Calculus of Variation and Integral Equation
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Lecture No. # 39

Welcome viewers to the lecture series on integral equation under the NPTEL lectures. In all preceding lectures, we have discussed about methods of solutions for integral equations of two types - that is volterra integral equations, and fredholm integral equation. In this lecture, we are going to discuss about two types of integro differential equations; that is fredholm integro differential equations, and volterra integral differential equations. You can see that some of the volterra, and fredholm integro differential equations can be converted into volterra integral equation, and fredholm integral equation respectively. And of course, using some other techniques like, adomian decomposition techniques or power series method, directly also will be able to solve those kind of integro differential equation. So, this lecture is completely devoted to integro differential equations.

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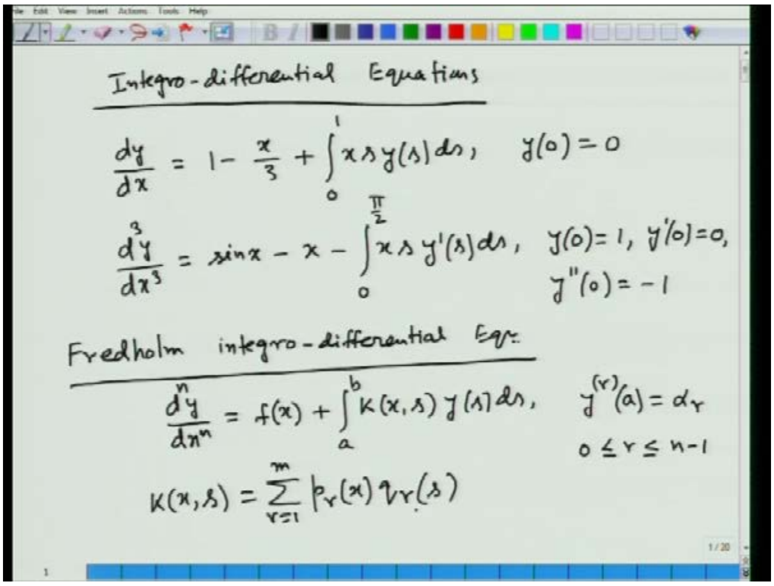

So we are going to discuss the method of solutions for integro differential equations. First of all, we will be considering fredholm integro differential equations. To such examples of fredholm integro differential equations are dy dx is equal to 1 minus x by 3 plus integral 0 to 1 x s y (s) ds with y 0 this is equal to 0. This is an example of an fredholm integro differential equations. Next we consider another this type of equation, d 3 y dx 3 this is equal to sin x minus x minus integral 0 to phi by 2 x s y dash (s) ds with y 0 - this is equal to 1 y dot 0 - this is equal to 0, and y double dot 0 - this is equal to minus 1. So, these are examples of some fredholm differential equation.

In the second example, you can see that apart from kernel, this integral also involve fast derivative of y; that is the unknown function. So, first of all you see the direct computation method for solving fredholm integro differential equations with a special type of kernel that is where you are going to consider separable kernels only.

So, general format is fredholm integro differential equation. We are going to consider solutions of equations of that type d n y dx n is equal to f (x) plus integral a to b k of (x) comma s y (s) ds with given initial conditions y r a, this is equal to alpha r for 0 less than equal to r, less than equal to n minus 1. And here, we are considering separable kernel, that is k(x,s) is given by sigma r runnings from 1 to m p r (x) q r (s).

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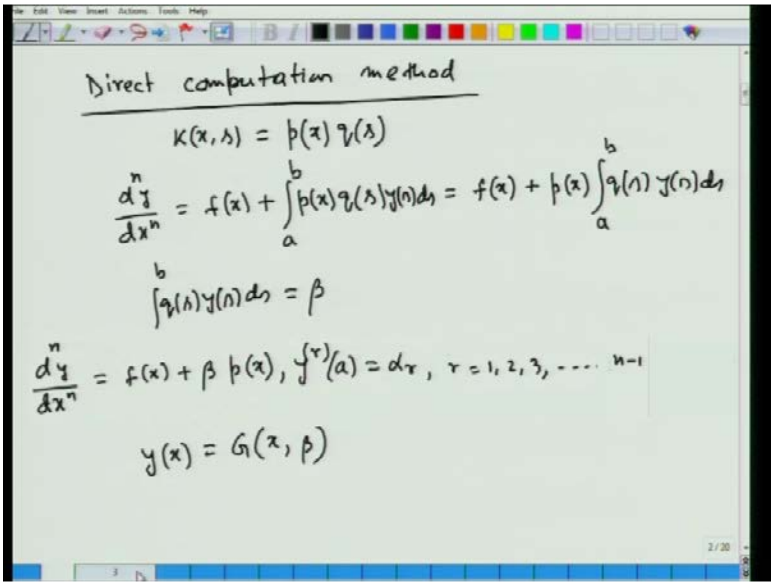

Now, first we discuss the direct computation method to solve this type of integro differential equation. And for simplicity of theoretical discussion, here we assume that k (x,s) is a separable kernel, but is the form p (x) into q (s). Just for simplicity, we are considering these types of separable format, but of course, with illustrative example will be considering kernel of the form sigma r runnings from 1 to m p r x q r s. So, with these type of kernel given integro differential equation becomes d n y dx n this is equal to f(x) plus integral a to b p (x) q(s) ds , as the range of integration is constant. And p (x) and q (x) these are in multiplicative format; so you can write, this is equal to f (x) plus p (x) integral a to b q (s) y (s) ds.

And now, if we define this unknown integral a to b q (s) y (s) ds; and assuming solution of this integral exist - such that y (x) is the continuous function. Then integral a to b q (s) y (s) ds, this exist and is a finite quantity, and we denote this quantity as beta. And then adobe equation becomes d n y dx n, this is equal to f (x) plus beta multiplied by p(x) with initial conditions y r a this is equal to alpha r, where r equal to 1, 2, 3 n minus 1. And you can see, this is an ordinary differential equation which is a initial value problem; so, assuming f (x) and p(x) these are continuous solution of these equation exist. And if we solve this equation using these n initial conditions, we assume that solution is given by or denoted by G (x,beta).

Of course, this G x will involve the given constants alpha 0, alpha ,1 alpha 2, up to alpha n minus 1, but for simplicity and from sake of sake of brevity. We can ignore all those arbitrary, not arbitrary constant even fixed constants here. And in a just intended to mention here, that y (x) will involve this beta. Now, once you have this expression for y (x) equal to G (x,beta), then in the above definition that is integral a to b, q (s) into i (s) ds equal to beta, you can substitute this expression for y (x). And substituting this expression you can integrate it, and then solving for beta you can find out the particular value of beta, and then substituting these beta in the final expression that is y (x) equal to capital G of (x,beta), you can find out solution of the given integro differential equation. So this is the method for separable kernel.

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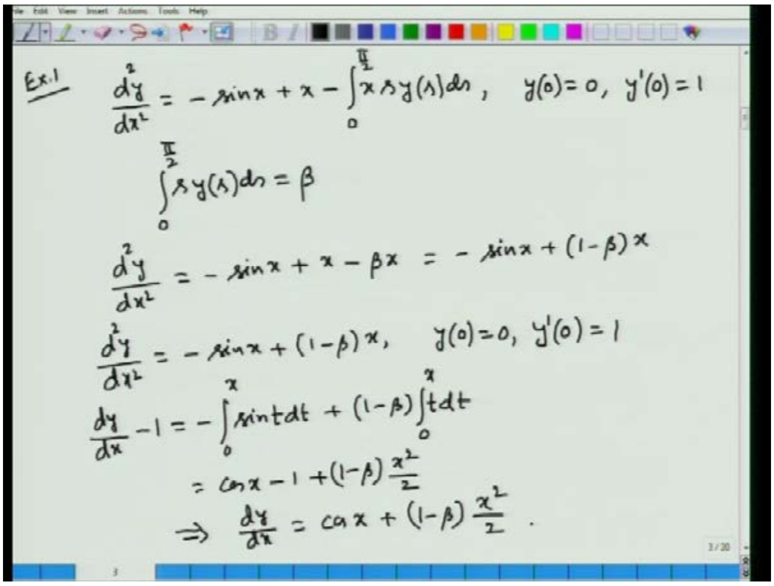

Now, we consider 2 examples: First example d 2 y dx 2 equal to minus sin x plus x minus integral 0 to phi by 2 x s y (s) ds with the initial conditions y 0 equal to 0, and y dot 0 this is equal to 1. So, assuming that integral 0 to phi by 2 s y (s) ds. This is equal to beta, we can write the different differential equation into the form d 2 y dx 2 is equal to minus sin x plus x minus beta x. So, this is equal to minus sin x plus 1 minus beta x. And now, we can solve this initial value problem that is d 2 y dx 2 - this is equal to minus sin x plus 1 minus beta x with the initial conditions y 0 equal to 0, and y dot 0 - this is equal to 1.

So, if we integrate both side of these equations. And then using the limit y dot 0 equal to 1, so that means we are integrating between the range 0 to x, then we can find dy dx minus 1 is equal to minus integral 0 to x sin t dt plus 1 minus beta integral 0 to x t dt, after integration this will comes out to be cos x minus 1 plus 1 minus beta multiplied with x square by 2. So, cancelling minus 1 from both sides from here, we can write dy dx, this is equal to cos x plus 1 minus beta x square by 2.

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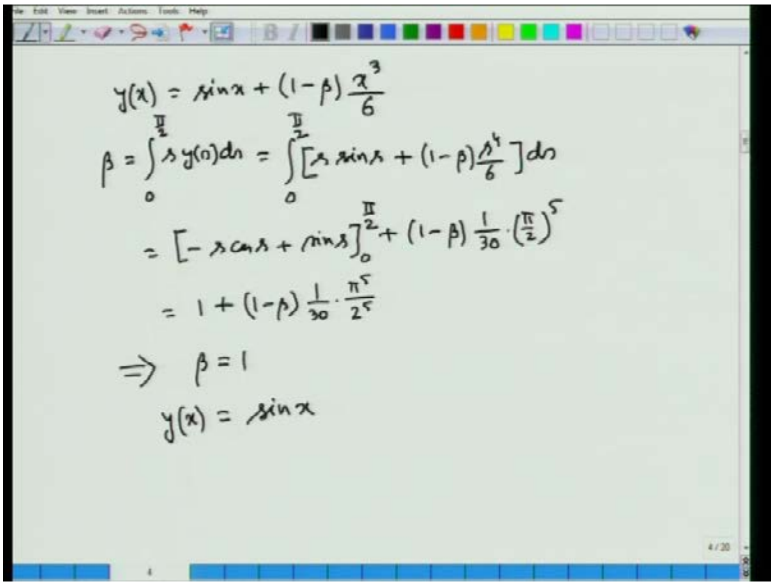

Again integrating both sides, and using the result y 0 equal to 0, and proceeding in a similar fashion, you can find y (x) equal to sin x plus 1 minus beta x cube divided by 6.

So now, if you compare this expression with the earlier notation that we have used, then sin x plus 1 minus beta x cube by 6 this is actually or capital G of (x,beta). So, y (x) equal to this 1. Now we recall the definition that is beta equal to integral 0 to phi by 2 s y (s) ds; so this is equal to integral 0 to phi by 2 s sin s plus 1 minus beta s to the power 4 divided by 6 ds this one. And after integration you can find minus s cosine s plus sin s, limit 0 to phi by 2 plus 1 minus beta times 1 by 30 into phi by 2 whole to the power 5. And from here, you will be having a 1 plus 1 minus beta multiplied with 1 by 30 into phi to the power 5 by 2 to the power 5, and you can verify these implies the solution for beta as beta equal to 1.

So already, we have y (x) equal to sin x plus 1 minus beta x cube by 6; and in these expression only beta was unknown. Now, we are able to find out the value of beta which is equal to 1. So substituting here, you can find the solution to the given problem is y (x), this is equal to sin x. And of course, by substituting this expression into the given integro differential equation, you can verify this is the solution of the given equation. And of course, this particular function y (x) equal to sin x, it satisfies the initial conditions that is y 0 equal to 0, and y dot 0 equal to 1.

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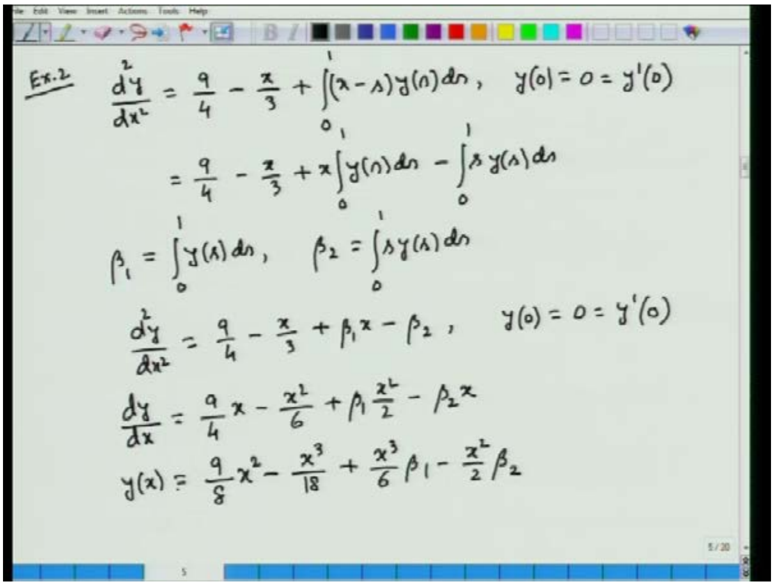

Next, we consider another example of this type of equation which is given by d 2 y dx 2 is equal to 9 by 4 minus x by 3 plus integral 0 to 1 x minus s y (s) ds with y 0 equal to 0, equal to y dot 0. So here, we are considering k (x,s) is x minus s; so this is of the form p 1 x q 1 s plus p 2 x into q 2 s. And we can write this equation into the form that 9 by 4 minus x by 3, then taking x out of the integral sin, it is integral 0 to 1 y (s) ds minus integral 0 to 1 s y (s) t s. Now, if we define to unknown quantity beta 1 which is defined by 0 to 1 y (s) ds, and beta 2 is equal to integral 0 to 1 s y (s) ds, then adobe differential equation becomes d 2 y dx 2 is equal to 9 by 4 minus x by 3 plus beta 1 x minus beta 2 with the initial conditions y 0 equal to 0 equal to y dot 0.

So integrating both sides within the range 0 to x, we can find dy dx this is equal to 9 by 4 x minus x square by 6 plus beta 1 x square by 2 minus beta 2 x, and again integrating within the limit 0 to x. We can find y (x) equal to 9 by 8 x square minus x cube divided by 18 plus x cube by 6 beta 1 minus x square by 2 beta 2. This is the expression for y (x). Now, if we substitute this expression for y (x) into the expression for beta 1 and beta 2, then we can find 2 linear equations involving beta 1 and beta 2. And solving those 2 linear equations will be able to find out the values of beta 1 and beta 2, and hence that will give you the solution for the given equation.

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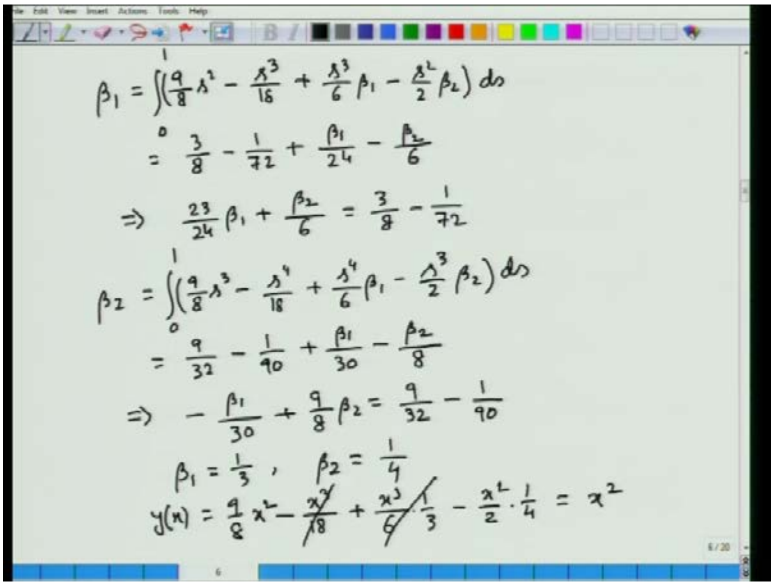

So if you substitute into the expression for beta 1, so beta 1 equal to integral 0 to 1 9 by 8 s square minus s cube divided by 18 plus s cube divided by 6 beta 1 minus s square by 2 beta 2 ds, and this will be equal to 3 by 8 minus 1 by 72 plus beta 1 by 24 minus beta 2 by 6. After rearranging these terms, from here you can get the first equation that is 23 by 24 beta 1 plus beta 2 by 6; this is equal to 3 by 8 minus 1 by 72. And imilarly, from beta 2 equal to integral 0 to 1, the previous integrant have to be multiplied by s, because beta 2 equal to integral 0 to 1 s y (s) ds .

So this will be 9 by 8 s q minus s to the power 4 divided by 18 plus s to the power 4 divided by 6 beta 1 minus s cube by 2 beta 2 ds . And this is equal to 9 by 32 minus 1 by 90 plus beta 1 by 30 minus beta 2 divided by 8. And from here, we will be having another equation that is minus beta 1 by 30 plus 9 by 8 beta 2, this is equal to 9 by 32 minus 1 by 90. Now solving these 2 linear equations, we find beta 1 that is equal to 1 third, beta 2 this is equal to 1 by 4. And after substituting these 2 values for beta 1 and beta 2 into the expression for y, you can find y (x) - this is equal to 9 by 8 x square minus x cube by 18 plus x cube by 6 into one third minus x square by 2 into 1 by 4. So this is equal to x square. Actually, this term cancels with this one, and this two will be simplified to x square. So y (x) equal to x square is solution of this integro differential equation.

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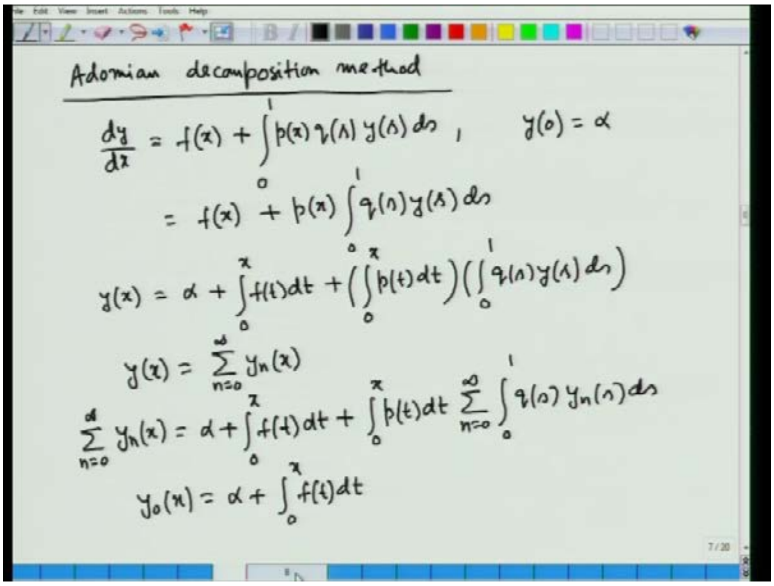

Next, we consider Adomian decomposition method.

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To solve similar type of fredholm integro differential equations. And here for simplicity will be discussing the method for fast order derivative involve in the left hand side, but of course, this result can be generalized for fredholm integro differential equation involving nth order derivative.

So here, we consider equation of the form dy dx is equal to f (x) plus integral 0 to 1. We are assuming that kernel is separable, and kernel is product of only 2 terms - that is p(x) and q (s). So equation is given by p(x) q (s) y (s) ds with given initial condition y 0, this is equal to alpha, this is the given equation. And as p(x) is under the integral sign, we can tic it out of the integral sin, and we can rewrite this as dy dx is equal to f (x) plus p(x) integral 0 to 1 q (s) y (s) ds - this is the expression for dy dx. Now, integrating this expression from 0 to x, we can find y (x) is equal to alpha plus integral 0 to x f (t) dt plus integral 0 to x p(t) dt, these expression multiplied by integral 0 to 1 q (s) y (s) ds .

Now, as usual what we have done earlier for adomian decomposition method. We are assuming solution of this equation can be expressed in the form y (x) equal to n running’s from 0 to infinity y n x. And here, we are assuming all the convergence criteria, and interchange ability of integral sin n, summation notation. So, substituting these expression for y in the adobe expression, we can find summation in running's from 0 to infinity y n x, this is equal to alpha plus integral 0 to x f (t) dt plus integral 0 to x p (t) dt, then sigma n running's from 0 to infinity, integral 0 to 1 q (s) y n (s) ds. And now, if we equate from left hand side y 0 x, with the term alpha plus integral 0 to x f (t) dt from the right, then we can find y 0 x, this is equal to alpha plus integral 0 to x f (t) dt. Then equating y 1 from the right with the term integral 0 to x p (t) dt integral 0 to 1 q (s) y 0 s ds , we can calculate y 1 x, because y 0 x is already known.

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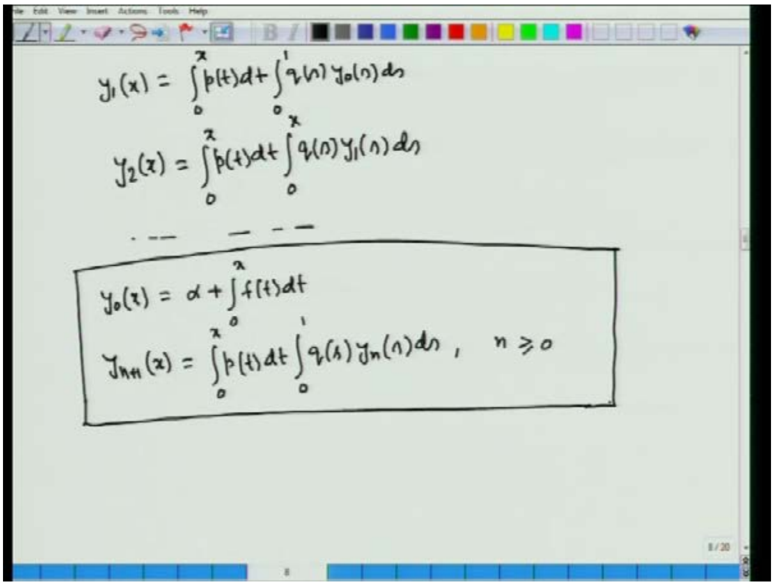

So therefore, y 1 x will be equal to integral 0 to x p (t) dt integral 0 to 1 q (s) y 0 (s) ds. Then using these result for y 1, we can calculate y 2 x is equal to integral 0 to x p (t) dt integral 0 to x q (s) y 1 (s) ds and so on. So, in this way we can find out y 0 x, y 1 x, y 2 x and so on. And after finding out the sum of that infinite series, we can find the solution to the given integro differential equation using this adomian decomposition method. So the iterates in the compact form can be written as that is y 0 x is equal to alpha plus integral 0 to x f (t) dt, and y n plus 1 x is equal to integral 0 to x p (t) dt, then integral 0 to 1 q (s) y n (s) ds. And these results is valid for n greater than equal to 0. So, this is the compact form for finding the quantities y 0, y 1, y 2, y 3 and so on. In order to find out solution of the fredholm integro differential equation using the adomian decomposition method.

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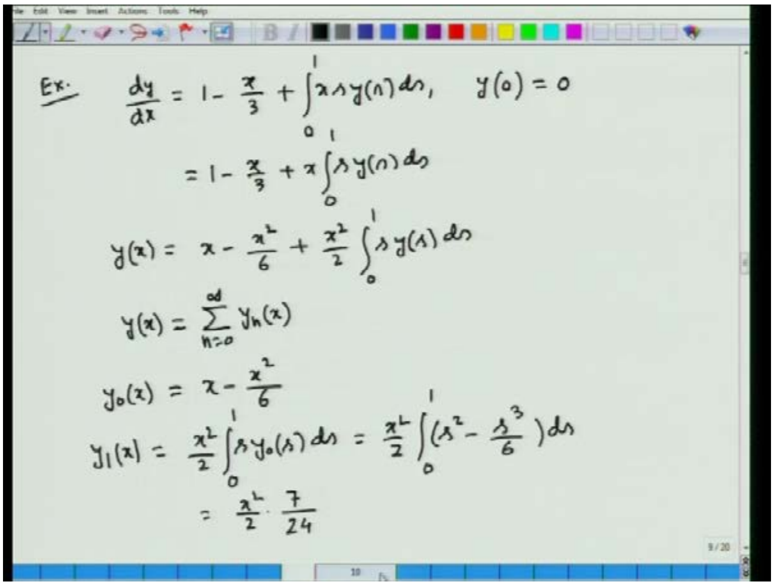

To understand this particular technique we consider one example. That is dy dx is equal to 1 minus x by 3 plus integral 0 to 1 x s y (s) ds with the initial condition y 0, this is equal to 0. And using the usual practice that taking x out of the integral sin, we can just write, this as 1 minus x by 3 plus x integral 0 to 1 s y (s) ds, and then integrating both sides from 0 to x. We can find y (x) this is equal to x minus x square divided by 6 plus x square by 2 integral 0 to 1 s y (s) ds. And now, we assume the solution of this equation can be expressed in the form y (x) equal to n running's from 0 to infinity y n x. And now we use the formula that is for computing y 0, y 1, y 2 and so on. So therefore, y 0 x is equal to x minus x square by 6 using this y 0 x. Now, we can calculate y 1 x this is equal to x square by 2 integral 0 to 1 s y 0 (s) ds .

And here you can note that this x square by 2 is nothing but integral 0 to x p (t) dt, that we have discussed in case of general formulation for solving this kind of equations using adomian decomposition method. And if you substitute this y 0 (s), so this will be x square by 2 integral 0 to 1 s square minus s cube divided by 6 ds , and this will be equal to x square by 2. And after evaluating these integral this constant will comes out to be 7 by 24. Now, we calculate 2, 3 more iterates in order to understand the structure of the solution of this particular equation.

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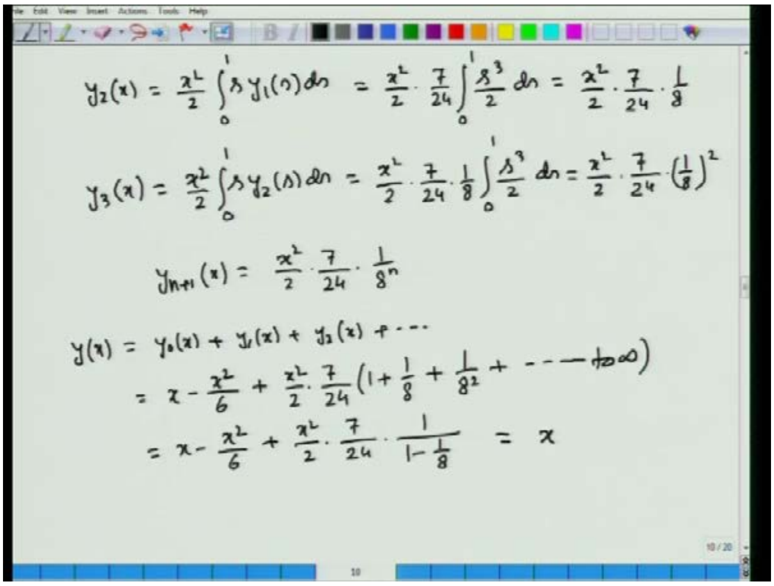

So with y 1 x equal to x square by 2 into 7 by 24, we can calculate y 2 x is equal to x square by 2 integral 0 to 1 s y 1 (s) ds, this will be equal to x square by 2 times 7 by 24 integral 0 to 1 s cube by 2 ds, and this is equal to x square by 2 into 7 by 24 into 1 by 8. If we calculate one more iterate y 3 x is equal to x square by 2 integral 0 to 1 s y 2 (s) ds; so this will be equal to x square by 2 into 7 by 24 into 1 by 8, and again integral 0 to 1 s cube divided by 2 ds. So, this will be equal to x square by 2 times 7 by 24 times 1 by 8 this whole square. So, using the trend for y 2 and y 3, we can write y n plus 1 x this will be equal to x square by 2 times 7 by 24 into 1 by 8 to the power n. And therefore, y (x) is equal to y 0 x plus y 1 x plus y 2 x and so on. This will be equal to x minus x square by 6, and then from rest of the terms, we can take x square by 2 into 7 by 24 common, then it will be multiplied with 1 plus 1 by 8 plus 1 by 8 square plus dot dot up to infinity.

And, this is nothing but a geometric series, and some of these series is 1 by 1 minus 1 by 8. So common ratio are satisfies the condition modulus are less than 1, and if you calculate this sum and after some algebraic manipulations, you can easily find this is 7 by 24 times 1 by 1 minus 1 by 8. So, this denominator will be 7 by 8. So, ultimately it will be multiplied by 8 by 7, and then you can find this is equal to x only. Because this minus x square by 6 term will cancelled with this particular term. And therefore, y equal to x is the solution.

So, this is the illustrative example, that how we can solve this kind of fredholm integro differential equation using the adomian decomposition method. Of course, in some of the problem instead of considering the entire quantity, that is alpha plus integral 0 to x f (t) dt, this quantity as y 0 x, as we have considered earlier if part of this particular function can be considered as y 0 x for getting rapid convergence or other iterates exactly equal to 0. But I am not going to that particular part. And before completing this kind of fredholm integro differential equation, we consider one more technique by which you can solve this kind of equation, that is we can convert this type of equation to fredholm integral equation only.

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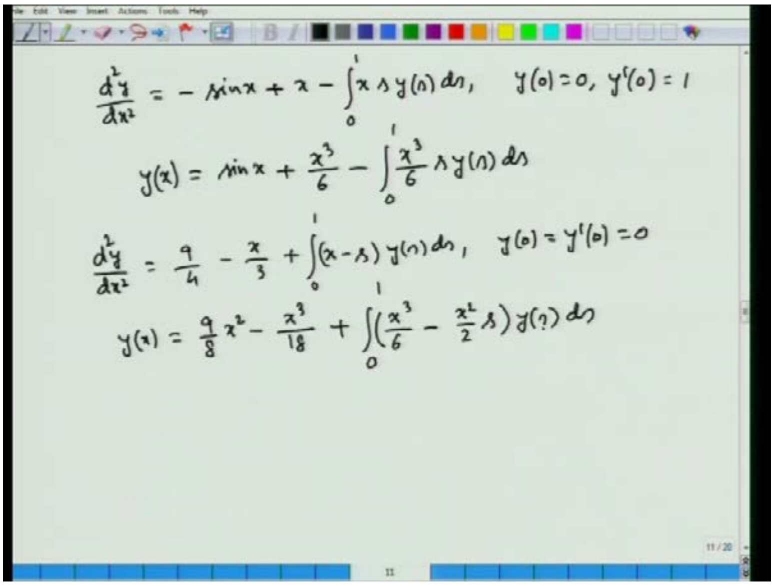

In this case, we consider this particular problem d 2 y dx 2 is equal to minus sin x plus x minus integral 0 to 1 x s y (s) ds with y 0 equal to 0, and y dot 0 this is equal to 1. This equation we have solved initially, now at that stage you can recall we have used that integral 0 to 1 s y (s) ds, this is equal to beta. Now, without replacing these 0 to 1 s y (s) ds equal to beta, you can find after twice integration from the limit 0 to 1, this y (x) equal to sin x plus x cube by 6 minus integral 0 to 1 x cube by 6 s y (s) ds. These results we have obtained using the conditions y 0 equal to 0, and y dot 0 equal to 1. Now, this equation y (x) equal to sin x plus x cube by 3 minus integral 0 to 1 x cube by 6 time s y s ds, is nothing but a fredholm integral equation of second kind, in homogenous fredholm integral equation. And using the earlier method, what we have considered for fredholm integral equation, you can also solve this equation by using any one of those techniques that we have already discussed.

And similarly, the second problem that we have solved for this kind of equations, that is d 2 y dx 2 this is equal to 9 by 4 minus x by 3 plus integral 0 to 1 x minus s y (s) ds with y 0 is equal to y dot 0, this is equal to 0. This can be also converted to the problem that is y (x) equal to 9 by 8 x square minus x cube divided by 18 plus integral 0 to 1 x cube divided by 6 minus x square by 2 s y (s) ds . So, actually you have to keep in mind that this type of conversion of fredholm integro differential equation - to fredholm integral equation is possible for the case when kernel is a separable kernel. And using the property of separable kernel or you can say the you can utilize that the kernel is separable, we can convert these integro differential equation to fredholm integral equation only. And then using the standard methods for solving this kind of equation, we can find out solution of these equations.

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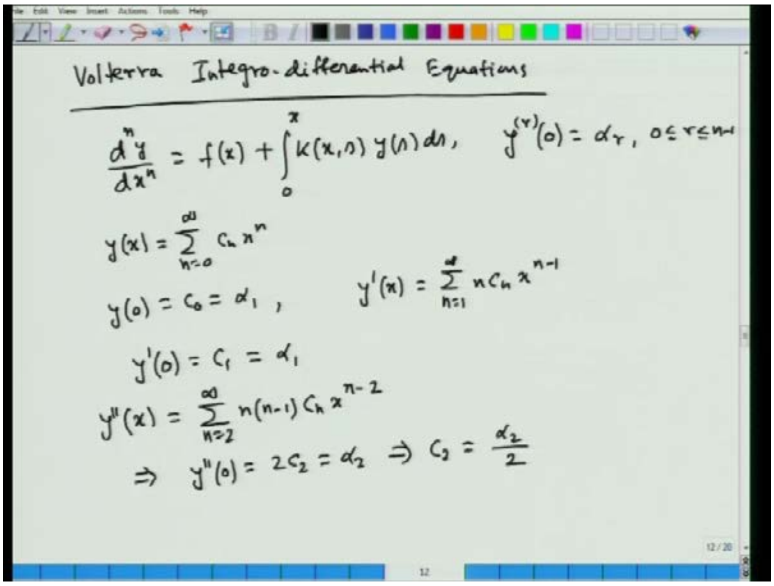

Next we consider volterra integro differential equations.

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And for volterra integral differential equations: We are going to discuss about the power series method of solutions for volterra integral integro differential equations. We consider equations of the form d n y dx n is equal to f (x) plus integral 0 to x k of (x) comma s y s(ds) with the given initial conditions, that is y r (0), this is equal to alpha r for r equal to 1, 2, 3, n minus 1. Now, we are assuming solution into the form y (x) is equal to sigma n running's from 0 to infinity C n x to the power n. And here, fast few constants of this power series can be obtained by using this initial conditions.

Because y 0 this is equal to C 0, and this will be equal to alpha 1. Similarly, y dot x is equal to sigma n running's from 1 to infinity n C n x to the power n minus 1, and using the initial condition we can find y 0 is equal to C 1, this is equal to alpha 1. Similarly, y double dot x is equal to sigma n running's from 2 to infinity n into n minus 1 C n x to the power n minus 2, and this will give us that y double dot 0, this is equal to 2 c 2 equal to alpha 2 implying c 2 equal to alpha 2 by 2.

So, first in constants of this particular power series can be obtained by using these initial conditions. And then we have to substitute this particular series into the equation, and we can find the solution by equating like powers from both sides. So that means, we can either find out the current formula or we can calculate first few constants apart from this C 0, C 1 up to C n. We can find out a power series solution for this kind of volterra integro differential equation.

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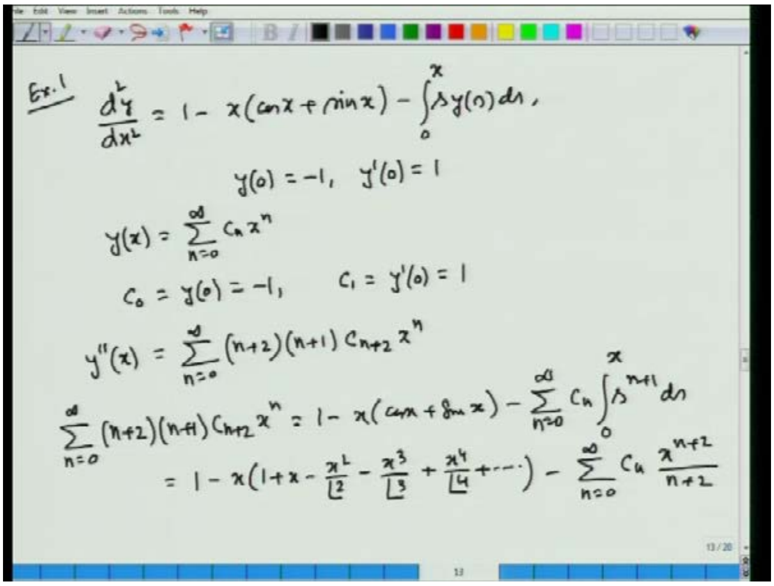

We consider one example with which we can illustrate the idea, that is d 2 y dx 2 equal to 1 minus x cos x plus sin x minus integral 0 2 x s y (s) ds - with the initial conditions y 0 equal to minus 1 y dot (0) this is equal to 1. So, we assume y (x) is equal to summation n running's from 0 to infinity C n x to the power n. So using the last 2 formulas, we can easily find that C 0 is equal to y 0 this is equal to minus 1, and C 1 this is equal to y dot 0 - this is equal to 1. Now, as y (x) equal to these one, then we can write y double dot x - this is equal to summation n running's from 0 to infinity n plus 2 times n plus 1 C n plus 2 x to the power n.

In the previous result, we can change the dummy variable n, and the range of summation from 2 to infinity to 0 2 infinity, we get this particular expression. Now that means, if we substitute y (x) equal to sigma n running's from 0 to infinity, C n x to the power n into the given integro differential equation will be having this result, that sigma n running's from 0 to infinity n plus 2 times n plus 1 C n plus 2 x to the power n, this is equal to 1 minus x into cos x plus sin x minus sigma n running's from 0 to n C n integral 0 to x s to the power n plus 1 ds . And this is equal to 1 minus x times 1 plus x minus x square by factorial 2 minus x cube by factorial 3 plus x to the power 4 by factorial 4 plus dot dot minus sigma n running's from 0 to infinity C n x to the power n plus 2 by n plus two.

Now, at this point I like to remark, that we are able to evaluate this integral that is integral 0 to x s to the power n plus 1 ds in this format, because here the given kernel k (x,s) is actually s only. So whenever, the given problem having separable kernel, then we can apply this method, and in case the separable kernel consist of the term that is p 1 x, p 2 x, p 3 x, and so on. Those are apart from the terms like x to the power n, if it contain some function like e to the power x or sin x or something like that. Then, we can use their Taylor series expansion in order to equate the like terms from the both sides. So here, we have this result that is summation n running's from 0 to infinity n plus 2 times n plus 1 into C n plus 2 x to the power n equal to this one.

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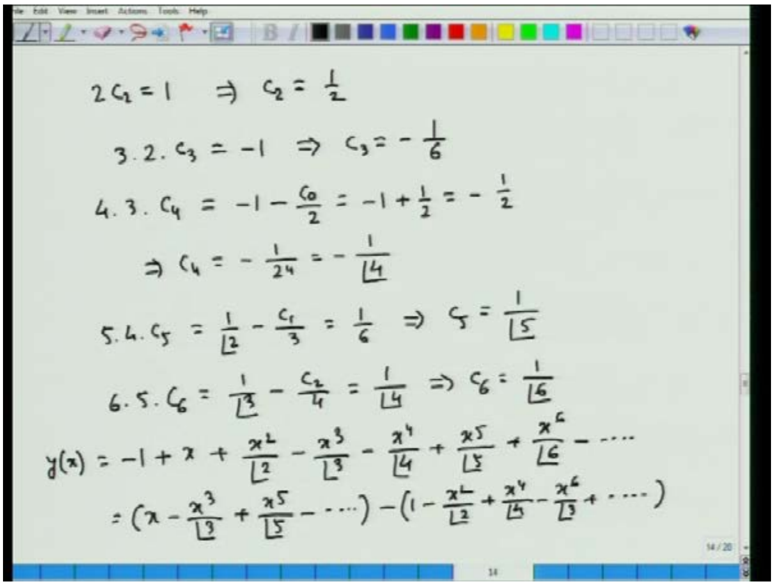

Now, if we equate the coefficient of constant term from both sides, then we can find 2 C 2 - this is equal to 1, because on the right hand side you can see, you are having the term one as the constant term, then minus x cos x plus sin x - these expression does not contains any constant term. And the last summation that is sigma n running's from 0 to infinity, this particular summation contains term from x square, and onwards. So therefore, equating the constant term from both sides; we can find 2 C 2 equal to 1 implying C 2 equal to half. Next, if we equate coefficient of (x) from both side on the left hand side substituting n equal to 1 in the summation, we can findx term. So therefore coefficient of x on the left hand side is 3 into 2 into C 3. This is equal to on the right hand side, we will be having x term only from the expression that is minus x into cos x plus sin x. And cos x plus sin x is 1 plus x minus x square by factorial 2 and so on.

So therefore, coefficient of x on the right hand side is equal to minus 1. So these implies C 3 - this is equal to minus 1 by 6. Next, if we equate the coefficient of x square from both sides, then we will be having 4 into 3 into c 4 - this is the contribution from the left hand side. And then from the right hand side, we will be having coefficient of (x) square is minus 1, this minus 1 is coming from minus x into cos x plus sin x. And another term will be having that is from this summation - that is minus C 0 by 2 into x square. Because into the summation n running's from 0 to infinity C n by n plus 2 into x to the power n plus 2, you can findx square term for n equal to 0. And therefore, 4 into 3 C 4 equal to minus 1 minus C 0 by 2, already we have the value for C 0 equal to minus 1. So this is equal to minus 1 plus half, and this is equal to minus half, and this implies C 4 is equal to minus 1 by 24, and we can write it has 1 by factorial 4.

If you calculate two more constant terms, that is equating the coefficient of x cube from both sides, you can find 5 into 4 C 5 is equal to 1 by factorial 2 minus C 1 by 3, substituting the value of C 1, you can find this is equal to 1 by 6. And this implies C 5 this is equal to 1 by factorial 5. And then equating the coefficient of x to the power 4, you can find 6 into 5 into C 6. This is equal to 1 by factorial 3 minus C 2 by 4 - substituting for C 2, you can calculate this is equal to 1 by factorial 4 and this implies C 6 is equal to 1 by factorial 6.

So, these constants if we right the expression for y (x), then we are having y (x) equal to minus 1 plus x plus x square by factorial 2 minus x cube by factorial 3 minus x to the power 4 by factorial 4 plus x to the power 5 by factorial 5 plus x to the power 6 by factorial 6 minus dot dot up to infinity. And then, we can rearrange the term as x minus x cube by factorial 3 plus x to the power 5 by factorial 5 minus dot dot, and then minus 1 minus x square by factorial 2 plus x to the power 4 by factorial 4 minus x to the power 6 by factorial 6 plus dot dot up to infinity. So, then you can see, the first one is the Taylor series expression for sin x, and second one is the Taylor series expansion for cos x.

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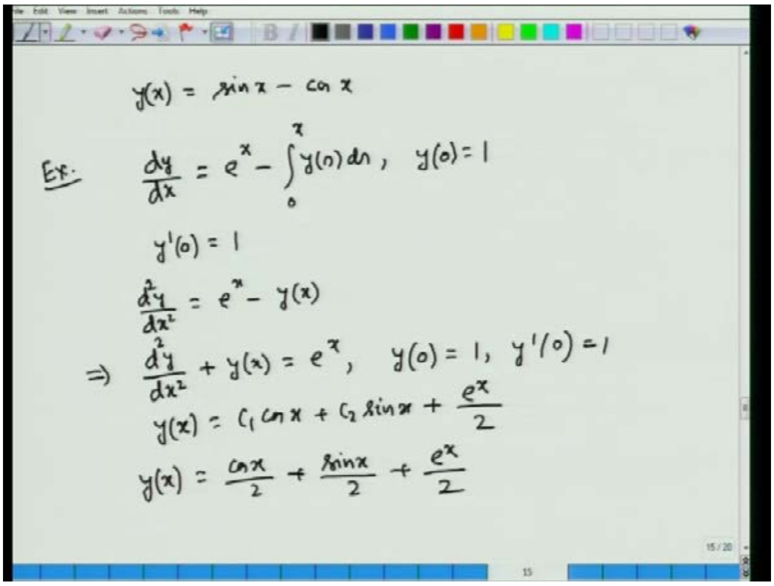

And hence, the given problem has the solution y (x) - this is equal to sin x minus cos x. This is the solution for the given problem. And of course, you can keep in mind that these type of technique is applicable for equations, where kernel is separable and depending up on the function involve with the problem, if it contains sin x cosine x kind of term. Then we need their Taylor series expansion in order to find out the unknown constants. And of course, this type of volterra integral equations can be converted into just volterra integral equation.

I am not going to discuss about that part, you can easily use the generalized replacement formula that we have used earlier, and discussed earlier, using that particular formula. You can convert the given volterra integro differential equations into volterra integral equation. Now, before concluding this part, I like to draw your attention that another method by which we can solve these kind of volterra integro differential equation, that is we can convert this equation into initial value problem.

That means volterra integro differential equation can be converted into ordinary differential equations. Without going into the theoretical discussion, we can only have a look at one example, that suppose we have to solve this problem dy dx - this is equal to e to the power x minus integral 0 to x y(s) ds - with the initial condition y 0, this is equal to 1. I have considered here very simple example. With this y 0 equal to 1, and this differential equation, we can differentiate both sides with respect to x and on the right hand side we can use the Leibniz formula, and before applying that we notice that y dot 0 is equal to 1. Then differentiating both sides with respect to x, we can find d 2 y dx 2 - this is equal to e to the power x minus y (x), applying Leibniz formula.

And this implies we are having this ordinary differential equation. That is d 2 y dx 2 plus y (x) this is equal to e to the power x, and we have two initial conditions - that is y 0 equal to 1, and y dot 0 this is equal to 1. So, that means the given volterra integro differential equation is now converted into an ordinary differential equation. And you can solve this equation directly - its general solution is given by y (x) is equal to C 1 cosine x plus C 2 sin x plus e to the power x by 2. First part C 1 cosine x plus C 2 sin x is the complimentary function, and e to the power x by to is the particular integral, and then using these two initial conditions you can find out solution of these problem as y (x) is equal to cos x by 2 plus sin x by 2 plus e to the power x by 2.

Of course, by substituting this expression for y (x) into the given integro differential equation, you can verify this is a solution of this particular problem. So, the method discussed in this lecture, for solving integro differential equations either of volterra type or fredholm type, and not exhaust it, there are many more other type methods by which you can solve this equations. But in case of separable kernel, these methods are very useful, and by which you can solve either volterra integro differential equation or fredholm integro differential equation. I am not going to proceed further on this particular topic. I stop today's lecture at this point. And in the next lecture, we will be considering some non-linear integral equation, and we will see that how those kinds of equations can be solved. So, thank you for your attention.