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Module No. # 01
Lecture No. # 37

Calculus Of Variations and Integral Equation

Welcome viewers, once again to the lecture series under NPTEL program on Integral Equation. In today's lecture, we are going to discuss about Hilbert Smith theory and its consequences such that, we can solve the non homogeneous Fredholm integral equations of second kind, by using the orthogonal functions associated with the corresponding homogeneous Fredholm integral equation. So, we are actually going to discuss the Hilbert Smith theorem and this related results will be used to solve the or find the solution of the Fredholm integral equation with symmetric kernel, this is very much important.

So, Hilbert Smith theory, in this lecture, we are surely concentrated on the property of the kernel, that kernel should be symmetric. So, before going to state the Hilbert Smith theorem, I am not going to prove the result, but before going to state the theorem, we need some relevant results related with the eigen values and eigen functions for the given problem.

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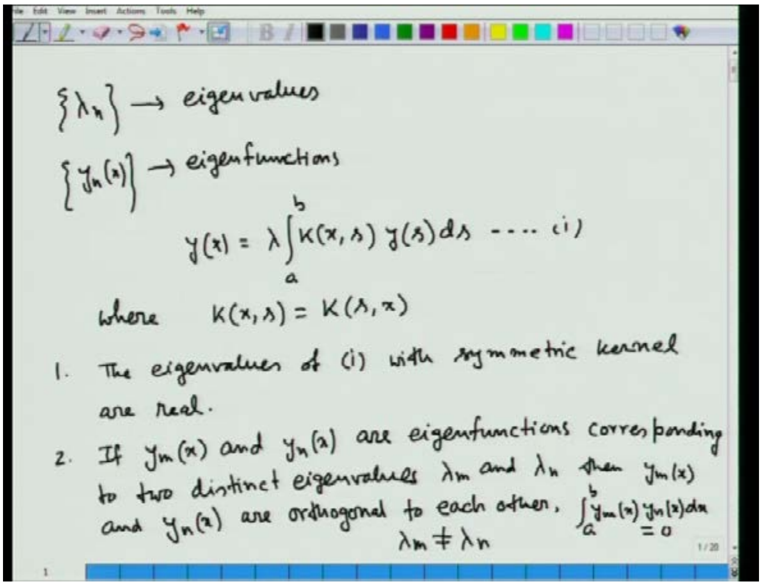

So, first of all we consider that lambda n is the eigen values, these are eigen values and y n x denotes the associated eigen functions, these are eigen functions and these eigen values are eigen functions are associated with the integral equation y x equal to lambda integral a to b K of x comma s y s d s, we call this particular equation as number 1, so this is a Fredholm integral equation, which is a homogeneous Fredholm integral equation. And the kernel K x comma s is symmetric that means, it satisfies the property K x comma s is equal to K of s comma x, now we state certain results related with this eigen values and eigen functions.

First of all the eigen values of 1 that means, this integral equation with symmetric kernel are real, so that means, if we consider this Fredholm integral equation y x equal to lambda integral a to b K x comma s y s d s, where this kernel K x comma s which is symmetric, then all eigen values of this particular problem will be real.

Second property, if y m x and y n x are eigen functions corresponding to two distinct eigen values corresponding to two distinct eigen values lambda m and lambda n, then y m x and y n x are orthogonal to each other that means, integral a to b y m x y n x d x, this is equal to 0, where lambda m this is not equal to lambda n. So that means, if we consider two eigen functions y m x and y n x corresponding to two distinct eigen values lambda m and lambda n, then integral a to b y m x y n x d x equal to 0, implying they are orthogonal to each other.

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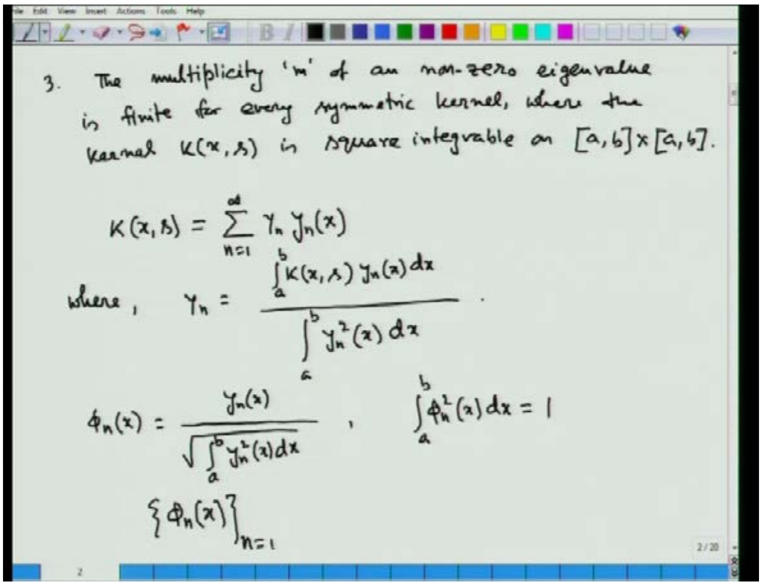

Number 3, the multiplicity m of an non zero eigen value is finite for every symmetric kernel, where the kernel K x comma s is square integrable on the square a comma b cross a comma b, so that means, if the kernel is symmetric and square integrable over the square a comma b cross a comma b, then every non zero eigen value having multiplicity m that means, multiplicity should be a finite quantity.

Now, with this and some other related results regarding the completeness of this set of eigen functions, we can find out the orthogonal expansion or fourier expansion of the kernel K x comma s in terms of the eigen functions. So first of all, we can write that K x comma s can be written as summation n running from 1 to infinity gamma n y n x, where this gamma n is defined by integral a to b K of x comma s y n x d x divided by integral a to b y n square x d x. Now, this representation can be simplified, if we use the orthonormal eigen functions instead of set of orthogonal eigen functions y n x.

So, if we define phi n x is equal to y n x divided by square root of integral a to b y n square x d x, then we can easily verify that integral a to b phi n square x d x, this is equal to 1 and therefore, we can expand this kernel K x comma s in terms of this orthonormal eigen functions phi n x, so that means, we can consider this set of orthonormal eigen functions phi n x n running from 1 to infinity.

Now, based upon all this observations and other related results, we can develop the Hilbert Smith theorem and this Hilbert Smith theorem is related with the expansion of f x, that is the inhomogeneous part of the Fredholm integral equation associated with the given Fredholm integral equation with symmetric kernel. The result states that, the result states that if we consider the integral equation of the form f x equal to lambda times integral a to b, K of x comma s y s d s, where this K x comma s is actually symmetric kernel, then f x can be expressed as an orthogonal expansion or you can say fourier series expansion in terms of the functions phi n x.

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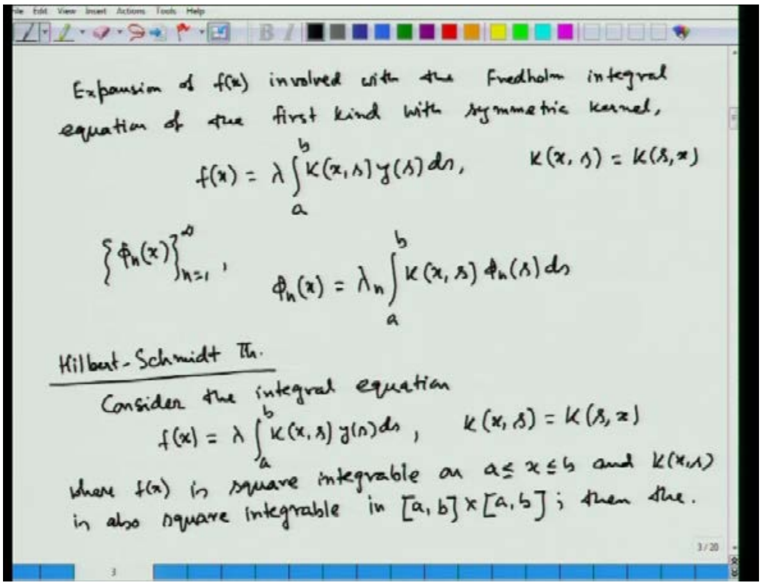

So, what we are going to do, we are actually intended to find out expansion of f x involved with the Fredholm Fredholm integral equation of the first kind with symmetric kernel, which is given by f x equal to lambda integral a to b K of x comma s y s d s, this is symmetric so that means, it satisfies the condition, this one. In terms of the orthonormal eigen functions phi n x and these phi n x is actually related by this formula, that is phi n x equal to lambda n integral a to b K of x comma s phi n s d s.

So that means, this phi n x, they are the eigen functions and lambda n they are the eigen values of the Fredholm integral equation, that y x equal to lambda integral a to b K of x comma s y s d s, so with this heads we can now state the Hilbert Smith theorem, this is the Hilbert Smith theorem. Considered the integral equation, f x equal to lambda integral a to b K of x comma s y s d s with symmetric kernel K x comma s is equal to K s comma x, where f x is square integrable on the closed interval a less than equal to x less than equal to b and the kernel K x comma s is also square integrable is also square integrable in the square a comma b cross a comma b.

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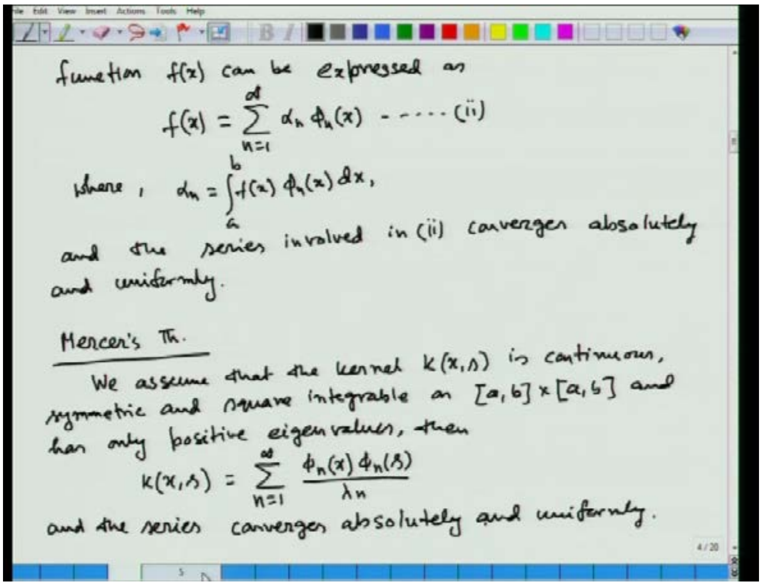

Then, the function f x can be expressed as f x can be expressed as f x equal to sigma n running from 1 to infinity alpha n phi n x we call this particular series as two, where alpha n are obtained from this formula integral a to b f x phi n x d x and the series involved in two converges, absolutely and uniformly this is actually Hilbert Smith theorem. So, if we just go through this theorem again, so first of all we are going to express f x, which is involved with the Fredholm integral equation of first kind with symmetric kernel, in terms of the set of orthonormal eigen functions obtained by solving the problem y x equal to lambda integral a to b K of x comma s y s d s, where this K x comma s is a symmetric kernel.

So, once we are able to find out the orthonormal set of eigen functions phi x then, this f x can be expresses as sigma n running from 1 to infinity alpha n phi n x, where each alpha n can be obtained from the formula integral a to b f x phi n x d x and here you have to keep in mind one important result, that this theorem is applicable whenever this f x is square integrable over the interval a comma b and apart from the symmetric kernel K x comma s K x comma s this kernel should be square integrable over the square a comma b cross a comma b.

These theorem and now we are going to define another theorem that is Mercer's theorem, they are essential to find out the solutions of the Fredholm integral equation with symmetric kernel, where the resolvent kernel of the Fredholm integral equation can be expressed in terms of this orthonormal eigen functions. So, now, we state another theorem this is Mercer's theorem, it states that, we assume that the kernel K x comma s is continuous symmetric and square integrable on a comma b cross a comma b and has only positive eigen values, then K x comma s that is the symmetric kernel can be expressed as sigma n running from 1 to infinity, phi n x phi n s divided by lambda n and the series involved with this expression, converges, absolutely and uniformly.

So that means, using two theorems, we can obtain the orthogonal series expansion or fourier series expansion for f x and the kernel K x comma s. Now, using this result now we can find out the solution of the Fredholm integral equation with symmetric kernel.

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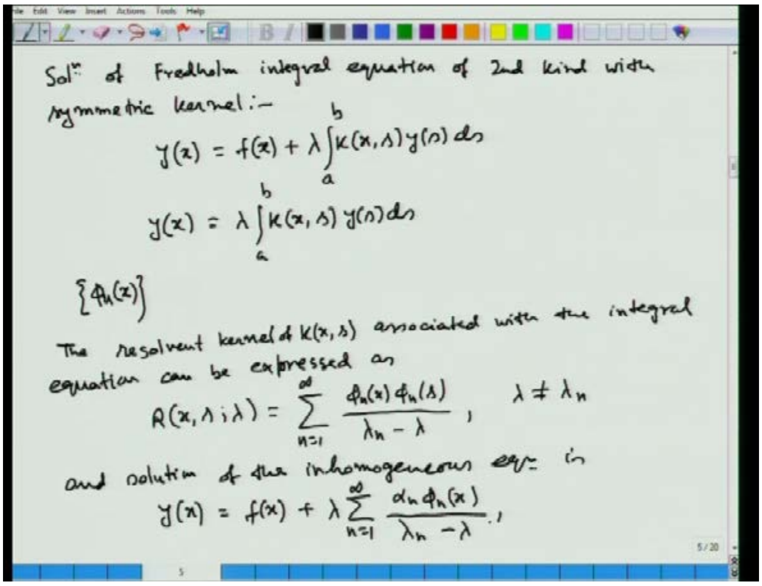

So, we are going to find out the solution of Fredholm integral equation of second kind with symmetric kernel, so that means, our target is to find the solution of this equation y equal to f x plus lambda integral a to b K of x comma s y s d s.

And in order to find out solution of this particular problem, first of all we have to calculate the orthonormal eigen functions from the associated homogeneous problem, that is y x is equal to lambda integral a to b K of x comma s y s d s. Actually the resolvent kernel are x s lambda corresponding to the symmetric kernel K x comma s can be expressed as the set of orthonormal eigen functions phi n x, corresponding to the set of eigen values lambda n associated with this homogeneous Fredholm integral equation.

The resolvent kernel of K x comma s associated with the integral equation associated with the integral equation can be expressed as R x s lambda this is equal to sigma n running from 1 to infinity phi n x phi n s divided by lambda n minus lambda, when lambda not equal to lambda n and hence the solution of the inhomogeneous equation can be written as solution of the inhomogeneous equation is y x is equal to f x plus lambda sigma n running from 1 to infinity alpha n phi n x divided by lambda n minus lambda, where alpha n is integral a to b f x phi n x d x.

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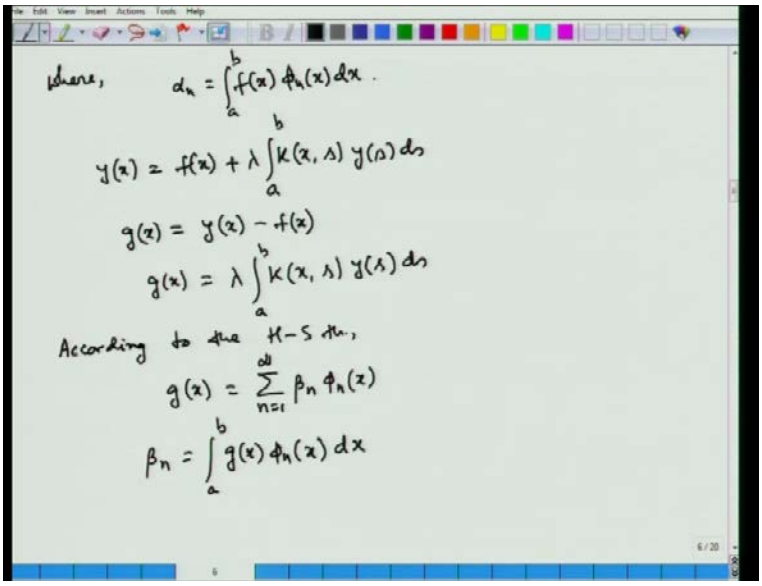

Now, we can try to derive this result using the Hilbert Smith theorem and other notations we have introduced earlier, that is we need the definition for this alpha n actually and then we can find out the solution to the Fredholm integral equation. So, given equation is y x equal to f x plus lambda integral a to b K of x comma s y s d s. Now, if we define the function g x equal to y x minus f x, then this given Fredholm integral equation inhomogeneous Fredholm integral equation of second kind can be put into the form, that g x equal to lambda integral a to b K of x comma s y s d s, now for this problem we can apply the Hilbert Smith theorem.

So, according to Hilbert Smith theorem, we can write g x is equal to sigma n running from 1 to infinity beta n phi n x, where beta n is integral a to b g x phi n x d x, this result we can write using Hilbert Smith theorem.

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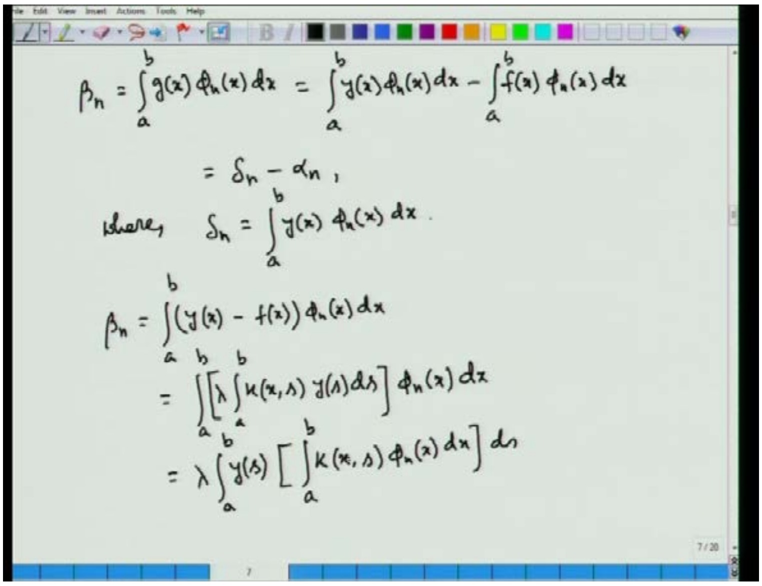

Now, this beta n, this is equal to integral a to b g x phi n x d x and this is equal to integral a to b y x phi n x d x minus integral a to b f x phi n x d x and this is equal to we can write delta n minus alpha n, where alpha n already we have defined as integral a to b f x phi n x d x.

So, here this delta n is equal to integral a to b y x phi n x d x. Now, for this problem y x is the unknown function so that means, delta n is also unknown, so our target will be replaced this delta n in terms of some known quantities such that, we can find out the solution of the Fredholm integral equation, and mainly for the given problem you can understand f x is given. Once f x is given, so if somehow we are able to relate this delta n with alpha n then, we can find out this beta n in terms of alpha n and hence we can find out the solution of the given problem.

So, for this purpose we can write this beta n is equal to integral a to b y x minus f x times phi n x d x, this is equal to integral a to b, we can substitute y x minus f x, this is equal to lambda times integral a to b k x comma s y s d s. So, from there we can write, this is lambda integral a to b K of x comma s y s d s, this expression with phi n x d x. Now, we can interchange the order of the integration, we can take lambda outside the integral sign, so this will be lambda integral a to b y s, then integral a to b K of x comma s phi n x d x d s and this is a very crucial step, because we are already familiar with the expression that, phi n x is equal to lambda integral a to b K x comma s phi n s d s with lambda equal to lambda n.

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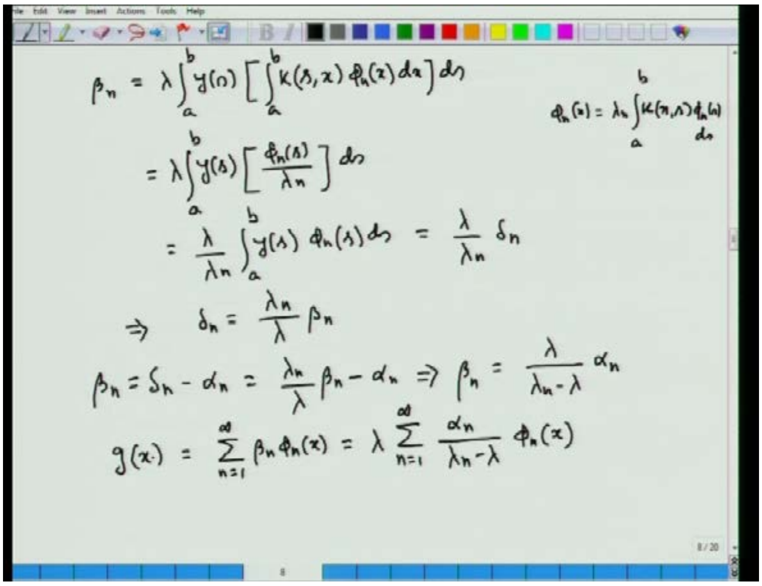

So, in order to apply that result, we can use the property of symmetric kernel to interchange the variables. And therefore, we can write this beta n is equal to lambda times integral a to b y s, then integral a to b K of s comma x phi n x d x d s, this is the result. Now, using that definition that is the result phi n x is equal to lambda n integral a to b K of x comma s phi n s d s, we can find that integral a to b K s comma x phi n x d x is nothing but, 1 by lambda n times phi n s.

So, therefore, this will be equal to lambda times integral a to b y s, the expression under the square bracket will be simply phi n s divided by lambda n d s, so this is equal to lambda divided by lambda n integral a to b y s times phi n s d s, this one (Refers Slide Time: 31:20). And this is equal to lambda divided by lambda n times delta n, because we have used the notation integral a to b y x phi n x d x equal to lambda n. And from here, we can write delta n this is equal to lambda n divided by lambda times beta n and therefore, beta n is equal to delta n minus alpha n equal to lambda n by lambda beta n minus alpha n.

These implies beta n, this is equal to lambda divided by lambda n minus lambda times alpha n. So, we have obtained this beta n and hence, this g x is equal to sigma n running from 1 to infinity beta n phi n x. So, that is equal to lambda sigma n running from 1 to infinity alpha n divided by lambda n minus lambda times phi n x.

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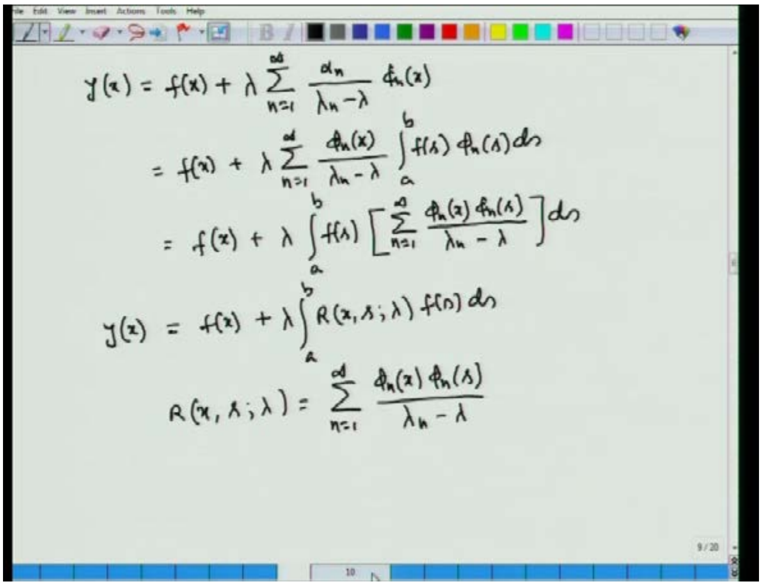

And therefore, using d x equal to y x minus f x from here, we can write y x this is equal to f x plus lambda sigma n running from 1 to infinity alpha n divided by lambda n minus lambda this into phi n x. Now, if we substitute the expression for alpha n, so this will be equal to f x plus lambda sigma running from 1 to n phi n x divided by lambda n minus lambda integral a to b f s phi n s d s.

Now, already we have discussed about the uniform convergence of this particular infinite series, that is sigma n running from 1 to infinity alpha n phi n x associated with the f x and therefore, interchanging the summation and integral sign, we can write this is equal to f x plus lambda integral a to b f s, then summation n running from 1 to infinity phi n x phi n s divided by lambda n minus lambda this d s.

And clearly you can recall that in terms of resolvent kernel, we have written solution of this particular problem as y x equal to f x plus lambda integral a to b R of x s lambda f s d s, so we have this expression in the same format and therefore, comparing this R x s lambda with this particular term, we can find that R x s lambda this is equal to sigma n running from 1 to infinity phi n x phi n s divided by lambda n minus lambda, so this is actually the target form of the solution that is, this is the resolvent kernel R x s lambda equal to summation n running from 1 to infinity phi n x phi n s by lambda n minus lambda.

And this expression is valid whenever, lambda not equal to lambda n, so this gives the solution, where this resolvent kernel R x s lambda can be evaluated in terms of the eigen values lambda n and set of orthonormal eigen functions phi n x, obtained from the associated homogeneous Fredholm integral equation. So, this is actually the use of Hilbert Smith theorem in order to find out the solution of the Fredholm integral equation.

Now, you have to keep in mind that this treatment is completely based upon the assumptions, that the parameter lambda is not equal to any one of the eigen values lambda n, so we have eigen values lambda 1, lambda 2, lambda 3, and so on. If this parameter lambda involved with the Fredholm integral equation is not equal to any one of this eigen values, then we can find out solution by this method.

And therefore, we can find this solution as y x equal to f x plus lambda integral a to b R x s lambda f s d s where R x s lambda is the resolvent kernel, and with help of these orthonormal eigen functions, we can find out this resolvent kernel given by the last formula. Now, we consider the case that, if lambda is equal to some of the eigen values say lambda K plus 1 and I am going to use this lambda K plus 1 in order to write the formula in a suitable format and to include the multiplicity of the eigen values.

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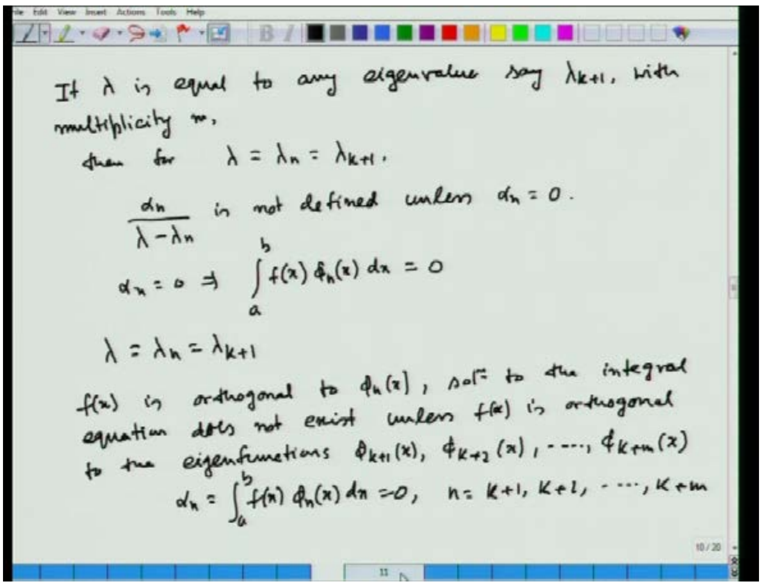

So, now if we assume that lambda is equal to any eigen value, say lambda K plus 1 with multiplicity m, then for lambda equal to lambda n is equal to lambda K plus 1, the coefficient that is lambda n by sorry alpha n by lambda minus lambda n is not defined, unless alpha n this is equal to 0. Now, we look at the result alpha n equal to 0, this actually implies integral a to b f x phi n x d x, this is equal to 0.

So that means, for lambda equal to lambda n equal to lambda K plus 1, the function f x is orthogonal to the associated eigen function phi K plus 1 x. If this happen and this alpha n equal to 0, then the quantity alpha n divided by lambda minus lambda n, this becomes indeterminate when lambda is equal to lambda n. And therefore, the condition alpha n equal to 0 leads us to the case that, alpha n becomes an arbitrary quantity. And therefore, if f x is orthogonal to phi n x solution to the integral equation does not exist, unless f x is orthogonal to the eigen functions, phi K plus 1 x phi, K plus 2 x up to phi K plus m x, because we have considered that the eigen value lambda equal to lambda K plus 1 is of multiplicity m corresponding to the eigen value that is K plus 1.

And that means, alpha n is equal to integral a to b f x phi n x d x, this is equal to 0 for n equal to K plus 1, K plus 2 up to K plus m. If these condition is satisfied, that is f x is orthogonal to each of these eigen functions phi K plus 1 x 2 phi K plus m x.

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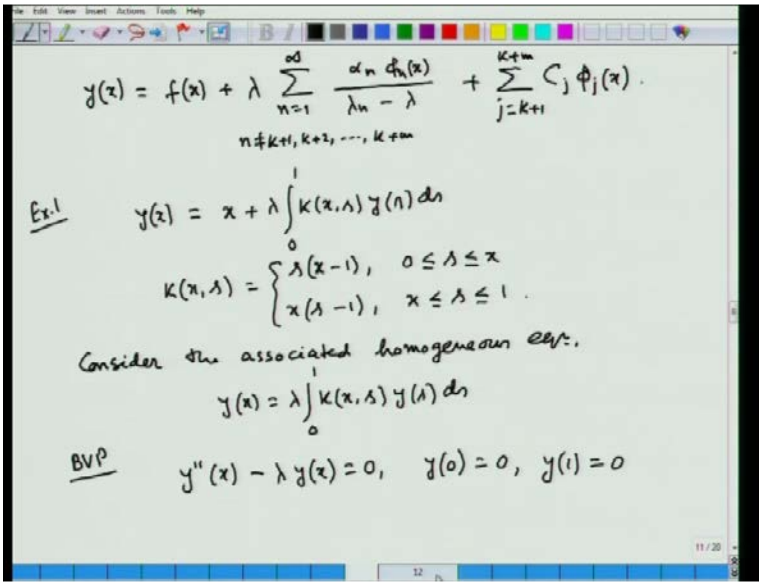

Then, we can write the solution to the given problem as y x equal to f x plus lambda times sigma n running from 1 to infinity n naught equal to K plus 1, K plus, 2 up to K plus m alpha n phi n x divided by lambda n minus lambda plus summation j running from K plus 1 2 K plus m C j phi j x, where all these C j's are actually arbitrary constants. And in this case, when phi x is orthogonal to the eigen functions phi K plus 1 x, phi K plus 2 x up to phi K plus m x, associated with the enfold eigen value lambda equal to lambda K plus 1, then we have infinitely many solutions of the given problem.

Finally, we consider two examples in order to understand these results that means, how this can be used to find out solution of the Fredholm integral equation, so first of all we consider the problem y x equal to x plus lambda integral 0 to 1 K of x comma s y s d s, where K x comma s is equal to s times x minus 1 for 0 less than equal to s less than equal to x and x times s minus 1 for x less than equal to s less the equal to 1, we have to solve this equation in terms of orthogonal eigen functions.

So first of all, we have to consider the associated homogeneous equation, that is y x equal to lambda integral 0 to 1 K of x comma s y s d s. Earlier we have discussed how this type of problem can be converted to boundary value problem. So, using the same tricks and the form of the kernel K x comma s, differentiating these equation twice, you can convert this problem to a boundary value problem, which is defined by y double dot x minus lambda y x, this is equal to 0 with the boundary conditions y 0 equal to 0 and y 1, this is also equal to 0.

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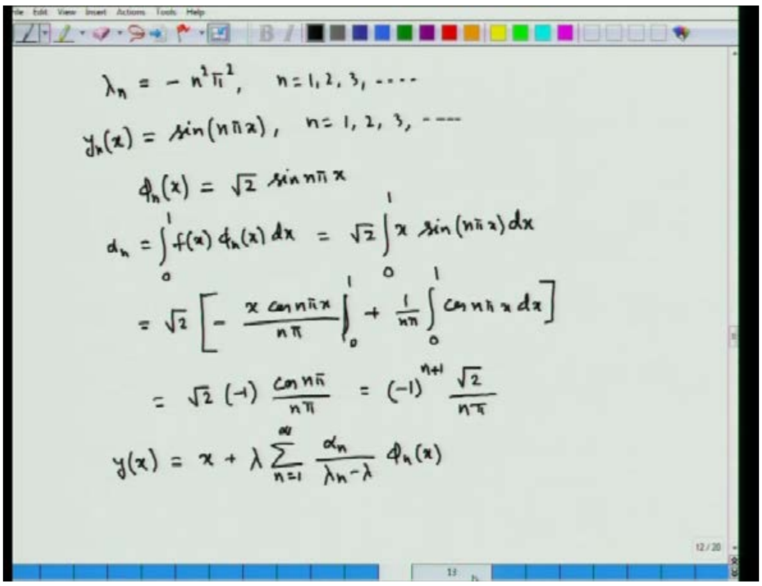

And using the standard procedure of eigen value eigen functions for this particular problem, you can calculate the eigen values of this particular problem exist, whenever lambda is negative. And in particular the eigen values are given by lambda n equal to minus n square pi square n equal to 1, 2, 3, and so on. And the corresponding eigen functions y n x is equal to sin of n pi x, where n equal to 1, 2, 3, and so on.

And for this eigen functions y n x is equal to sin n pi x, we can calculate the corresponding set of orthonormal eigen functions, that is phi n x this will be root 2 sin of n pi x. Now, with this pi n x where n ranging from 1, 2, 3, up to infinity we can calculate this alpha n is equal to integral 0 to 1 f x phi n x d x for the given problem the non homogeneous part is x, so therefore, f x equal to x, so this is equal to root 2 integral 0 to 1 x sin n pi x d x. And using the formula for integration by parts we can derive this is root 2 times minus x cosine n pi x whole divided by n pi limit 0 to 1, then plus 1 by n pi integral 0 to 1 cosine n pi x d x and this will be equal to root 2 times minus 1 cosine n pi divided by n pi.

Last integral will be exactly equal to 0 and if we substitute it to the first term x equal to 0, that is lower limit that will be also 0, so you will be survived only with the term root 2 times minus 1 cos n pi divided by n pi now cos n pi is equal to minus 1 to the power n, so this is equal to minus 1 whole to the power n plus 1 times root 2 divided by n pi. So, this is actually value for alpha n and then we can find out the solution to the given problem, that y x is equal to x plus lambda sigma n running from 1 to infinity alpha n divided by lambda n minus lambda times phi n x.

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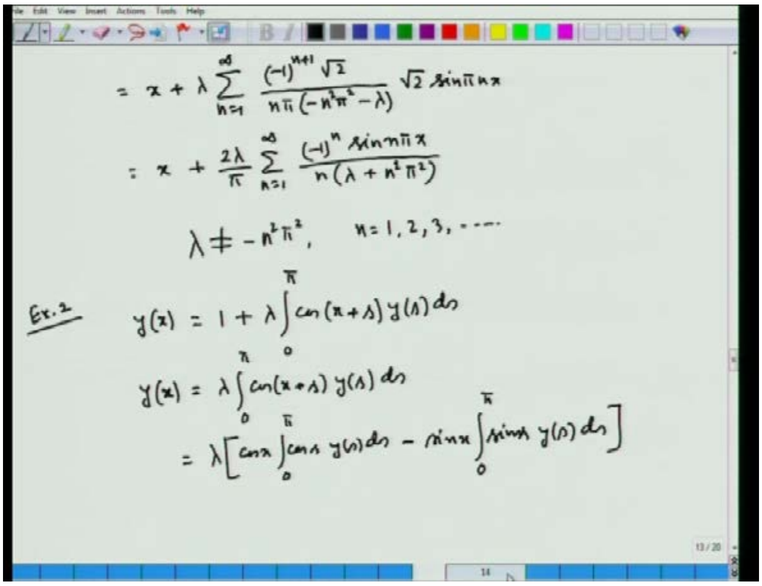

So, substituting we can find x plus lambda sigma n running from 1 to infinity minus 1 to the power n plus 1 root 2 divided by n pi minus n square pi square minus lambda multiplied by root 2 sin n pi x. Now, if we take minus 1 common from the denominator and this 2 by pi outside the summation sin, so therefore, we can find solution of this problem as y x equal to 2 lambda divided by pi summation n running from 1 to infinity minus 1 to the power n sin n pi x divided by n into lambda plus n square pi square.

So, that means, this solution y x equal to x plus 2 lambda divided by pi sigma n running from 1 to infinity minus 1 whole to the power n sin n pi x divided by n into lambda plus n square pi square, this is a valid solution whenever lambda is not equal to minus n square pi square for n equal to 1, 2, 3, and so on.

If lambda is not equal to any one of this eigen values, then we have this particular solution. Next, we consider another example this is very interesting example, where we can show that depending up on values of lambda we have three situations that is unique solution, no solution and infinite remaining solutions. The problem is y x equal to 1 plus lambda integral 0 to pi cos of x plus s y s d s.

So, first of all we have to find out eigen values and eigen functions for y x equal to lambda integral 0 to pi cos of x plus s y s d s and using the procedure of separable kernel, we can calculate the eigen values and we can recall that, we can rewrite this expression as cos x integral 0 to pi cos s y s d s minus sin x integral 0 to pi sin s y s d s and then defining this first integral 0 to pi cos s y s d s as c 1 and 0 to pi sin s y s d s as c 2 we can find out the eigen values and eigen functions for this particular problem.

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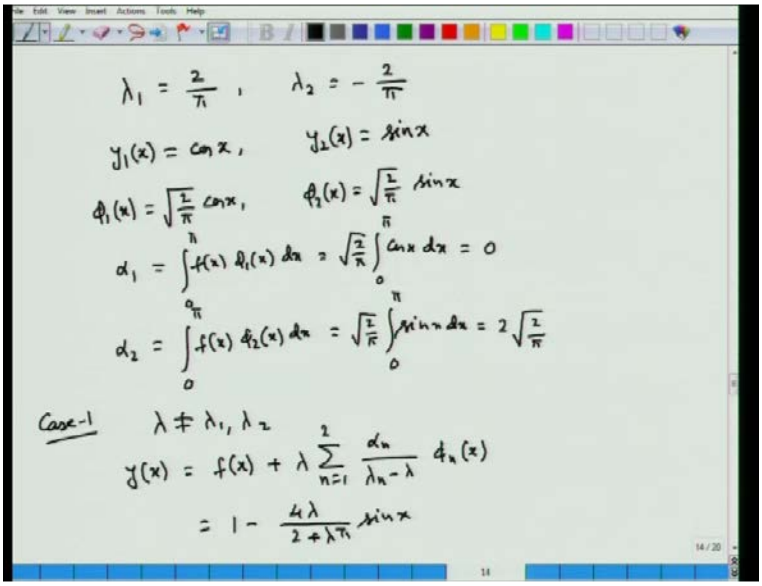

I am not going to solve that part and if you solve it then you can find lambda 1 is equal to 2 by pi and lambda 2 this is equal to minus 2 by pi. And associated eigen functions will be y 1 x this is equal to cos x and y 2 x this is equal to sin x, so these are eigen values and eigen functions. Now, if we use the ortho-normalization condition, then we can find orthonormal eigen functions that is phi 1 x is equal to root of our 2 by pi cosine x and phi 2 x, this is equal to root of our 2 by pi sin x and from here this phi 1 and phi 2, we can calculate the constants alpha 1 and alpha 2, because here we have only two eigen functions, this root of at 2 by pi cosine x and root of at 2 by pi sin x, so we have to calculate only two constants alpha 1 and alpha 2.

So, alpha 1 is equal to integral 0 to pi f x phi 1 x d x, this is equal to root of at 2 by pi integral 0 to pi co sin x d x, this is equal to 0 and alpha 2 this is equal to integral 0 to pi f x phi 2 x d x, so this is equal to root of at 2 by pi integral 0 to pi sin x d x, this is equal to 2 into root of at 2 by pi.

So, with these result case 1, if we consider that lambda not equal to lambda 1 comma lambda 2, then y x will be equal to f x plus lambda sigma n equal to 1 to 2 alpha n divided by lambda n minus lambda phi n x. Now, alpha 1 is equal to 0 and alpha 2 we have obtained here, so after substituting you can find this is equal to 1 minus 4 lambda divide by 2 plus lambda pi times sin x, this is the unique solution to the given problem. When lambda not equal to either lambda 1 or lambda 2.

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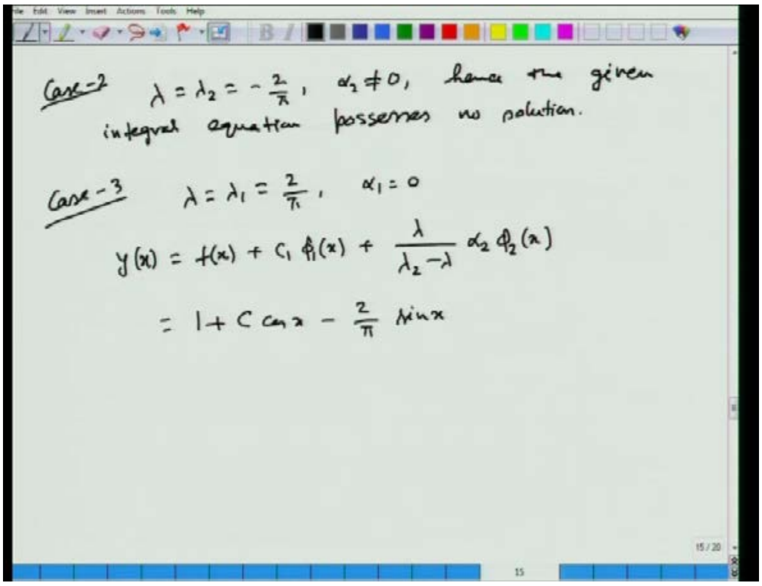

Case 2, if lambda equal to lambda 2 is equal to minus 2 pi and has lambda 2 not equal to 0, hence the given equation, given integral equation possesses no solution, there is no solution for this problem. And case 3, if lambda equal to lambda 1 is equal to 2 by pi alpha 1 equal to 0 and in this case problem have infinitely many solutions, those are given by y x equal to f x plus c 1 phi 1 x plus lambda divided by lambda 2 minus lambda alpha 2 phi 2 x and this will be equal to 1 plus c cos x minus 2 by pi sin x, this c is the arbitrary constant, so this is your infinite number of solutions.

So, these example we have explained that depending upon lambda, whether it is equal to lambda 1 or lambda 2 and if these are not equal to either of this eigen values of the problem then we have unique solution, if lambda equal to lambda 2 then the given problem does not possesses any solution and in case of lambda equal to lambda 1, then we have infinitely many solutions.

So, this illustrates the method we have described, to find out the solution of the Fredholm integral equation with the help of Hilbert Smith theorem. So, I can stop this lecture at this point. Thank you for your attention.