are assumed to be continuous on (a,b) and r x is also assumed to be continuous on (a,b). So, this would imply that these a j x and b x are continuous on (a,b). Then we have this Picard's theorem, it says that this let say this is we will call it 4.1 now, this is the fourth lecture so we will call it 4.1 here. So, this equation 4.1 as a unique solution passing through the point that let say alpha 1, alpha 2, alpha n where alpha n's are like this, where alpha j's are such that y at x 0 equal to alpha 1, y prime x 0 equal to alpha 2 and up to y n minus 1 x 0 equal to alpha n. Here this x 0 is a point in (a,b). So that is the Picard's theorem. How do we see this actually?

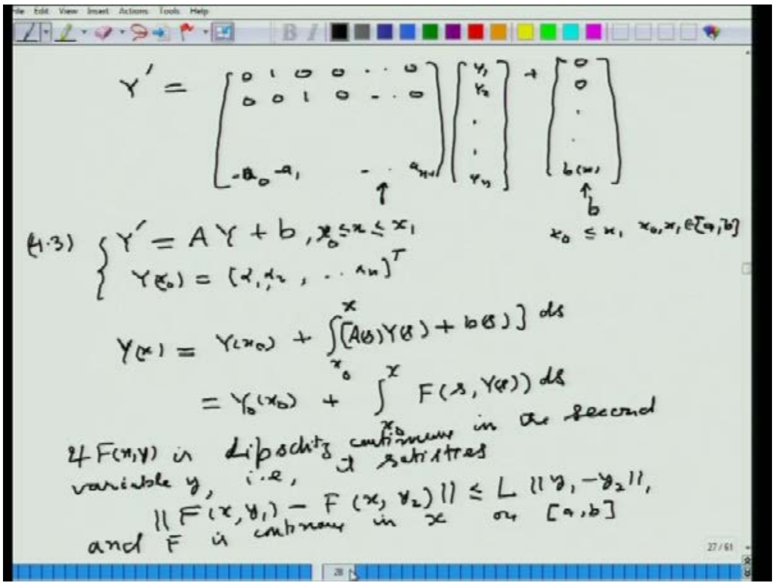
(Refer Slide Time: 09:59)



We have this, we introduce these variables y 1 equal to y, y 1 prime equal to y 2 equal to y 1 prime which is nothing but y prime, similarly y 3 equal to y 2 prime which is y double prime and so on. So, like this we will have y n as y n minus 1 prime. So, the equation let these variables we taken like this then we can see that then 4.1 can be re-written as y n prime equal to we have like this. So, take all the things on the other side this will be...

So, this system we will write like this, y 1 prime equal to y 2, y 2 prime equal to y 3 and so on from this and so y n prime equal to rather y n minus prime equal to y n, and then y n prime equal to minus… So that is first term is b x minus then you have these terms taken on the other side, so like this a 0 y 1, minus a 1 y 2 and minus 1 to you have a n minus 1, n minus 2 rather n minus 1 and then y n minus 1, so a 0. So, like this you have a n minus 1, this is y n here sorry this should be n minus 1 y n, a 0 y 1 a 1 y 2 and so on up to a n minus 1 y. So, this is the system of first order equation, each one is a first order equation, so this is the… So, this is 4.2. So, 4.1 can equivalently be written as a system of these first order equations, and if you introduce this y equal to y 1, y 2, y n transpose that is writing it an as a column vector then and this matrix A will be…

(Refer Slide Time: 13:37)

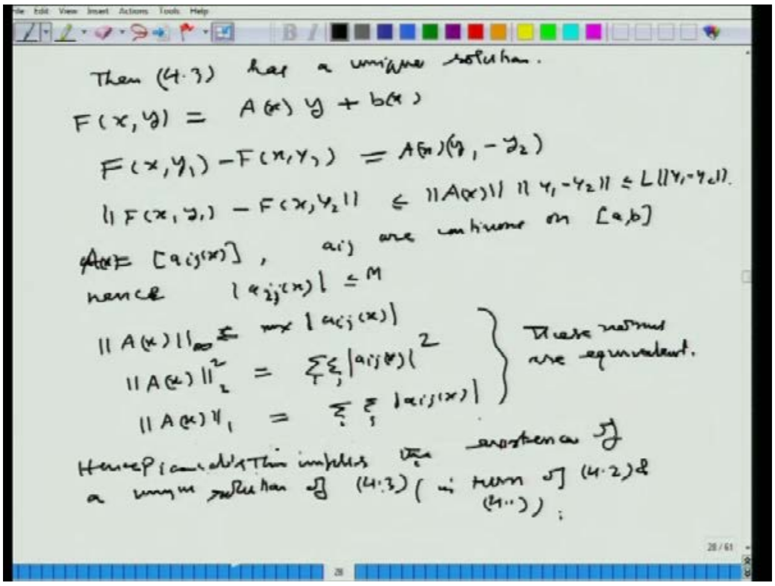


Then we can write this like this that y dash equal to this matrix where you have this is 0, the first term is 0 here because there is no y 1 here, and then 1 here, and then 0 0 so on, and here then 0 0 1 and 0 like this, and then the last one would be last one would be these terms will come there minus is sorry a 0, minus a 1, up to minus a n minus 1. So, here y 1, y 2 and y n plus you have 0 0 and up to b x. So, this system is written like this, y prime equal to this matrix AY plus some vector b like this. So, where A is this matrix and b is this vector here. So, this is the system can be written in a matrix form, a vector it is called the matrix form of the equation 4.1 which was equivalently written as 4.2 is a system of first order equation and so in matrix form this can be written like this.

And the initial point, so this question is to be satisfied where 0 less than equal to x rather x 0 less than x equal to something, it can be x 1 where x 0, x 1 lie in that interval, the x 0 less than equal to x 1 and here x 0, x 1 both belong to (a,b) and y at x 0 is given by the vector alpha 1, alpha 2, alpha n transpose. So, this is the system of first order equations which is equivalently written from the 4.2 which was in turn written from 4.1. So, this 4.1 is equivalent to 4.2 that the system of first order equation and 4.2 can be then in the matrix form can be written equivalently like this 4.3. Now, here this is reduced equivalently to the integral equation like this y x equal to y at x 0 plus integral x 0 to x, here you have this AY s plus b s ds. So, this is like you you have this A is a because all these A s, A s are functions of x. So, this actually A s also function of x. So, we will write it like this A s Y s like this.

Now, this is a particular case of this one Y at x 0 plus x 0 to x something like F s Y s ds. So, here the Picard's theorem states that if this function F where F is this form here in this particular case. So, Picard's theorem said that if this F is Lipschitz continuous in the second argument variable. So, let us write the dependence here if this F(x,y) is Lipschitz continuous in the second variable y that is it satisfies this norm is the R n norm, some Lipschitz constant L such a conditions called Lipschitz continuity which is stronger than the continuity and and F is continuous in x on this interval (a,b)

(Refer Slide Time: 19:44)



Then 4.3 has a unique solution. This 4.3 is actually initial value problem, because here this is the initial point and from this point the solution should evolve satisfying this equation. So, that is what is called the initial value problem. So, this has a unique solution. So, we have to verify only that this F(x,y) here is in this case we have this thing

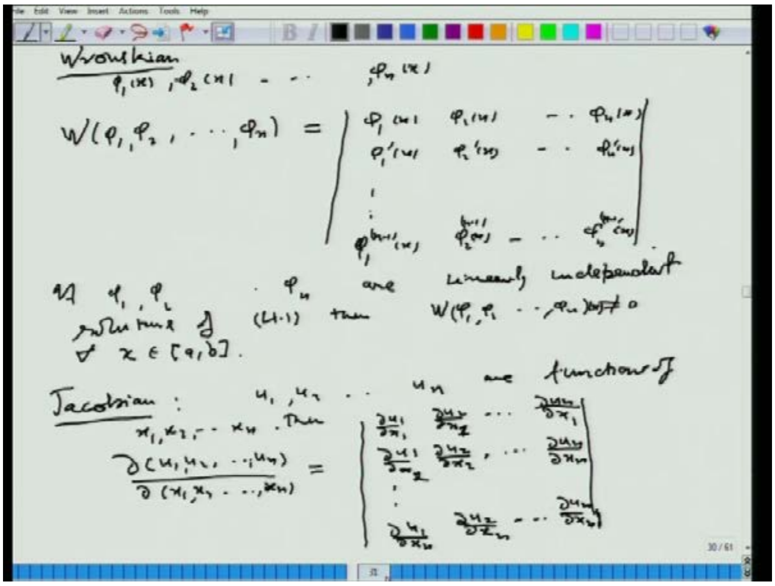
here A (x) y plus b (x). So, we have to check only the Lipschitz continuity. So, here because here A and b… Since the components of this A are continuous so therefore the matrix is continuous. And F(x,y 1) minus F(x,y 2) will be A (x) y 1 minus y 2, here this will cancel and so norm of this so norm of this F(x,y 1) minus F(x,y 2) will be less than equal to this is matrix norm into this.

Now, since A i this… Let say A is of the form a i j (x) where a i j are like this. Most of these are constants only the last row has these as functions of x and they are continuous, and a i j are continuous on (a,b). Therefore, hence this absolute value a i j (x) is bounded by some number m and so A this norm A… This is dependent on x. So, norm A (x) is less than equal to… Here each one is less than that so if we take the… This is defined as maximum of a i j (x). You can define any norm those are equivalent here for the matrix either this or if we define summation a i j or or the Euclidean norm. So, various norms are there, this norm let say this is called infinity norm and there are other norms A (x) 2 that is summation a i j square, square of this summation, the doubles sum over i and j or the other norm this one norm that is double sum a i j (x) mod a i j (x) summation over i j.

So, all these norms are equivalent, these are norms are equivalent on… So, you can use any norm here on this matrix and therefore… Here therefore, this will be bounded by some number like it will be any square, I mean in each or this will be in this if you use this maximum norm then it will be bounded by m times y 1 or some number L we can take whichever we take so there is some number L where L is dependent on any of those are norms. It is independent of x in this case and therefore, we have this Lipschitz continuity as well as since a i j s are continuous and this b - the components of b are all continuous functions and therefore, this F will be continuous in the first variable x. So…

Hence, Picard's theorem implies the existence of a unique solution of 4.3 in turn of 4.2 and 4.1. So, that is where we have the existence of this. We need to verify only that the coefficients are continuous function in an and the right hand side the non homogeneous function is continuous on the given interval, and we will have the existence of initial value at any point starting and any point lying in the interval.

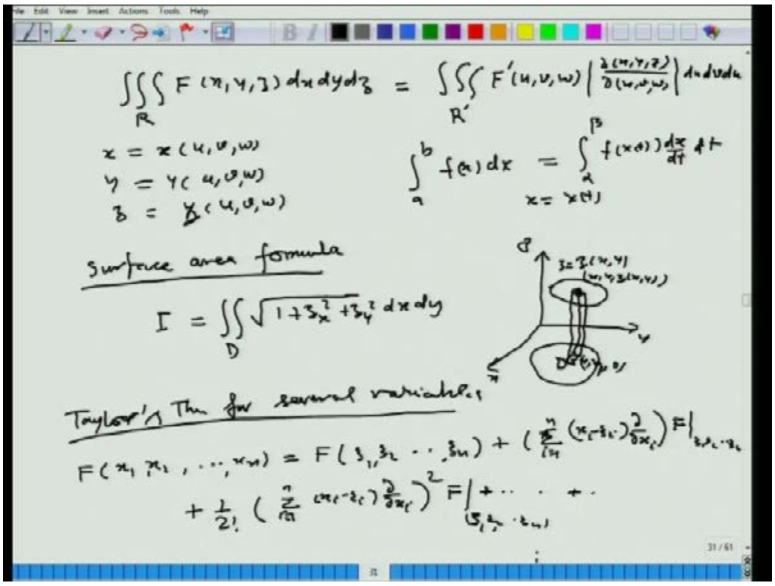
(Refer Slide Time: 25:46)



So, next we have a these functions. So, here let us consider end functions phi 1 x, phi 2 x and so on and phi n x. So, Wronskian of this phi 1, phi 2, phi n is called Wronskian. So, this is the determinant like this phi 1 x, phi 2 x and phi n x, and second row we get phi 1 prime x, phi 2 prime x and phi n prime x, and like this we continue, and then phi 1 n minus 1 th derivative, phi 2 n minus 1 th derivative and so on to phi n n minus 1. So, this determinant is called the Wronskian of n functions. And we see that if we have the result if phi 1, phi 2 and phi n are linearly independent solutions of 4.1. Then this Wronskian is n w of phi 1, phi 2 is not 0. It is a function all it attacks, is not 0 for all x in (a,b). And converse is if this Wronskian vanishes at any point x in the interval then these phi 1, phi 2 are linearly dependent. So, that is what we have in this Wronskian case then we comes under the Jacobian also.

The Jacobian here what we will consider is the functions u 1, u 2, u n, and these are functions of x 1, x 2, x n. So, these are functions x 2, x 1, x 2 and so on x n, then this Jacobian of or it is denoted like this del u 1, u 2, u n over del x 1, x 2, x n. So, this is defined by the following determinant that is del u 1 over del x 1, del u 2 over del x 2, and sorry del x 1, del u n over del x 1, and then the second row del u 2 over del x 1, del u 2 over sorry u 1 and this x 2, so del u 2 over del x 2, del u n over del x n and so on and then last one will be del u 1 over del x and del u 2 over del x n and so on. It can be written in the row wise or column wise, because we know that A, A transpose of the same determinant. So, it does not matter which way be write, we can as well write this in the rows as columns. So, we get the same thing here and so it can be written row wise or column wise. So, the this is what is called the Jacobian of this and functions u 1, u 2 and u n which are assume to be differentiable partially with respect to x 1, x 2, x n. So, the Jacobian of this n functions with respect to x 1, x 2, x n will be defined like this.

(Refer Slide Time: 31:06)

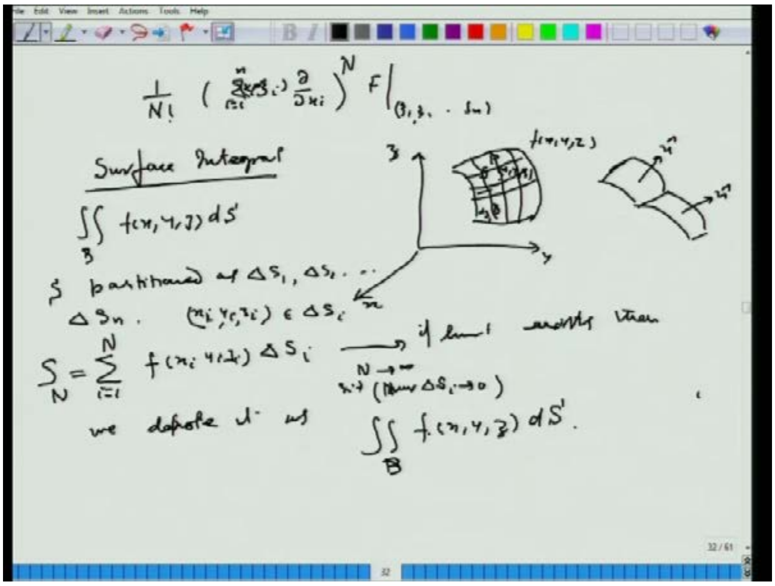


In particular, here when we have the change of transformation here, so this integral let us say the triple integral if we consider over certain region R here, then F(x,y,z) dx dy dz. So, if we change x to let say x is a function of (u,v,w), y is a function of (u,v,w), z is a function of (u,v,w) like this. So, then this can be written as, the region R will be changed to R dash and this F here will be actually let say this is F dash (u,v,w). And then you will have the absolute value of the Jacobian del(x,y,z) over del(u,v,w). So, this is the Jacobian and you take the absolute value, the bar is for the absolute value of this. So, we should if it comes out to be negative we remove that negative sign, if it is positive we written it as it is. So, you have du, dv, dw. So, that is what we have the change of rule like it is an extension for the one-dimensional case, supposing that you are taking a to b f(x) dx and x is a taken as a function of t then this changes to alpha to beta f(x(t)) like this, and dx by dt d t. So, that is the change of the variable rule and it is the generalization of this of the one-dimensional case in the general three variables or any n variable one can extend it.

Then we have we have already mentioned this surface area formula that is… Here we have this integral which we have mentioned already earlier over this x, y, z, and this is a domain in the x, y plane, and let say the surface over this is defined by z, x, y. So, here this is any point here is x y 0, z component is 0 here, and let us say this is here. So, this is (x,y,z) which is a function of (x,y). So, here are we take this element area here dx dy and over this we form this beam kind of a thing, and so this beam will have certain area here on top like this shaded area and with then we vary these variables x, y, 0 here over this domain and we get the whole surface area which is given by this double integral D square root of 1 plus z x square plus z y square dx dy. It will turn out to be a particular case of general surface integral, we will see in a short while.

Then also we have this Taylor theorem for general and variable case for several variables. So, suppose that this F is a function of x 1, x 2, x n and here we write it as the in the neighborhood of this that is say we have psi 1, psi 2, psi n. So, we want to expand this in the neighborhood of this fix point psi 1, psi 2, psi n. So, this will be expanded like this plus, you have x, so summation here summation over i equal to 1 to n x i minus psi i and then you have del over del x psi of this F, and the next term will be would be plus 1 upon a factorial 2 and same thing here summation of this i equal to 1 to n x i minus psi i del over del x i square. Here a square will be taken in algebraic sense and of this operator, evaluate F and like that we go. And this at to be finally evaluated at psi i coordinates. So, that is what we will write it as evaluated at psi 1, psi 2, psi n, if F this thing evaluated at psi 1, psi 2, psi n plus 1.

(Refer Slide Time: 37:37)



And like this will have 1 over any fact integer n factorial n like this summation earns x i minus psi i i equal to 1 to n del upon del x i power n and F then evaluated at psi 1, psi 2, psi n. Here this will be the powers will be taken in algebraic sense you have, for example, if you have only two terms then A plus like del x over del over del x 1 del x over del x 2 is square of that will have in that operator sense, this square A square plus the B square plus twice AB like that. So that the general formula for the Taylor series expansion.

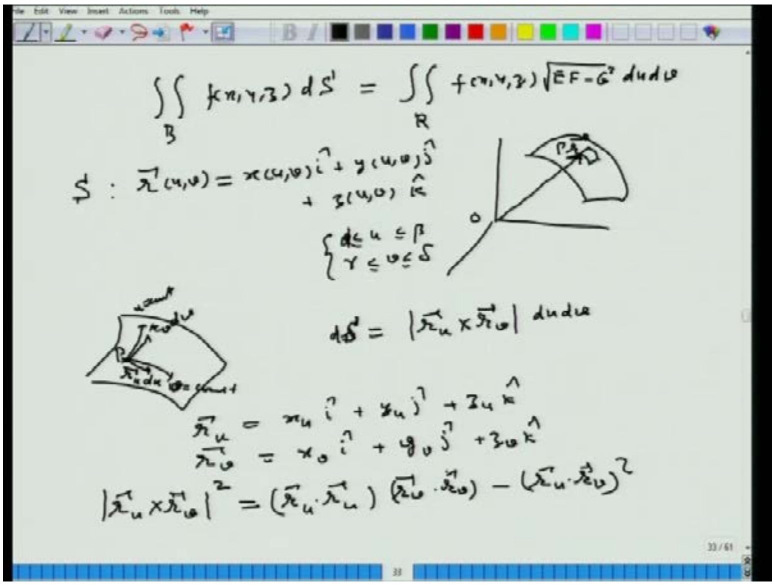
Now, we come to the surface integral - general surface integral. Now, here what we have is like this some surface s is given here like this and we want to here again the function F is given at each point on the surface. So, here x, y, z, point p is moving on the surface and there is a function F(x,y,z) defined at each point on this, then we would like to consider what do you mean by this integral over the surface, let say that is B surface. So, over B of F(x,y,z) ds, this is capital S.

Now, what do you mean by this? This is like a partition this surface in this way. To partition this surface in this way, and let us say each. So, S is partition as delta s 1, delta s 2 that is say this is delta s 1, delta s 2 and so on, like this we (( )) them. And here say (x i,y i,z i) is a point in delta s i. So, then we consider this sum S n that f(x i,y i,z i) i equal to 1 to capital N times delta s i. Now, if this limit… So, here these surfaces are partition in such a way in these surfaces such that the maximum surface area goes to 0. So, that is what is to be ensured here. So that we do not do only partition this or and then leave at certain other elements that should not be done. Each element should be partition so that area goes to 0.

So, this this limit as N tends to infinity such that this maximum surface area maximum, so that maximum or delta s i goes to 0 that is what we have to ensure and if this limit exist, if limit if limit exist than we define it we denote it as over this B f(x,y,z) ds. So, one has to ensure what are they sufficient conditions that because this is a (( )) sum and we need to expect that this sum should not depend, the limit should not depend on the way we partition it.So therefore, I will see that the sufficient conditions are F s to be continuous on this surface as well as on the boundary of this surface here, and this boundary should be piecewise smooth. That means at each point here, here with the on the surface it should be such that the normal should be continuous in each patch like you could have surfaces like this and like this. So, in these two patches the normal n should be… So, here everywhere normal n cap will be continuous here, it will be continuous here.

So, this is what it should happen on that surface. So, surface should be a partition in such a way that on each subsurface partitioned area the normal should be continuous. And this F is continuous everywhere then we can see that this limit will not depend upon the manner which we partition it and so that limit will be independent of the way we define the partition. So, this is what is called the surface area of this B and where the function surface integral of the function F over the surface B. Now, how do we evaluate this? Evolution of this will be done as we know how to evaluate the integrals over two-dimensional areas on a plane. So that is what we want to use here.

(Refer Slide Time: 44:29)

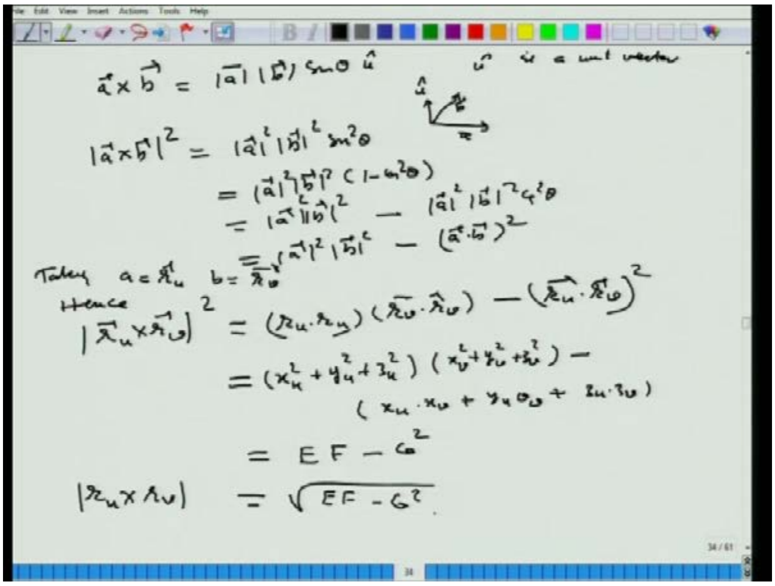


We will actually see that this surface area will be the surface integral B F(x,y,z) ds is actually written out to be here R is the projection of the surface on the u v plane and it will be F(x,y,z) and you have square root of EF minus G square and du dv where these x, y, z will be functions of u and v. We can see that actually surface S is represented as... So, here the surface S is like this and any at any point here the position vector let us say this o p is r is the position vector of this. So, this r is a function of (u,v) which will be like x(u,v) i plus y(u,v) j plus z(u,v) k. So, this surface can be described by two parameters u, v. So that position vector on this surface o p, let the vector like this will be given by these two parameter functions (x,y,z) like this x i plus y j plus z k where x, y, z are functions of u and u and v. So that is what where you will be having certain a to or alpha, beta, and we will have gamma, delta ranges like this.

So, here if we see that how actually this comes out to be like this. So, on the surface we have like this. So, if you consider this as v equal to constant will give you, let us say u curve so this is v equal to constant. When we restrict parameter to a constant we get a curve on the surface and like this it will be… This is v equal to constant that is u curve and let us say u equal to constant will give you another curve like this, and then the surface will be actually spanned by this kind of curves.

So, here we can see that the tangent this is a point p here and tangent is given by r u the derivative and then element area element length will be like this t u. And similarly, here tangent here will be r v dv. So, this curved element area at this point like this, this is the surface ds. So, ds is actually ds will be given by the area of the parallelogram found by this o p that is given by r u cross r v du dv, so absolute value of this and d u d v is anyway positive. So, it comes out of that. So, that is what we will have to see that this ds comes out to be a cross product absolute value. We know that area of the parallelogram found by a and b is given by the absolute value of the cross product. And so, here if you calculate r u like this, so that is x u i plus y u j plus z u k, similarly r v is x v i plus y v j plus z v k, and so r u cross r v will be we know that this we use this formula here is this is square will be actually equal to r u dot r u into r v dot r v minus r u dot r v square.

(Refer Slide Time: 49:29)



This can be seen because we know that for any given given vectors a and b - a cross b is mod a mod b a sin theta and then some unit vector in this direction let us say u cap in this direction where u cap is a unit vector. So, this is a, b here and u cap is perpendicular to this plain found by a cross b. So, this is a, this is b, so this is the plane spanned by a and b vector. So, here this a cross b is perpendicular to that plane. So, a unit vector if you take, so like this. So, absolute value of this square will be a square b square sin square theta, so because unit vector has a magnitude 1, so we get this. And so, this can be written as 1 minus cos square theta and so we get a b square of these minus this is a square mod a square mod b square times cos square theta which is nothing but the dot product. So, we get… So, this is a dot b the square of that. So, that is what we use here. So, we got the same thing here for this we can use this 1. Now…

So therefore, hence this r u cross r v square, we got here this… So, replacing a and b by r u r v. So, here you got r u dot r u like this dot and b is r v dot r v minus this is r u dot r v square. So, taking a equal to r u and b equal to r v. So, we get this. Now, this r u is nothing but so… So, this is x u square plus y u square plus z u square, here x v square plus y v square plus z v square and minus this is, so here x u dot x v plus y u dot y v plus z u dot z v. So, this is what we got which is nothing but in our… This is denoted as E and this denoted as F and this is a minus G square so E F minus G square. So therefore, we get r u cross r v this as square root of EF minus G square. So, that is what… So, this is E, this is F and this is G square and that is what we have here.

So, in particular for the flat surfaces this will reduce to our the earlier case which we had here in the this case. We will see that this also comes out from the our formula general formula for the surface integral like this here. So, with this I stop my lecture and next we will consider some more concepts and finally, will move on to the introduction of main topics on the calculus of variation. Thank you very much for viewing this lecture.