Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

Lecture #38

Welcome viewers, once again to the lecture series on integral equation, and a NPTEL courses. In all proceeding lectures, we have considered the two types of integral equations; that is Volterra integral equation, and Fredholm integral equation. And we have considered different methods, how to solve those kind of integral equation. This lecture is completely devoted to the discussion on singular integral equations. So, and this singular integral equations will be considering two types of singular integral equation; one is called just singular equation, and others are weakly singular equations. And will be considering two different methods how to solve those kind of integral equations.

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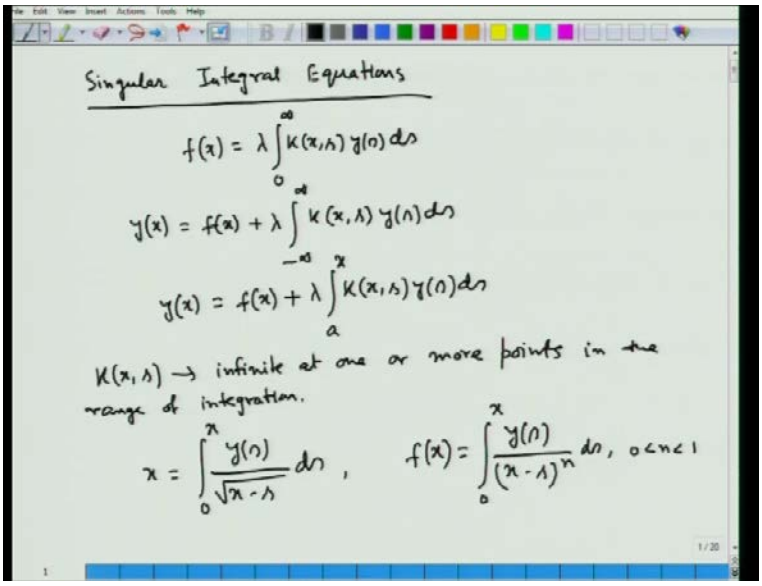

So in this lecture, we are going to discuss about singular integral equations. So first of all, we have a look at some singular integral equations f(x) equal to lambda integral 0 to infinity k(x,s)y(s)ds; this is an singular integral equation, because upper limit is infinity.

Another singular integral equation is y(x) equal to f(x) plus lambda integral minus infinity to plus infinity k of x comma s y(s) ds. So, in case here both upper limit, and lower limit are both of them are infinite, so this is again and singular integral equations.

And another type of integral equation is given by that y(x) equal to f(x) plus lambda times integral a to x k(x,s)y(s)ds, where this k(x,s) that is kernel, this becomes infinite at one or more points more points in the range of integration. So, these are actually three specific types of singular integral equations, and this type of integral equations we have to solve in order to find out solutions of the different physical, and other type of problems. In different different modeling approaches, and in different problems of physics; most of the time we encountered these type of singular integral equations, and there are several methods to find solution of this kind of integral equations; that is analytical solutions as well as numerical solutions.

Of course within this lecture series, we are not going to address about the numerical methods to solving this kind singular integral equations. Now, before proceeding further we just have a look to some particular examples, which are this type of singular integral equations. One example is, x equal to integral 0 to x y(s) divided by root over x minus s ds. Here, at s equal to x, this kernel that is 1 by root over x minus s this becomes infinite; so this is an example of an singular integral equation. Another well known singular integral equation, that is Abel's integral equation is f(x) equal to integral 0 to x y(s) divided by x minus s whole to the power n ds, where 0 less than n less than 1, this is another example of integral equation.

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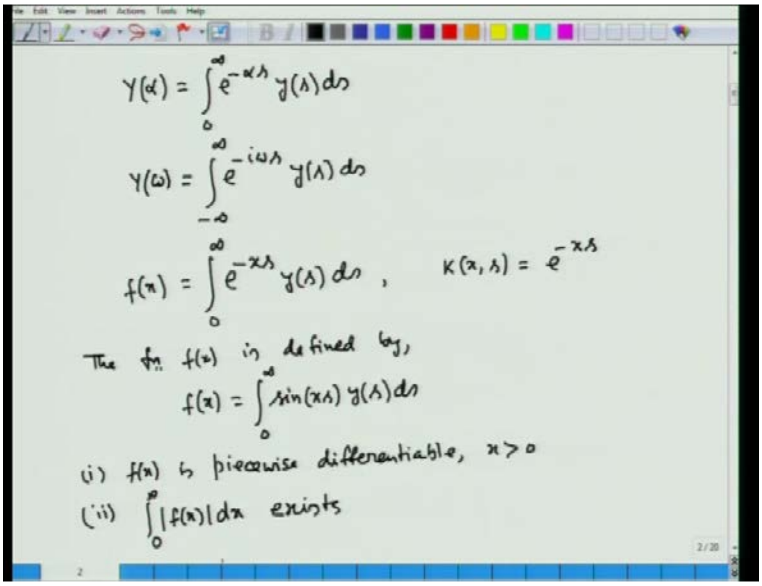

Now, we recall two important transforms, that is Laplace transform of a function is define by Y alpha this is equal to integral 0 to infinite e to d power minus alpha s y(s) ds; this is the definition for Laplace transform of a function. And Fourier transform is defined by Y omega is equal to integral minus infinity to plus infinity e to d power minus i omega s y(s) ds, actually these type of 2 transforms somehow related to the singular integral equations, because we can rewrite this type of transform formula by this way that we are considering transform of a function y, and we are taking the Laplace transform. So, instead oF alpha, here we can write f(x) equal to integral 0 to infinity e to d power minus x s y(s) ds, and clearly you can see this is an integral equation which is singular integral equation with kernel k(x,s) is equal to e to the power minus x s.

So that means for these type of integral, that is integral 0 to infinity e to the power minus x s y(s) ds. Suppose f(x) is known, then we are intended to find out the function y(x), that means for which function y(x); f(x) is going to be its Laplace transform, considering this x as the variable of the Laplace transform. Then solution of this singular integral equation will serve the purpose, that means solution of this integral equation will gives us the answer, that s if we take the Laplace transform of y(s), then will be having f(x) as its Laplace transform with the variable x. And similar result holds for this kind of Fourier transform, because instead of Y omega we can write here, some g(x) equal to integral minus infinity to plus infinity e to the power minus i x s y(s) ds.

So, considering the complex (( )) kernel, that is k(x,s) is equal to e to the power minus i x s, we can say that solution of that particular equation, that particular singular integral equation will gives us the answer, that if we take the Fourier transform of the function y(x), then will be having f(x) as the result of the Fourier transform. Now, we can considered another important transform, that is sin transform, and in these case the function f(x) is define by this formula f(x) is equal to integral 0 to infinity sin of xs y(s) ds.

So, clearly f(x) is nothing but the Fourier sin transform of y(s). Now with these particular example, we are going to discuss the fact, that in case of singular integral equation for a particular Eigen value, we can find an infinite set of linearly independent Eigen functions. So, first of all we assume that f(x) satisfies 2 properties: Number one f(x) is piecewise differentiable, for x greater than 0; and number two integral 0 to infinity modules of f(x) dx, these exists. If these two conditions are satisfied, then we can use the inversion formula for Fourier sin transform.

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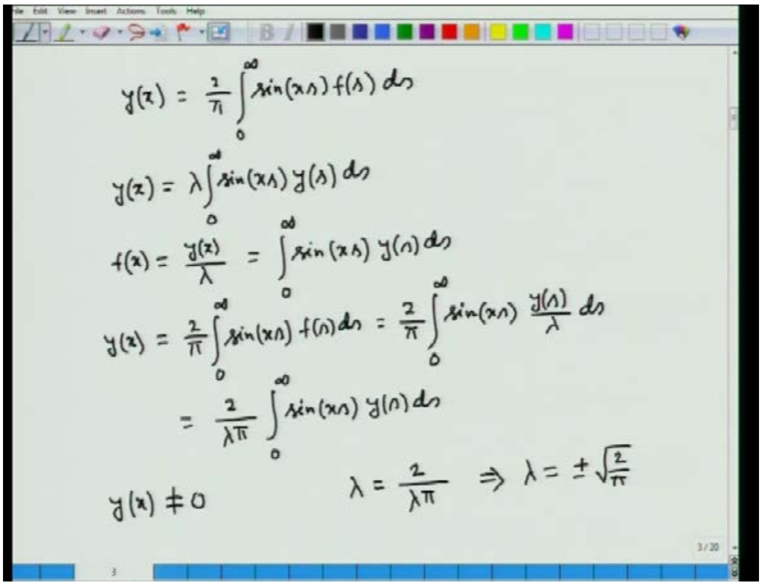

So using the inversion formula for Fourier sin transform, we can find y(x) is equal to 2 by phi integral 0 to infinity sin of x s f(s) ds. Now using the property, we can considered the integral equation which is given by y(x), this is equal to lambda integral 0 to infinity sin of x s y(s) ds. So clearly, this is an singular integral equation. Now, if we choose f(x) that is equal to y(x) by lambda from (( )) expression, then we can find y(x) by lambda this is equal to integral 0 to infinity sin of x s y(s) ds. So, considering this f(x) equal to integral 0 to infinity sin of x s y(s) ds, and using the inversion formula just we have discussed; we can write that y(x) equal to 2 by phi integral 0 to infinity sin of x s f(s) ds, that is equal to 2 by phi integral 0 to infinity sin of x s y(s) by lambda ds. So this is equal to 2 by lambda phi integral 0 to infinity sin of x s y(s) ds.

So, we have started with the expression y(x) equal to lambda integral 0 to infinity sin of x s y(s) ds, and using the concept of Fourier sin transform of a function, and its related inversion formula. We arrived at y(x) equal to 2 by lambda phi integral 0 to infinity sin of x s y(s) ds. So, of course y(x) equal to 0 this is a trivial solution for both this equation, that is y(x) equal to lambda integral sin x is y(s) ds, and y(s) equal to 2 by lambda phi integral 0 to infinity sin x s y(s) ds. Now, if we assume that y(x) not equal to 0. Now, we are assuming y(x) not equal to 0, and then we are intended to look at the possibilities of existence of certain solution, that is some non trivial solution of this equations. Then these two expressions will be comfortable, if and only if, lambda this is equal to 2 by lambda phi, and which gives lambda equal to plus minus root over 2 by phi.

So, these are two Eigen values, and if we just start from y equal to lambda integral 0 to infinity sin x s y(s) ds, then we can prove using some other methods, that these are the Eigen values. Now here, we are going to establish that lambda equal to plus minus root over 2 by phi is the Eigen values for this particular problem. We can take help of an result from an integral.

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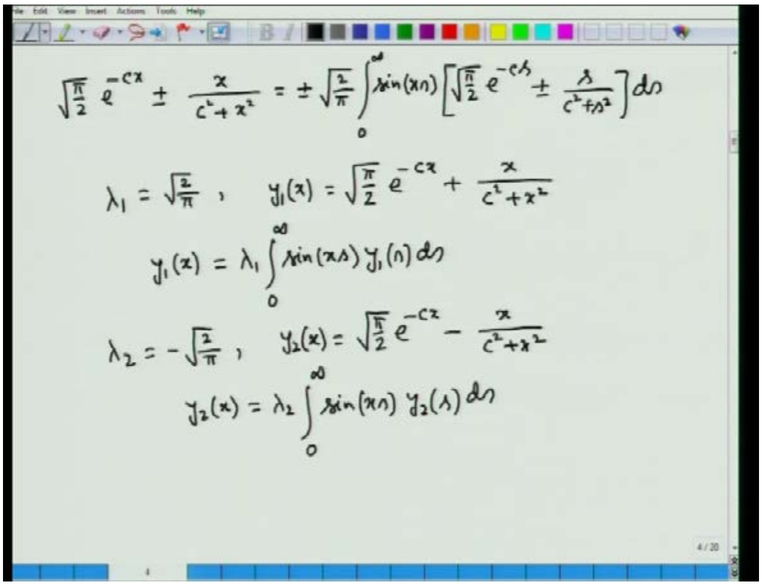

And that result is we can use this result that root over phi by 2 e to the power minus c x plus minus x by c square plus x square, that is equal to plus minus root over 2 by phi integral 0 to infinity sin of x s multiplied by root over phi by 2 e to the power minus c s e to the power minus c s plus minus s by c square plus s square ds, this is the expression. Now, if we write lambda 1 is equal to root over 2 by phi, and y 1(x) this is equal to root over phi by 2 e to the power minus c s plus x by root over c square plus x square. So that means in the (( )) expression, we are considering only the plus sin ignoring the minus sign from (( )) expression, we can write the result, that is y 1(x) is equal to lambda 1 integral 0 to infinity sin of x s y 1(s) ds. So, clearly this lambda 1, and y 1; they satisfies the equation y(x) equal to lambda integral 0 to infinity sin x s y(s) ds, and similarly for lambda 2 equal to minus root over 2 by phi.

And y 2 x this is equal to root over phi by 2 e to the power minus c x minus x by c square plus s square; again we can verify y 2(x) this is equal to lambda 2 integral 0 to infinity sin of x s y 2(s) ds. So, that means for lambda 1 equal to root over 2 by phi, y 1(x) equal to root over phi by 2 e to the power minus c x plus x by c square plus x square; these are the Eigen values, and Eigen functions. And similarly, for lambda 2 equal to minus root over 2 by phi, this y 2(x) is the Eigen function. Now, here c is actually obituary constant; so whatever value of c, if you take c equal to 1, c equal to 2, and so on. So any particular value of c will satisfy the result; that is y 1(x) equal to lambda 1 integral 0 to infinity sin x s y 1(s) d s.

So for each value of c, you will be having a particular expression for y 1(x), and 2 unequal values of c, we can prove that the functions root over phi by 2 e to the power minus c x plus x by c square plus x square, they are linearly independent to each other. So, ultimately will be having for two characteristics values of lambda, that is root over 2 by phi, and root over 2 by phi with pre multiple by minus 1. We they are of infinite multiplicity, as each of them are corresponds to infinitely many linearly independent Eigen functions. And these particular result in contrast with the non singular fredholm integral equation, where we have mention that if lambda is a multiple Eigen value, then its multiplicity should be finite.

So this is one nice example, that is one part it serves an example of a singular integral equation, and again with the help of this result we can verify that with singular integral equation, if we are able to find out Eigen values, Eigen functions. Then for a particular Eigen value, we can find infinitely many linearly independent Eigen functions.

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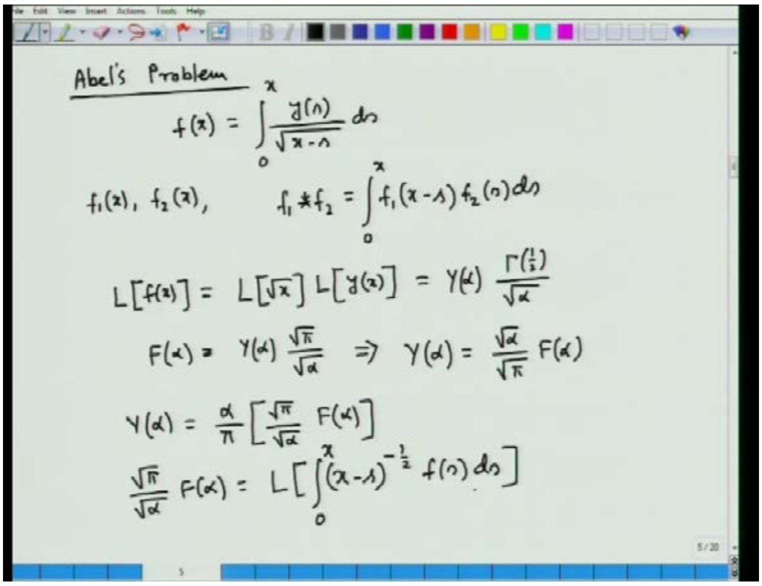

Next we considered Abel's problem; this Abel's problem it is define by f(x) equal to integral 0 to x y(s) root over x minus s ds. In first lecture, we have discuss the origin of this particular problem. Now, we are going to solve this equation using the method of Laplace transform, you can recall for 2 functions f 1(x), and f 2(x); convolution of these 2 function is denoted by f 1 star f 2, and its defined by integral 0 to x f 1(x) minus s f 2(s) ds. So using this result, that is the concept of convolution of 2 functions, we can understand that right hand side of the Abel's problem, that is integral 0 to x y(s) divided by root over x minus s ds; this is nothing but the convolution of two functions, that is root over x and y(x).

So, therefore considering the Laplace transform on the both sides of the given problem, we can write L of f(x) is equal to L of root over x times L of y(x). Now, if you denote the Laplace transform of y(x) as Y alpha; so will be having this is equal to Y alpha times gamma half divided by root over alpha. This gamma half by root over alpha is result of the Laplace transform of root over x. And therefore, we can write that F alpha is equal to Y alpha times root over phi divided by root over alpha, which implies Y alpha - this is equal to root over alpha divided by root over phi multiplied by F alpha.

This is actually the expression for Y alpha. Now from here, we can write this Y alpha is equal to alpha divided by phi times root over phi divided by alpha times F alpha, now root over phi by root over alpha, this is actually Laplace transform of 1 by root over x. And therefore, we can write after taking the writing invest Laplace transform formula are using the Laplace transform of the convolution of two functions, we can write that root over phi divided by root over alpha - F alpha is actually equal to Laplace transform of integral 0 to x x minus s whole to the power minus half times f(s) ds, this is the result.

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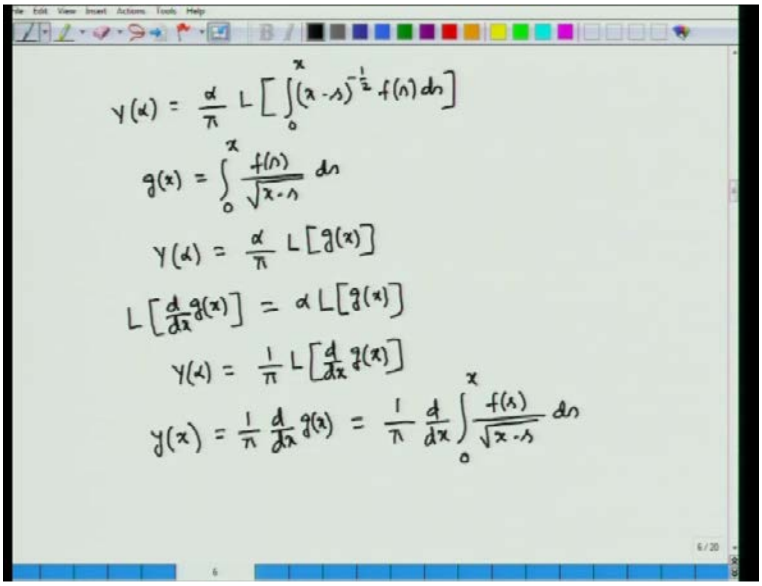

And therefore, Y alpha - this is equal to alpha divided by phi times Laplace transform of integral 0 to x x minus s whole to the power minus half f(s) ds. And, if we denote this function as g(x), that is g(x) equal to integral 0 to x f(s) divided by root over x minus s ds, then this given expression becomes Y alpha is equal to alpha by phi L of g(x). Now we can use this result for Laplace transform that L d dx of g(x); this is equal to alpha times Laplace transform of g(x). So, if we use this particular result, then will be having Y alpha this is equal to 1 by phi Laplace transform of d dx of g(x). Now, taking invest Laplace transform of both sides, we can find solution of this integral equation as y(x) is equal to 1 by phi d dx of g(x), and we have define g(x) equal to integral 0 to x f(s) by root over x minus s ds. This is equal to 1 by phi d dx of integral 0 to x f(s) by root over x minus s ds. So, this is actually solution to the Abel's problem.

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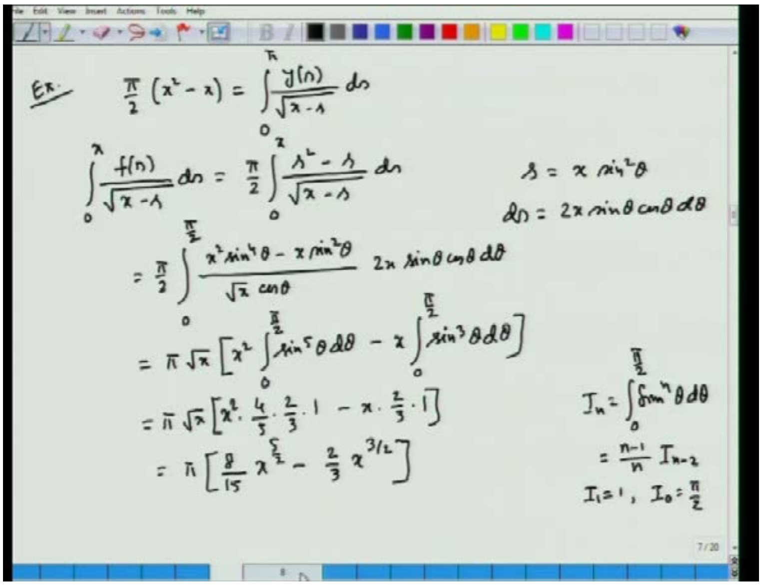

Now we just have a look towards a particular example, that how to solve this kind of problem. We can try to find out solution of this equation, that is phi by 2 into x square minus s is equal to integral 0 to phi y(s) divided by root over x minus s ds. So, clearly here f(x) equal to phi by 2 into x square minus x. So first of all, we have to evaluate this integral; that is integral 0 to x f(s) divided by root over x minus s ds. And this is equal to phi by 2 integral 0 to x s square minus s divided by root over x minus s ds, now we can use the substitution s equal to x sin square theta. So, ds equal to 2 x sin theta cos theta d theta, and from here we will be having phi by 2 limit will be integral 0 to phi by 2, because when s equal to 0 then theta equal to 0, and at s equal to x sin square theta is 1.

So therefore, theta equal to phi by 2, and therefore we will be having x square sin to the power 4 theta minus x sin square theta divided by root over x sin theta, this multiplied with 2 x sin theta cos theta d theta. And therefore, after rearranging this terms, we can find this will be equal to phi times root over x, then x square integral 0 to phi by 2 sin to the power phi theta d theta minus x integral 0 to phi by 2 sin cube theta d theta. In the previous step, it will be cos theta, this one. Now, you can use the reduction formula that is if I n is equal to integral 0 to phi by 2 sin to the power n theta d theta, where n is a positive integer, this is equal to n minus 1 by n I n minus 2; with I 1 equal to 1, and I 0 equal to phi by 2.

You can evaluated this kind of integral, and then substituting we can find phi root over x multiplied with x square into four by 5 into 2 third into 1 minus x into 2 third into 1; so this expression results in phi times 8 by 15 x to the power 5 by 2 minus 2 third x to the power 3 by 2. So, this is actually result of this integration, that is integral 0 2 x f(s) by root over x minus s ds.

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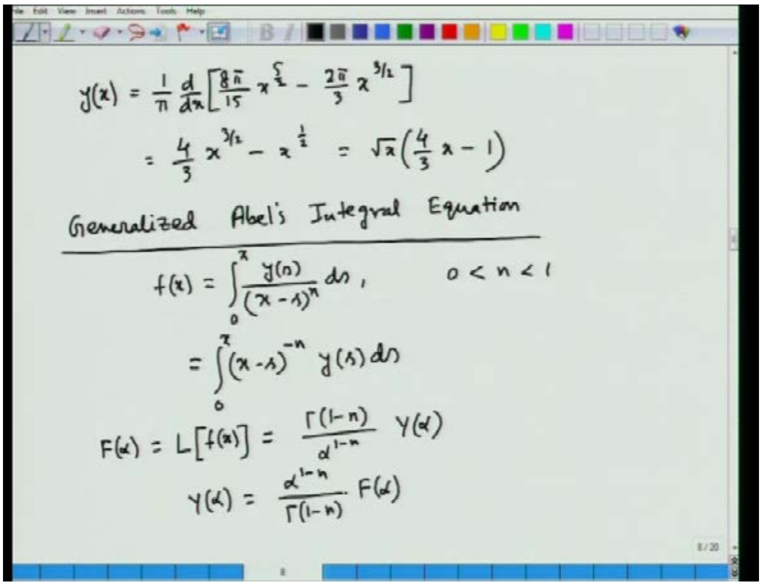

Then required result to the given problem, that means solution of the given integral equation is 1 by phi d dx of 8 phi by 15 x to the power 5 by 2 minus 2 phi by 3 x to the power 3 by 2. And after differentiation, and cancelling this term phi, we can get the solution as 4 by 3 x to the power 3 by 2 minus x to the power half; so this is equal to root over x times 4 by 3 x minus 1. So, this is the solution of the given singular integral equation. So, that means, if you have this Abel's type singular integral equations, that is f(x) equal to integral 0 to infinity.

Sorry 0 to x f(s) by root over x minus s ds; its solution will be given by y(x) equal to 1 by phi d dx of integral 0 to x f(s) by root over x minus s ds; this is the solution. Now, existence of the solution in closed form depends upon the existence of the integral in the closed form, and sometimes we have to take help of some table of integration. In order to find out this integral, that is involved with integral 0 to x f(s) by root over x minus s ds.

Next, we consider the generalized, Abel's integral equation. This generalized Abel's integral equation is defined by f(x) equal to integral 0 to x y(s) divide by x minus s whole to the power n ds, where 0 less than n, less than 1. Now, the pervious example that is the Abel's integral equation what we have consider, that was for n equal to half. So, now we can write this expression as integral 0 to x x minus s whole to the power minus n times y(s) ds, I have written this expression only for the reason, again we are going to solve this problem using the Laplace transform method.

So taking Laplace transform the both sides, we can find F alpha that is equal to L of f(x), and this is equal to Laplace transform of convolution of these two functions; that is x to the power minus n and y(x), where n is ranging between 0 and 1. And then we can find the result as gamma 1 minus n divided by alpha to the power 1 minus n, these multiplied with Y alpha. From here, we can write Y alpha this is equal to alpha to the power 1 minus n divided by gamma 1 minus n multiplied with F alpha. And here again, we are going to apply the same type of procedure, in order to find out the solution of this particular problem.

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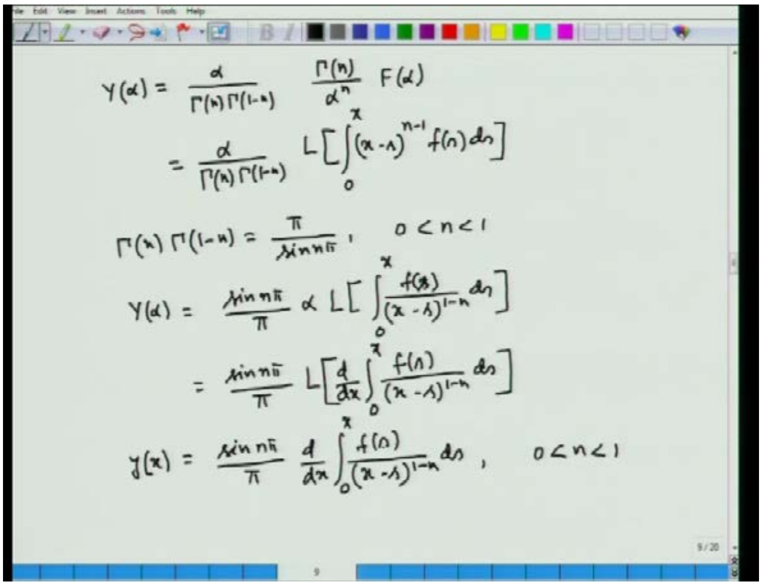

So we can rewrite this expression as, Y alpha this is equal to alpha divided by gamma n multiplied with gamma 1 minus n; then will be having gamma n divided by alpha to the power n, this multiplied with F alpha. And this is equal to alpha by gamma n into gamma 1 minus n, then Laplace transform of integral 0 to x x minus s whole to the power n minus 1 times f(s) d s. And then using the result that gamma n into gamma 1 minus n equal to phi divided by sin n phi, where 0 less than n less than 1, we can write Y alpha this is equal to sin n phi divided by phi times alpha into Laplace transform of integral 0 to x f(s) divided by x minus s whole to the power 1 minus n ds. And using the same procedure, that means L of g dot x is equal to alpha L of g(x), we can write this is equal to sin of n phi divided by phi, then Laplace transform of d dx of integral 0 to x f(s) divided by x minus s whole to the power 1 minus n ds.

And then taking this inverse Laplace transform on both sides, and noting the fact that sin n phi by phi is a constant. We can find y(x) is equal to sin n phi divided by phi d dx of integral 0 to x f(s) divided by x minus s times 1 minus n ds. Now, we can simplify this result further by considering the integral involved with this result, that is y(x) equal to sin n phi by phi d dx of integral 0 to x f(s) by x minus s 1 minus n and ds; keeping in mind that n is ranging between 0 and 1. formula.

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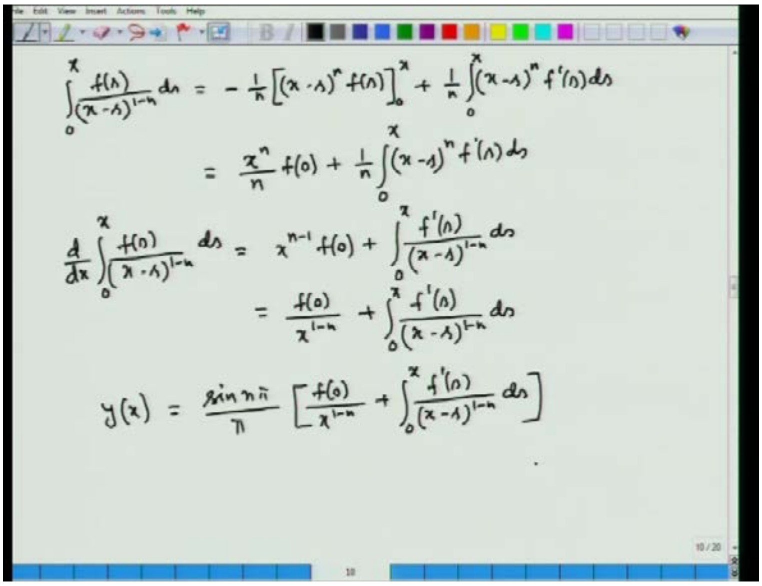

So, considering this integral, and using the Biper's formula, we can write that integral 0 to x f(s) divided by x minus s whole to the power 1 minus n ds; this is equal to minus 1 by n x minus s to the power n f(s), limit 0 to x, then plus 1 by n integral 0 to x x minus s to the power n times f dot s ds. And substituting the limit, we can find this is equal to x to the power n divided by n f(0) plus 1 by n integral 0 to x x minus s whole to the power n f dot s ds. Now, this integral that is integral 0 to x x minus s whole to the power n f dot s ds, this is not an in proper integral.

And therefore, we can apply here Leibniz rule, and then we can write taking derivative on both sides, that is d dx of integral 0 to x f(s) divided by x minus s whole to the power 1 minus n ds; this is equal to x to the power n minus 1 f(0) plus this n will cancels with this one, integral 0 to x f dot s divided by x minus s to the power 1 minus n, then ds. And this is actually equal to, we can write to f(0) divided by x to the power 1 minus n plus integral 0 to x f dot s divided by x minus s whole to the power 1 minus n ds.

And therefore, substituting this expression for d dx of integral f(s) by x minus s to the power 1 minus n ds into the pervious step, we can find solution to the given problem is y(x) is equal to sin n phi divided by phi times f(0) divided by x to the power 1 minus n plus integral 0 to x f dot s divided by x minus s whole to the power 1 minus n ds, this is the result. Now, before concluding this type of singular integral equation, here will be considering one more example, where we can show that directly will not be able to find out solution to this problem into the closed format, but we can find out an approximate solution of the given problem. And this approximation is depending upon the assumption, that x is very small - x is greater than 0, but x is very small.

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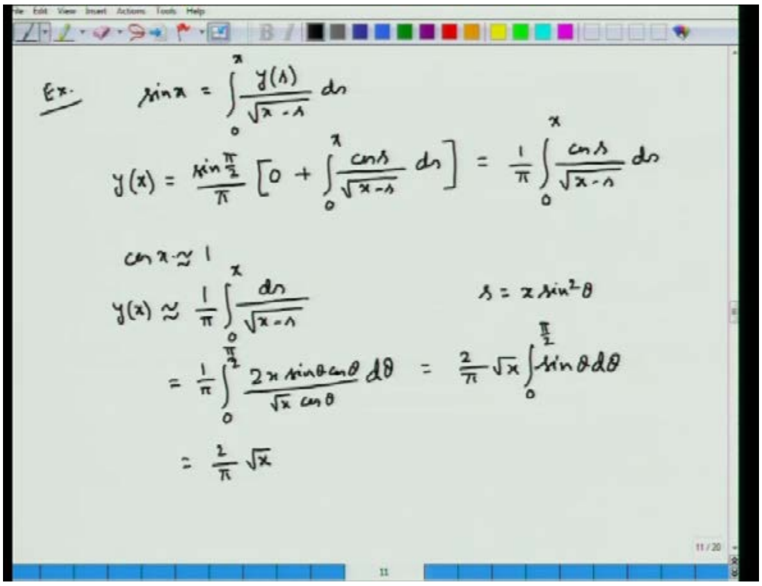

And the particular example is… Let us try to solve this equation sin x is equal to integral 0 to x y(s) divided by root over x minus s ds; according to this last formula, we can write the solution will be sin x times sin phi by 2 divided by phi multiplied with 0 plus integral 0 to x cos s divided by root over x minus s ds. In the given problem f(x) equal to sin x, f(0) equal to 0, f dot s is cosin x, so therefore, solution comes out to be 1 by phi integral 0 to x cosin s divided by root over x minus s ds. Now, in this case evaluation of this integral is little bit problematic, and you will not able find out solution of this integral into the close format.

But we can make an attempt to find an approximate solution, where this cos x is approximated by 1. So therefore, approximate solution y(x) is approximated to 1 by phi integral 0 to x ds by root over x minus s, and again using the same substitution s equal to x sin square theta; we can find this is equal to 1 by phi integral 0 to phi by 2 2 x sin theta cos theta divided by root over x cos theta d theta. So finally, will be having this result 2 by phi root over x integral 0 to phi by 2 sin theta d theta; so this is equal to 2 by phi root over x.

So that assuming, x is very small up to the level when cos x can be approximated by 1, we can find the 2 by phi root over x is an approximate solution of this integral equation. So, if we assume that x is not that much small, such that second power of x can be neglected. So considering the smallness of x such that third, and that higher power of x can be neglected, we can approximate cos x by 1 minus x square by 2, and accordingly after evaluating the integral, we can find a better approximation as a solution of this particular integral equation.

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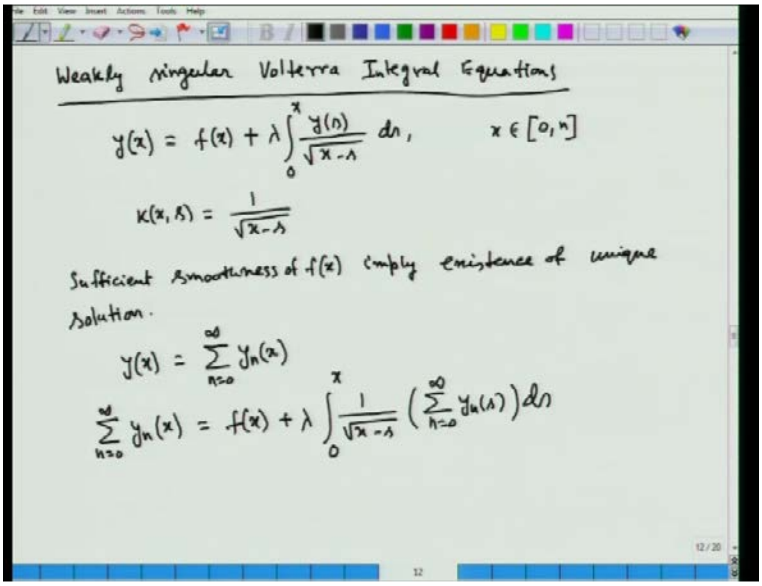

Next we consider weakly, singular Volterra integral equations. There are several analytical, and numerical methods to solve this kind of equations, but you are already familiar with adomian decomposition method. So here, we are we considering adomian decomposition method to solve this weakly singular Volterra integral equation. This weakly singular Volterra integral equation is given by y(x) equal to f(x) plus lambda integral 0 to x y(s) divided by root over x minus s ds, where x belongs to 0 to n, where n is a finite positive number.

And this problem, where actually considering for this particular kernel k(x,s) is equal to 1 by root over x minus s. Of course, there are several other types of kernel, and other type of weakly singular integral equation, but in this lecture series we are considering only one such example. And sufficient smoothness - sufficient smoothness of the f(x) actually implies the existence of unique solution for the given integral equation, imply existence of unique solution. I am not going to prove about this uniqueness and other things, and also not considering the convergence, but just describe how to solve this kind of equations.

So as usuall, we can assume y(x) equal to sigma n runnings from 0 to infinity y n(x) is the possible form of solution of this particular equation. And substituting this expression into the integral equation, and assuming the interchangeability of the summation, and the integral sign, we can find that sigma n runnings from 0 to infinity y n(x); this is equal to f(x) plus lambda times integral 0 to x 1 by root over x minus s summation n runnings from 0 to infinity y n(s) d s.

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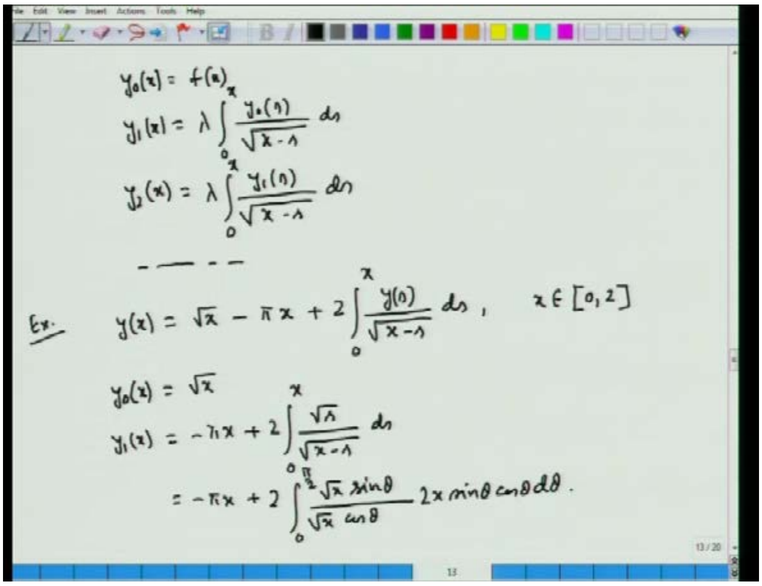

Then, we can find out the successive eta rates by equating y 0(x) is equal to f(x), we can find y 1(x) is equal to lambda integral 0 to x y(0) s divided by root over x minus s ds, y2(x) is equal to lambda integral 0 to x y 1(x) divided by root over x minus s ds, and so on. As we have assumed y(0) equal to f(x), and depending upon the evaluation of this integral; theoretically this y 1, y 2 all this successive eta rates are actually exists. So once, we have the expression for y 0, y 1, y 2, and so on. Then, summing up this series actually gives the solution to the given problem, but one thing you have to keep in mind, that sometimes instead of considering y0 equal to f(x), we can decompose f(x) into two parts; say f 1(x) plus f 2(x). And considering y 0(x) equal to f 1(x) will give us quickly, what is going to be the solution. Because a clever choice of y 0(x) equal to f 1(x) instead of, it is exactly equal to f(x); sometimes we can immediately find, other eta rates are exactly equal to 0. And we can illustrate this concept with help of an example, is example is consider this equation y(x) is equal to root over x minus phi into x plus 2 integral 0 to x y(s) divided by root over x minus s ds; this is the integral equation.

So here f(x) is actually root over x minus phi x. Now, instead of considering y 0 equal to root over x minus phi x; first we consider y 0(x) - this is equal to root over x. And then, actually we are considering here f(x) equal to f 1(x) plus f 2(x), then next eta rate y 1(x) will be equal to minus phi x plus 2 integral 0 to x root over s divided by root over x minus s ds. Now for this particular problem, x belongs to (0,2), this is required for the convergence of this particular series. And now, if we evaluate this integral, then it will be minus phi x plus 2 integral 0 to phi by 2; similar as a earlier, it will be root over x sin theta divided by root over x cos theta; this multiplied with 2 x sin theta cos theta d theta.

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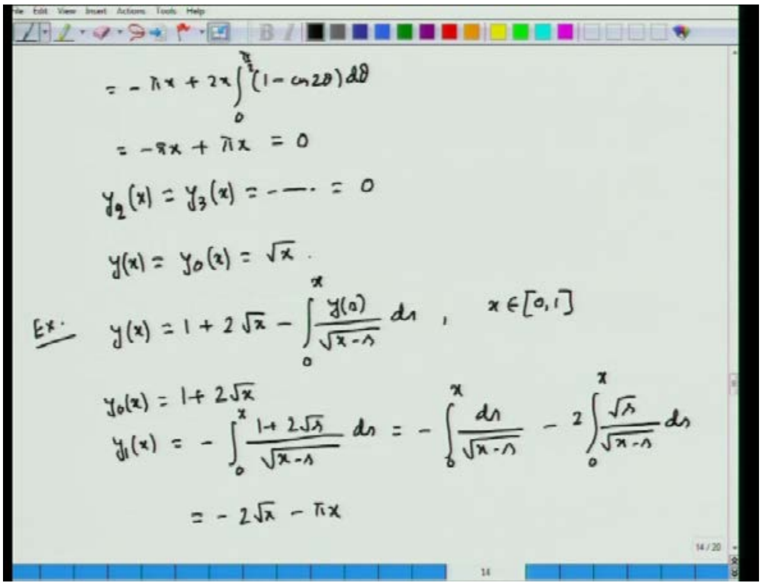

And after evaluating this integral will be having minus phi x plus 2 x integral 0 to phi by 2 1 minus cos 2 theta d theta; this is the integral. And after evaluating this integral, we can find this is minus phi x plus will be having phi x, the second integral is that is integral 0 to phi by 2 cos two theta d theta is equal to 0, so this is equal to 0. So with the choice of y 0(x) equal to root over x, we have arrived at y 1(x) equal to 0, and therefore clearly y 2(x), y 3(x), and so on.

All these quantities are exactly equal to 0; and therefore, solution to this particular problem is given by y(x) is equal to y 0(x), and that is equal to root over x. You can easily verify that y(x) equal to root over x is a solution to the given problem. Now, we consider one more example of this type, if we consider this equation y(x), this is equal to 1 plus 2 root over x minus integral 0 to x y(s) divided by root over x minus s ds, here x belongs to the interval 0 to 1. Then, if we choose y 0(x), this is equal to 1 plus 2 root over x.

Then y 1(x) will be equal to minus integral 0 to x 1 plus 2 root over s divided by root over x minus s ds, and just for your understanding. Here, I am dividing this integral into 2 parts; that is minus integral 0 to x ds divided by root over x minus s minus 2 integral 0 to x root over s divided by root over x minus s ds. And using the similar approach, if you solve this integral, then it will results in minus 2 into root over x minus phi x. So, assuming y 0(x) equal to 1 plus 2 root over x, you are getting y 1(x) equal to minus 2 root over x minus phi x.

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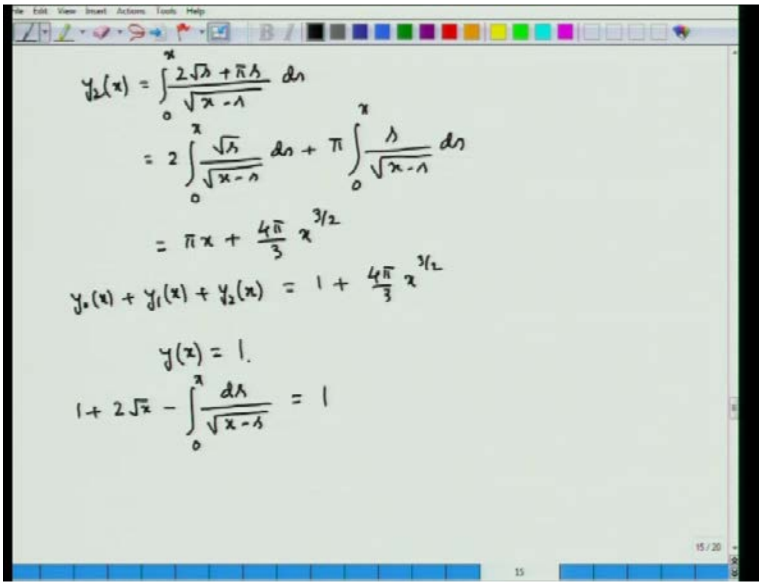

So with this y 1(x), if you calculate y 2(x), then y 2(x) will be equal to integral 0 to x 2 root over s plus phi s divided by root over x minus s ds; so this is equal to 2 integral 0 to x root over s by root over x minus s ds plus phi integral 0 to x s by root over x minus s ds. Now, you can see in this y 2(x) we are having 2 integral 0 to x root over s by root over x minus s ds, and in the expression for y 1(x), we had minus 2 integral 0 to x root over s divided by root over x minus s ds. So that means, if you consider this sum, then second term of y 0(x) will cancel with fast term of y 1(x), and second term of y 1(x) will cancel with second term of first term of y 2(x), and so on. So as intense to infinity after summing up, and using the condition that x belongs to 0 to 1. So you can see, some higher powers of x will come up, if you calculate the further eta rate for y n.

And in this case, this will be equal to phi x plus 4 phi by 3 times x to the power 3 by 2. So this expression y 0(x) plus y 1(x) plus y 2(x), ultimately results in 1 plus 4 phi by 3 times x to the power 3 by 2. And this power of x will increase, if you calculate further eta rates, and other terms will cancels with each other, and ultimately as intense to infinity you will be landed at the solution y(x) is equal to 1. And you can easily verify that 1 plus 2 root over x minus integral 0 to x ds divided by root over x minus s; this is equal to 1.

So, that means y(x) equal to 1 is a solution; I have consider this example only for the reason, if you try to solve this equation by considering y 0(x) equal to 1, and if you do not take this 2 root over x within the consideration for y 0(x), then you can find y 1(x) will be equal to 0, and all other eta rates will be exactly equal to 0. So, that means for a clever choice of y(x) will give you the solution quickly, this depends upon the fact that whether the solution of the given problem will exists in a closed format or it contains, if finite number of terms in x or not.

If the actual solution does not exists in a closed format, and if it be a infinite series of x, then there is no way for this clever choice for y(0). It only give you some idea, that in case of closed form solution or solution having finite number of terms in x, sometimes this clever choice give you quickly the complete solution, because other eta rates are comes out to be exactly equal to 0. So today, we can conclude at this particular point, we are not going to discuss anything more on the singular integral equation, but of course, there are lots of other theories, and techniques dealing with the solution of singular integral equation. In the next lecture, we will be considering intrigue differential equation of both the type, that is Volterra integral equation as well as Fredholm integral equation type with intrigue differential approach. So, thank you for your attention.