Calculus Of Variations and Integral Equation

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Welcome viewers, once gain to the lecture series of NPTEL on the topic Integral Equation. In the last lecture, we were discussing the successive approximation or eternity method for solving non homogeneous Fredholm integral equation of the second kind. And we have considered one example in the last lecture, to find out a solution using that particular method. Now in these lecture we are again going to address the same eternity method in order to define the resolvent kernel, and in terms of resolvent kernel we are going to describe solution of the non homogeneous Fredholm integral equation of the second kind.

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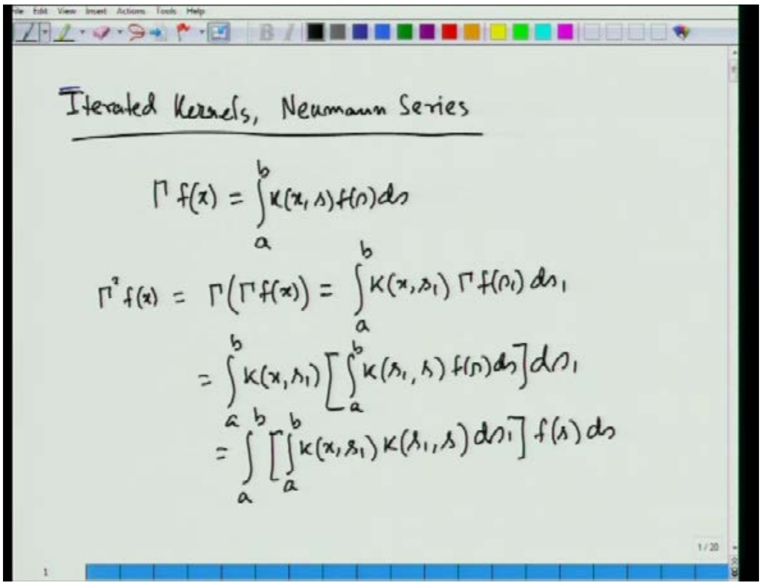

So, in these lecture, we are going to consider the topic that is iterated kernels, which ultimately leads us to Neumann series which is will be used to solve the Fredholm integral equations. In as per the previous discussion you can recall, we have introduced this notation for integral operator that is capital gamma f x is equal to integral a to b k of x, s f s d s, we have introduced this notation. Now, in order to obtain the iterated kernels that we have done for Volterra integral equations, we can write these gamma 2 f x is nothing but, the integral operator gamma is operating upon gamma f x, so that means, this integral operator gamma is operating upon gamma f x and therefore, we can write this is equal to integral a to b k of x comma s 1 gamma f of s 1 d s 1.

So, here this f s is repressed by gamma f s 1 and we have considered this dummy variable as s 1, in order to define the integral operator gamma on gamma f x, and then using the definition for gamma s 1, we can write integral a to b k of x, s 1 then integral a to b k of s 1, s f s d s then d s 1. Now, rearranging the terms that means, interchanging the order of integration we can write this is actually integral a to b k of x, s 1 then k s 1, s d s 1 these result can be integrated from a to b multiplied with f s then d s.

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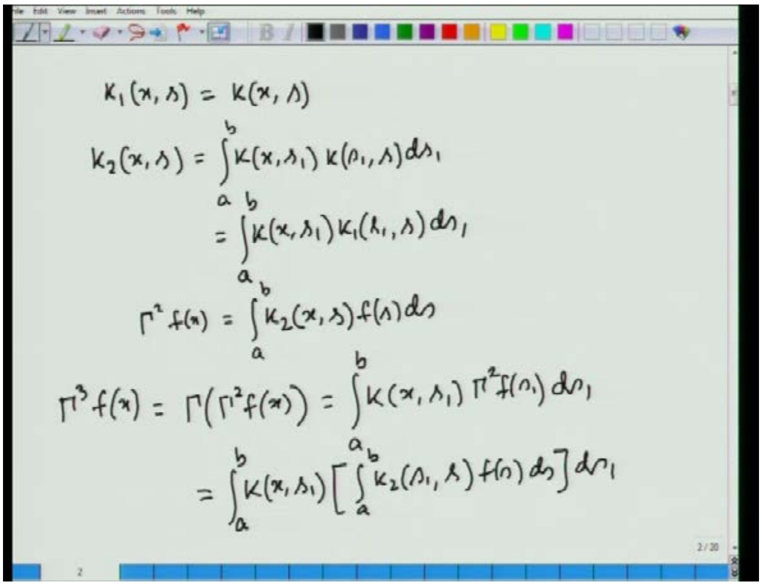

Now, if we define that k 1 x, s this stands for k x s same as we have done in case of Volterra integral equations therefore, k 2 x, s can be defined by integral a to b k of x, s 1 then k s 1, s d s 1, so this actually integral a to b k x, s 1, now repressing this k s 1 s by k 1 s 1, s d s 1 we get the second iterated kernel k 2 x, s.

And therefore, gamma 2 f x comes out to be integral a to b k 2 x, s f s d s this is the expression for gamma 2 f x, next if we calculate gamma 3 f x in terms of iterated kernel, then we can find this gamma is operating upon gamma 2 f x similarly, as previous what we have done that is integral a to b k of x, s 1 gamma 2 f of s 1 d s 1. Now, from here we can write gamma 2 f s 1 this will be equal to integral a to b k of x, s 1 then integral a to b k 2 s 1, s f s d s this with d s 1.

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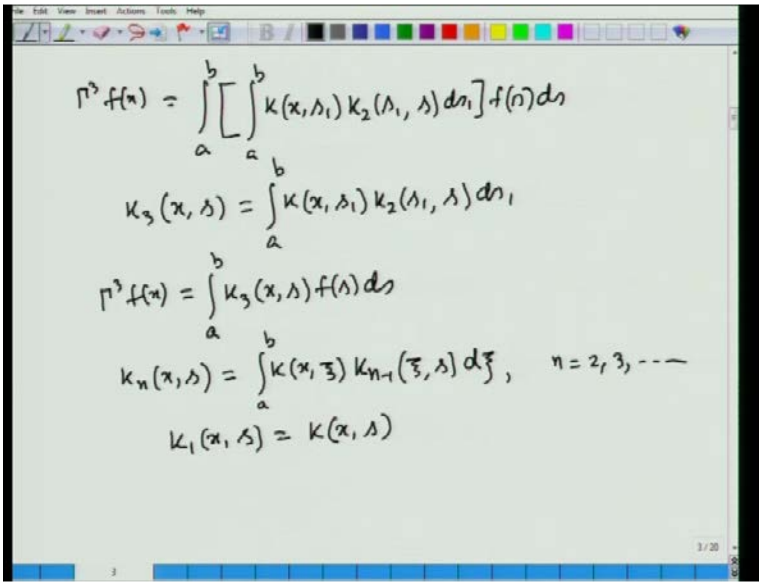

Again interchanging the order of the integration, we can write gamma 3 f x this is equal to integral a to b then integral a to b k of x, s 1 then k 2 s 1, s d s 1 multiplied with f s d s and now, if we define that k 3 x, s is equal to integral a to b k of x, s 1 then k 2 s 1, s d s 1, so therefore, gamma 3 f x will be equal to integral a to b k 3 x, s f s d s.

So, proceeding in this particular way, we can find nth iterated kernel k n x, s that is equal to integral a to b k of x, xi k n minus 1 xi, s d xi in all this definition for k 2 x s k 3 x, s here, these dummy variable s 1 can be replaced by xi, so that means, in general k n x, s equal to integral a to b k x, xi k n minus 1 xi, s d xi and these particular result holds for n equal to 2, 3 and so on. And where k 1 x, s is exactly equal to k of x, s and therefore, you can recall the solution for the Volterra integral equation, what we have considered in the last lecture that was…

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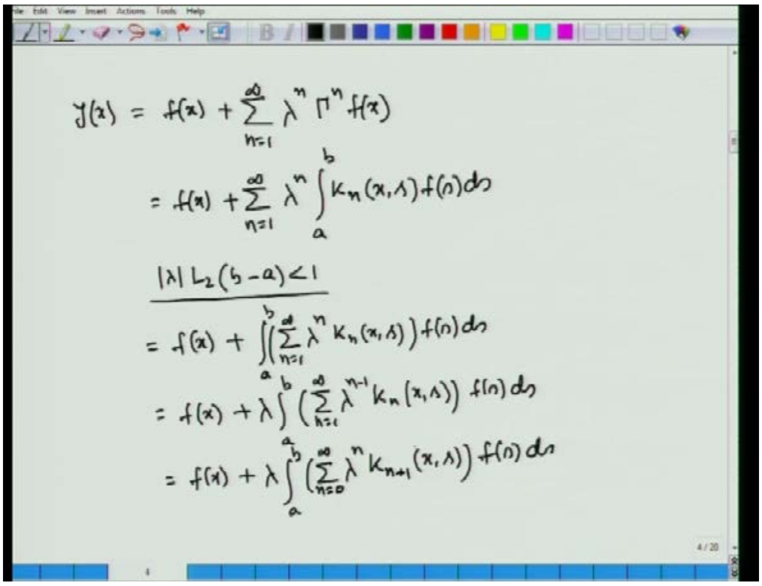

y x is equal to f x plus sigma in running’s from 1 to infinity lambda to the power n gamma n operated upon f x, this was the result of the solution integral equation; that this is the solution for the Fredholm integral equation. And now, using these earlier results that is for gamma 3 f x gamma 2 f x and in general, you can write also this gamma n f x in terms of this nth order iterated kernel, we can write this is equal to f x plus sigma in running’s from 1 to infinity lambda to the power in integral a to b k n x, s f s d s this is the result (Refer Slide Time: 09:17).

And assuming satisfaction of this condition that is modulus lambda L 2 b minus a less than 1, assuming this condition hold where L 2 is actually maximum value of the kernel k x, s it is modulus within the interval a, b cross a, b that is within a square therefore, we can interchange this summation and integral sign. Because, in the last lecture we have already proved the uniform convergence of this infinite series and therefore, this is equal to f x plus integral a to b sigma n running’s from 1 to infinity lambda to the power n k n x, s this entire expression multiplied with f s d s.

Now, taking one lambda outside the integral sign we can write, this is equal to f x plus lambda integral a to b sigma n running’s from 1 to infinity lambda to the power n minus 1 k n x, s this f s d s. And now, changing the range of variation for in we can get this is equal to f x plus lambda integral a to b sigma n running’s from 0 to infinity then it will be lambda to the power n k n plus 1 x, s this f s d s, so therefore, this infinite series that is sigma n running’s from 0 to infinity lambda to the power n k n plus 1 x, s, this is actually resolvent kernel.

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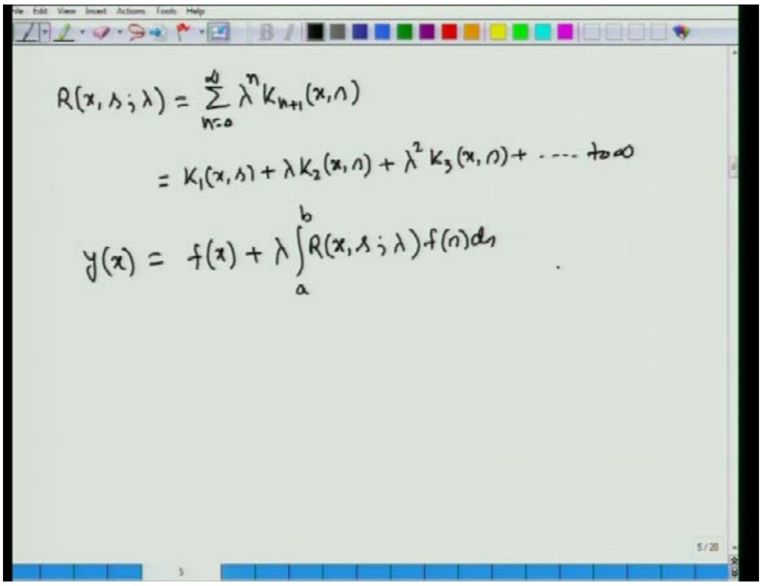

And this resolvent kernel it is denoted by R x s lambda and that is equal to sigma n running’s from 0 to infinity lambda to the power n k n plus 1 x, s, so that is actually equal to k 1 x, s plus lambda k 2 x, s plus lambda square k 3 x, s plus dot dot up to infinity. And therefore, with this resolvent kernel R x, s semi colon lambda we can write solution of the Fredholm integral equation is y x equal to f x plus lambda integral a to b R x s lambda f s d s this is actually solution to the given problem; and this series that is k 1 x s plus lambda k 2 x s plus lambda square k 3 x s plus dot dot up to infinity this series actually call the Neumann series. And this is the solution of this Fredholm integral equation, of the second kind which is a non homogeneous equation in terms of the resolvent kernel.

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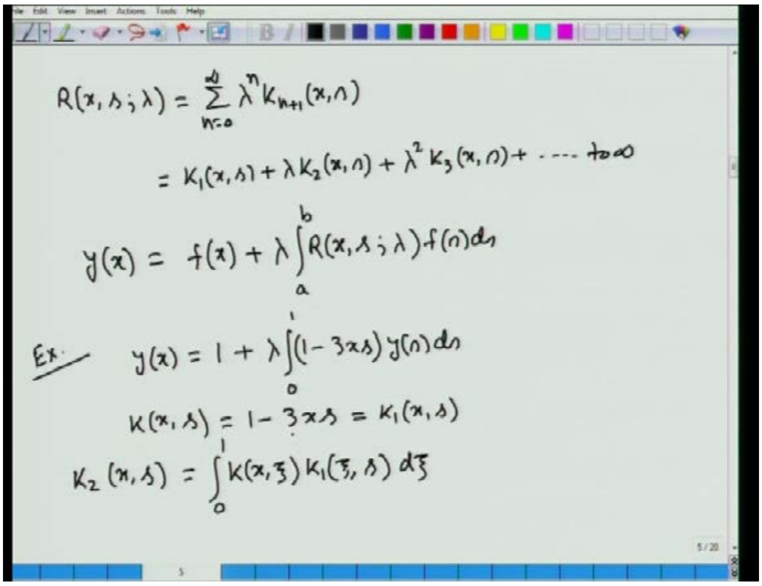

Now, we consider one interesting example, this example you can find in many books for example, the book by Karneval as well as Hildebrand in different books you can find this very famous example, and these example will address again in some later lectures in order to compare the different methods by for the solution of Fredholm integral equation.

Now, here we are considering the problem that is y x is equal to 1 plus lambda integral 0 to 1 1 minus 3 x s y s d s we have to solve this problem. So, therefore, our kernel k x, s this is equal to 1 minus 3 x s. Now, first we calculate few initial iterates that is k 2 x, s k 3 x, s and so on, and then using the Neumann series we can calculate that resolvent kernel and then in terms of resolvent kernel we write down the solution for the given problem. So, here this k x, s is nothing but, your k 1 x, s next we have to calculate this k 2 x, s, by definition this is integral 0 to 1 k x, xi multiplied with k 1 xi, s d xi, so with this definition that is k x s equal to 1 minus 3 x s and k 1 x s equal to 1 minus 3 x s.

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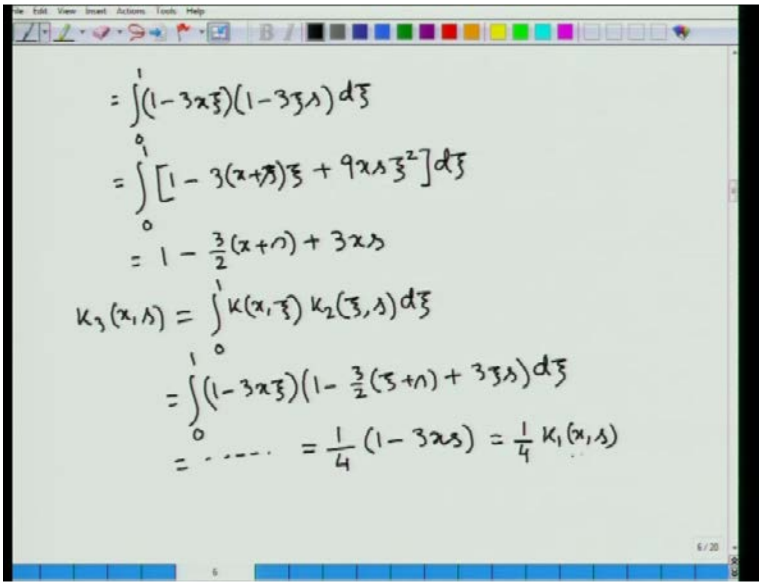

We can write this is equal to integral 0 to 1 1 minus 3 x xi this multiplied with 1 minus 3 xi s d xi, so this is equal to integral 0 to 1 1 minus 3 x plus s, this multiplied with xi plus 9 x s xi square d xi this one, and after integration we can find this will be equal to 1 minus 3 by 2 x plus s plus 3 x s this will be the result, so this is actually our second iterated kernel k 2 x, s. Using this definition for k 2 x, s not definition this is actually we have derived k 2 x, s, so this expression we can calculate k 3 x, s.

So, k 3 x, s by definition integral 0 to 1 k x, xi then k 2 xi, s d xi this is equal to integral 0 to 1 1 minus 3 x xi this multiplied with 1 minus 3 by 2 xi plus s plus 3 xi s d xi and after with respect to xi you can arrive at this result, this will be equal to 1 by 4 1 minus 3 x s. So, these result is very much important, because from here you can observe this k 3 x, s is nothing but, 1 by 4 k 1 x, s, so what, we have assumed k 1 x, s and that is actually your given kernel k x, s.

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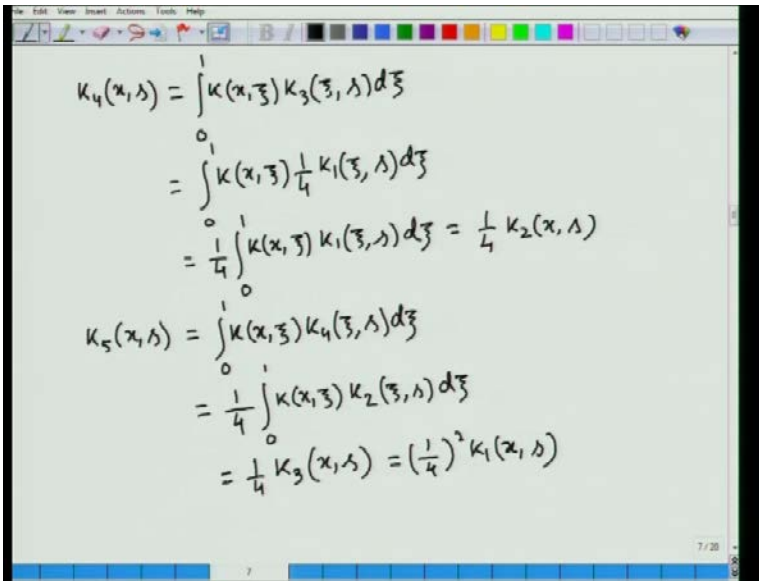

So, with these result that is k 3 x, s is equal to 1 4 multiplied with k 1 x, s you can calculate k 4 x, s, now k 4 x, s is equal to integral 0 to 1 k x, xi then k 3 xi, s d xi, so this is equal to integral 0 to 1 k x, xi times 1 4 k 1 xi, s d xi, so that is equal to 1 by 4 integral 0 to 1 k x, xi k 1 xi, s d xi, so this is equal to 1 by 4 k 2 x, s because, 0 to 1 k x xi k 1 xi is d xi is nothing but, k 2 x, s.

So, similarly, if you calculate k 5 x s this will be equal to integral 0 to 1 k x, xi k 4 xi, s d xi, now k 4 x, s is equal to 1 4 k 2 x, s, so using this result you can write this is equal to 1 by 4 integral 0 to 1 k x, xi k 3 xi, s d xi this result you can obtain, this will be equal to sorry it will be 2, this one, so this is nothing but, 1 by 4 k 3 x, s this will be the result for k 5. Now, already we have obtained that k 3 x, s is equal to 1 4 k 1 x, s, so this is equal to 1 by 4 whole square k 1 x, s, so with these few results, we can claim that in general will be having this recursive formula.

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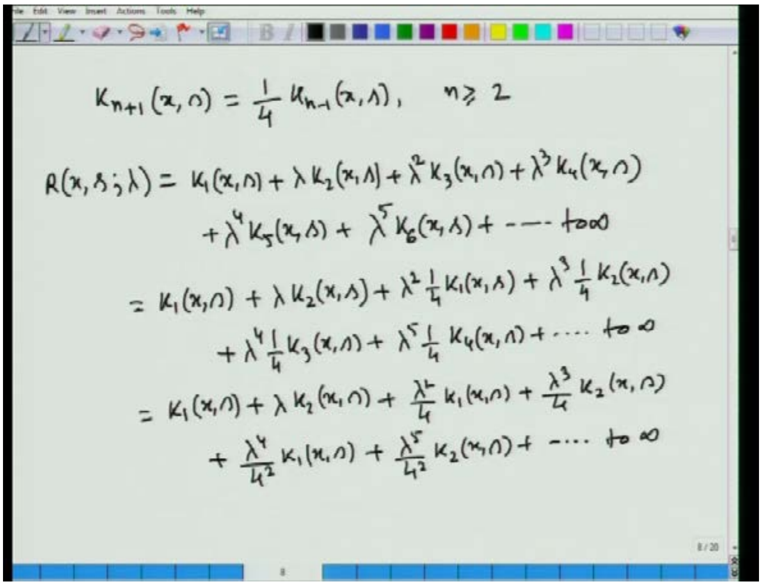

That is k n plus 1 x, s this is equal to 1 by 4 k n minus 1 x, s this result is valid for n greater than equal to 2, so this is actually one important step that we have obtained. So, from here, we can write R x s lambda that means, with this recursive relation and with some few initial itter iterates of the kernel, we can calculate the dissolvent kernel R x, s lambda for the given problem.

So, this is equal to k 1 x, s plus lambda k 2 x, s plus lambda square k 3 x, s plus lambda q k 4 x, s plus lambda to the power 4 k 5 x, s plus lambda to the power 6, it will be lambda to the power 5 not 6, lambda to the power 5 k 6 x, s plus dot dot up to infinity.

And now, we can use this result for eternity kernels and some initial results to get this will be equal to k 1 x, s as usual there is no change, no change for k 2 x, s then lambda square it will be 1 by 4 k 1 x, s plus lambda q it will be 1 by 4 k 2 x, s and then lambda to the power 4 1 by 4 k 3 x, s plus lambda to the power 5 1 by 4 k 4 x, s plus dot dot up to infinity.

Then using the result in last two terms, that is k 3 x, s is equal to 1 by 4 k 1 x, s and k 4 x, s equal to 1 by 4 x 2 x, s we can write, this is equal to k 1 x, s plus lambda k 2 x, s plus lambda square by 4 k 1 x, s plus lambda q by 4 k 2 x, s plus lambda to the power 4 by 4 square k 1 x, s plus lambda to the power 5 by 4 square k 2 x, s plus dot dot up to infinity, so we have one set of term where k 1 x, s is there and other set of terms involving k 2 x, s.

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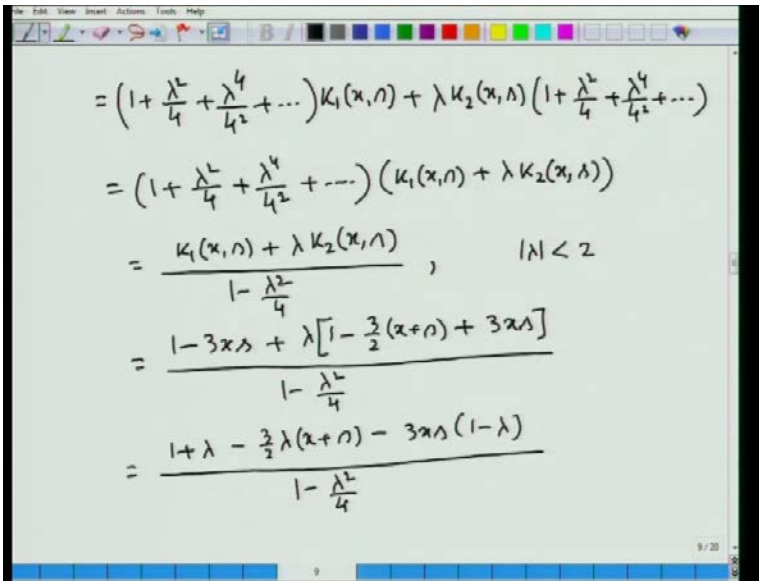

So, these expression is equal to 1 plus lambda square by 4 plus lambda to the power 4 by 4 square plus dot dot, this multiplied with k 1 x, s and for the rest of the term, if you take common lambda and k 2 x, s then this will be multiplied with 1 plus lambda square by 4 plus lambda to the power 4 by 4 square plus dot dot. So, ultimately we are having this expression that is 1 plus lambda square by 4 plus lambda to the power 4 by 4 square plus dot dot up to infinity these multiplied with k 1 x, s plus lambda k 2 x, s this pre multiplied infinite series you can easily observe this an geometric series, and this geometric series with first term 1 and common ratio lambda square by 4.

So, this will be equal to k 1 x, s plus lambda k 2 x, s these divided by 1 minus lambda square by 4 and criteria for convergence is given by modulus lambda less than 2 and after substituting the expression for k 1 x s and k 2 x s, you can find this is 1 minus 3 x s plus lambda into 1 minus 3 by 2 x plus s plus 3 x s this whole divided by 1 minus lambda square by 4. So, that means, this is equal to 1 plus lambda minus 3 by 2 lambda times x plus s minus 3 x s multiplied with 1 minus lambda divided by 1 minus lambda square by 4, so this is actually the sum for the Neumann series, and also this is the expression for the resolvent kernel.

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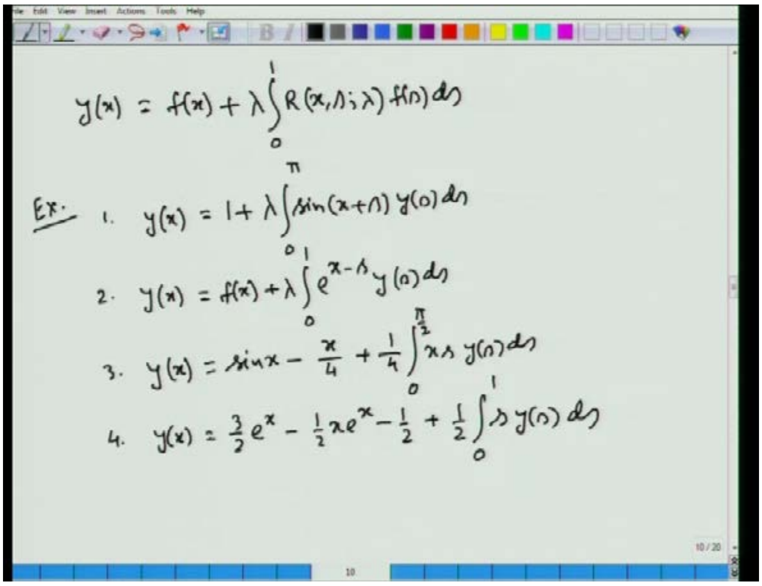

So, with these resolvent kernel, if you substitute into the expression that is y x is equal to f x plus lambda integral 0 to 1 R of x s lambda f s d s then you will be having solution to the given Fredholm integral equation. And now, before going to the next part, I am giving some exercise for your practice, you can solve this problems first one, y x is equal to 1 plus lambda integral 0 to pi sin of x plus s y s d s, second problem y x is equal to f x plus lambda integral 0 to 1 e to the power x minus s y s d s.

Number 3, y x this is equal to sin x minus x by 4 plus 1 by 4 integral 0 to pi by 2 x s y s d s and number 4, y x this is equal to 3 by 2 e to the power x minus half x e to the power x minus half plus half integral 0 to 1 s y s d s, so all these problems you can solve by the method of dissolvent kernels.

Now, before going to the next topic I discuss briefly, an interesting result that is involved with the resolvent kernel, and where we can show that resolvent kernel is actually satisfy an integral equation of Fredholm type. But, that will be in terms of two variables x and s, where f x can be replaced by the given kernel and deduction is very straight forward.

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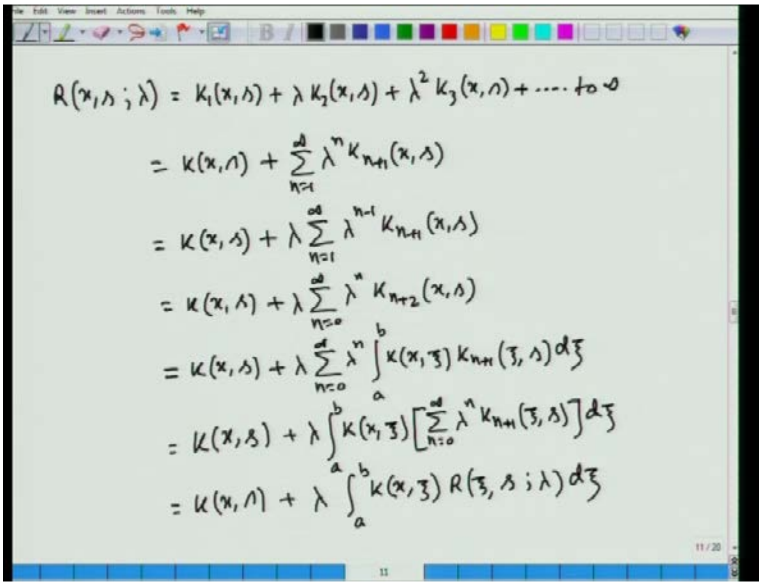

We start with a definition, that is r x s lambda this is equal to k 1 x, s plus lambda k 2 x, s plus lambda square k 3 x, s plus dot dot up to infinity; and we can write this is equal to since, k 1 x, s you know this is equal to actually k x s, then we can write this is plus summation in running's from 1 to infinity lambda to the power n k n plus 1 x s. So, that means, this is the rest of the part is written under the summation notation, and now if you take one lambda outside the summation notation, this will be k x plus s plus lambda sigma n equal to 1 to infinity lambda to the power n minus 1 k n plus 1 x, s.

Now, when you are substituting n equal to 1, so first index of lambda is going to be 0, so changing this limit of the sum, we can write this is k x, s plus lambda sigma n equal to 0 to infinity then it will be lambda to the power n k n plus 1 will be converted into k n plus 2 x, s. And now, here for k n plus 2 x, s we can write the formula for iterated kernel, so that means, this will be equal to k x, s plus lambda sigma n equal to 0 to infinity lambda to the power n integral a to b k of x, xi then k n plus 1 xi, s d xi here, we are just writing the formula for iterated kernel of k n plus 2 x, s is equal to integral a to b k x, xi k n plus 1 xi, s d xi.

Now, already we have proved the uniform convergence of these part, so therefore, we can interchange the summation and integral sign, so after interchanging you will have k x, s plus lambda then integral a to b k of x, xi then sigma n running's from 0 to infinity lambda to the power n k n plus 1 xi, s this d xi. Now, this n running's from 0 to infinity lambda to the power n k n plus 1 xi, s is nothing but, our resolvent kernel written in terms of xi and s. So, therefore, we can write this is equal to k x, s plus lambda integral a to b k of x, xi then R of xi, s lambda d xi, so if you look at the final expression, so that means, we have obtained R x, s colon lambda is equal to k x, s plus lambda integral a to b k x, xi r xi s lambda d xi.

So, that means, the solution of the Fredholm integral equation given equation was y x equal to f x plus lambda integral a to b k x, s f s d s this was the solution of the Fredholm integral equation.

Now, here this y is replaced by R x s lambda and f is replaced by k x, s, so therefore, you can see this dissolvent kernel satisfies a similar type of integral equation, this is one important observation. Now, we are going to consider an algebraic method where you can see, we have to solve a system of linear equations, and by solving that system of linear equations by some technique.

We can find out the solution of the Fredholm integral equation which is a non homogeneous Fredholm integral equation and with degenerate kernel, so that means, kernel is separable. And in that case, we can see the solvability condition depends upon the solution or uniqueness of the solution for the system of linear equation, so first of all we described this method and then we will consider a simple example.

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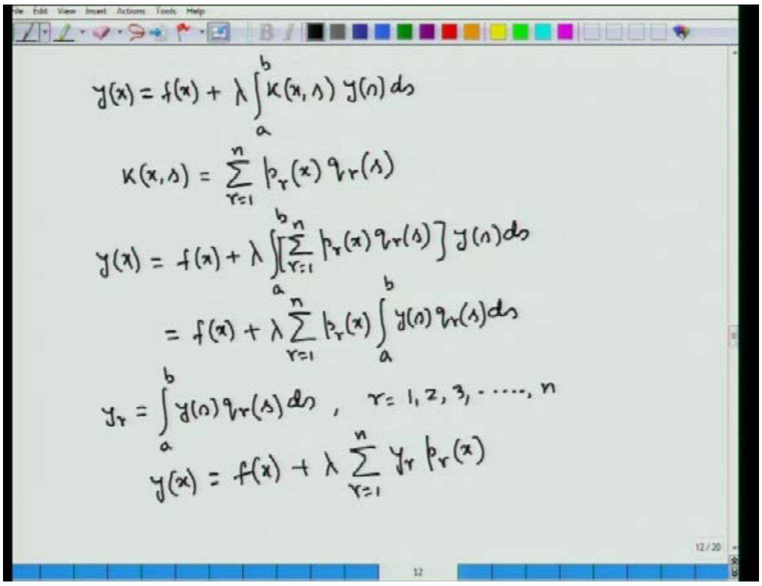

So, we are considering equation of the form y x equal to f x plus lambda integral a to b k of x, s y s d s this is the given equation, kernel is separable, so that means, k x, s this is equal to sigma r running's from 1 to n p r x q r s, this is a separable equation. And if we substitute these expression k x, s into this integral, under this integral sign then you will be having y x this is equal to f x plus lambda integral a to b sigma r running's from 1 to n p r x q r s, this expression multiplied with y s d s as the kernel is separable.

So, we can take this p r x outside the integral sign and therefore, will be having this expression f x plus lambda sigma r running's from 1 to n p r x integral a to b y s q r s d s, now this kernel is separable, so that means, p r x q r s they are known, whenever r ranging from 1 to n, but y is unknown quantity.

So, if we introduce the notation that is y r this stands for integral a to b y s q r s d s where r equal to 1, 2, 3 dot dot up to n, then these expression y x equal to f x plus lambda sigma r running's from 1 to n p r x integral a to b y s q r is d s comes out to be y x, this is equal to f x plus lambda sigma r running's from 1 to n y r times p r x.

So, now you can see by some how we are able to calculate this scalar quantities y r, where r ranging from 1 to n, then immediately will be having solution to this problem, because y x equal to f x plus lambda times sigma r running's from 1 to n y r p r x. In order to find this solution, we can do one thing q r s where r ranging from 1 to n this is known, we can multiply both sight of this equation by q m where m is taking any value within the range 1 to n.

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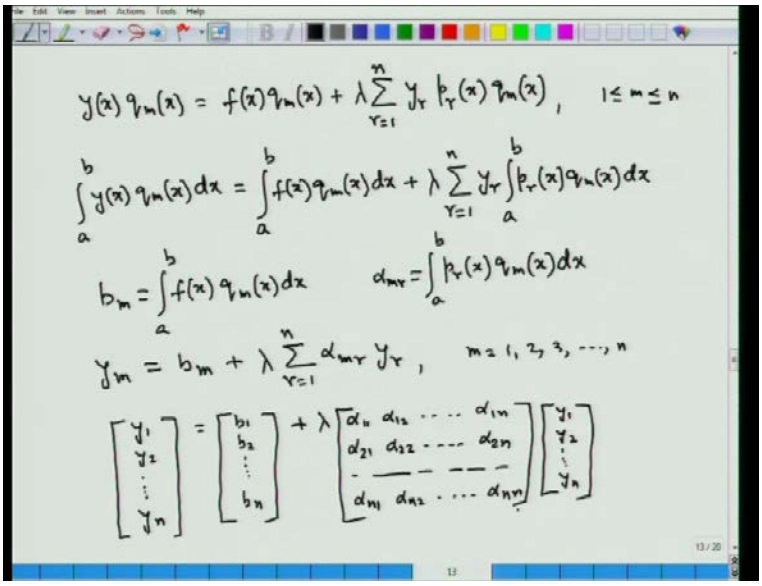

So, therefore, we can write y x q m x this is equal to f x q m x plus lambda sigma r running's from 1 to n y r p r x q m x, this is the result we are getting by multiplying q m; where 1 less than equal to m less than equal to n. And then integrating from the range a to b, we can find a to b y x q m x d x this is equal to integral a to b f x q m x d x plus lambda sigma r running's from 1 to n y r integral a to b p r x q m x d x, this one.

Now, we need two notations for integral a to b f x q m x d x and integral a to b p r q m x d x, if we denote by b m that is the integral a to b f x q m x d x this is the definition for b n, and this integral a to b p r x q m x d x this is defined by alpha m r this one, then this result that is integral a to b y x multiplied with q m x d x equal to integral a to b f x q m x d x plus lambda integral summation r running's from 1 to n y r integral a to b pr x q m x d x, can be written as y m is equal to b m plus lambda sigma r running's from 1 to n alpha m r y r. Now, when you multiplied the expression y x equal to f x plus lambda summation r running's from 1 to n y r p r x by q m x, then we have mentioned that m is ranging from 1 to n.

So, that means, we can find these type of n equations which have given by y m equal to d m plus lambda times summation r running's from 1 to n alpha m r y r, where m equal to 1, 2, 3 up to n and therefore, we are having a system of equations which can be written as y 1 y 2 up to y n, that is into a matrix form this is equal to b 1, b 2 up to b n plus lambda multiplied by alpha 1 1, alpha 1 2, up to alpha 1 n, then alpha 2 1, alpha 2 2, up to alpha

2 n proceeding this way, last row will be alpha n 1, alpha n 2, up to alpha n n; this multiplied with y 1, y 2 up to y n, now this matrix equation is nothing but, a system of linear equation.

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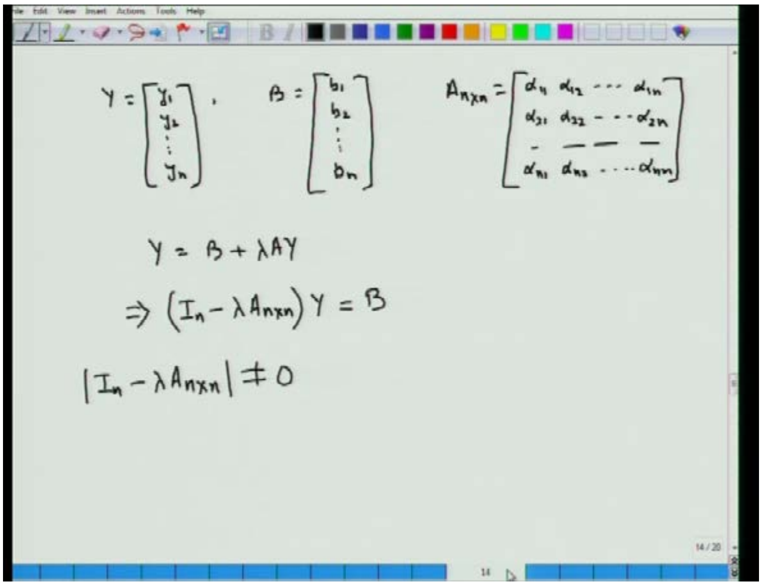

If we introduce the notations that is capital Y is equal to y 1 y 2 up to y n this one, then capital B equal to this column matrix b 1, b 2 up to b n, and capital A which is an n cross n matrix, this stands for alpha 1 1, alpha 1 2 up to alpha 1 n, in this way alpha 2 1, alpha 2 2, up to alpha 2 n finally, alpha n 1, alpha n 2 up to alpha n n this is a n cross n matrix. And therefore, the matrix equation can be written as Y equal to B plus lambda A Y, now this Y is simply rewritten as I n times Y that is identity matrix, so that means, from here we are having a system of equation I n minus lambda, where a is an n cross n matrix, this matrix multiplied with Y this is equal to capital B.

So, if this matrix I n minus lambda A n cross n is invertible, then we will be having unique solution, so that means, whenever determinant of I n minus lambda A n cross n this is not equal to 0, then we will be having unique solution. And if this is equal to 0, that means, if determinant I n minus lambda A n cross n equal to 0, then we will be having either infinite number of solution or no solution, that we will be discussing the next lecture.

But, the point is that if we are able to find out some hallows of lambda, such that this determinant is non 0, so therefore, we can find unique solution for this system of linear equations, and once we are able to find out unique solutions y 1, y 2, y 3 up to y n, these has the unique solutions, so then the expression y x equal to f x plus lambda sigma r running's from 1 to n y r p r x this is uniquely determined and this is nothing but, the solution of the given Fredholm integral equation (Refer Slide Time: 46:31).

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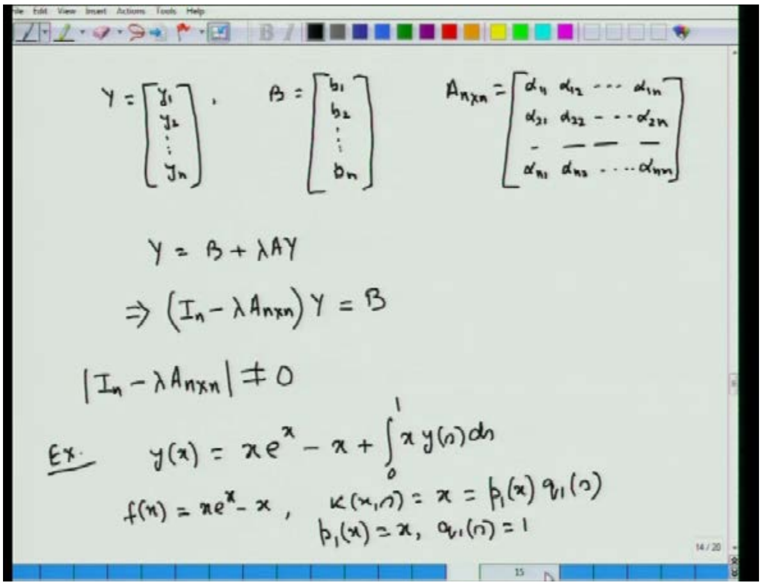

So, now, we consider one example, here we consider the example, we can solve by this method, this is a very simple example, that is y x is equal to x e to the power x minus x plus integral 0 to 1 x y s d x, so just for your understanding this f x is equal to as usual x e to the power x minus x. Now, kernel K x, s this is equal to x, so therefore, this will be equal to as per our notation p 1 x q 1 s where p 1 x this is equal to x and q 1 s this is equal to 1, so now, if we just solve this equation by the method.

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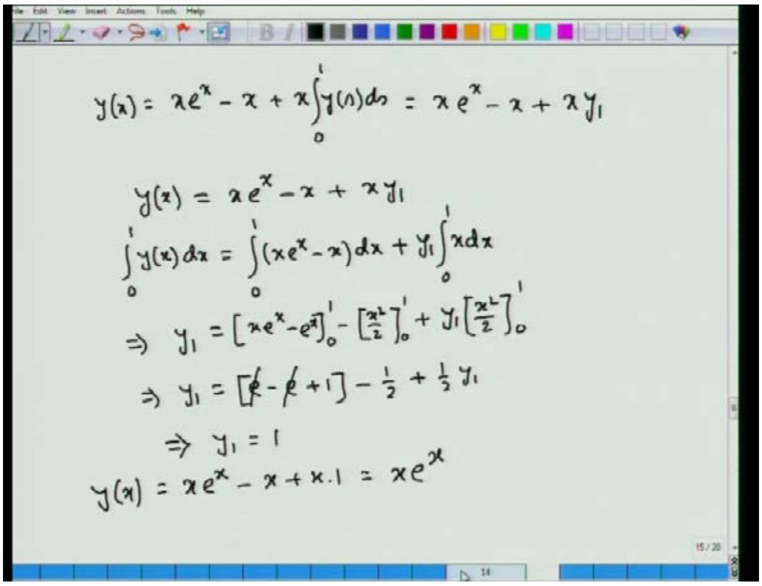

We have just discussed, this becomes y x is equal to x e to the power x minus x then we can take x outside the integral sign, so this is integral 0 to 1 y s d s; now just see, this integral 0 to 1 y s d s can be think about this is nothing but, integral 0 to 1 y s q 1 is d s, because q 1 is here 1. So, with our notation that we have introduced this is equal to x e to the power x minus x plus x y 1, so that means, this x y 1 is actually continuation from the expression lambda sigma r running's from 1 to n y r p r x.

So, in that stage we have multiplied both side by q m x and then we have integrated, here we have only one q that is q 1 x, so q 1 x is going to be 1, so that means, we have to integrate this result y x equal to x e to the power x minus x plus x y 1 both sides with respect to x.

So, we are multiplying this expression both sides with respect to x from 0 to 1, means we are actually multiplying this equation by q 1 x and then integrating from 0 to 1, so therefore, we are having integral 0 to 1 y x d x this is equal to integral 0 to 1 x e to the power x minus x d x plus y 1 is a constant here then integral 0 to 1 x d x. So, 0 to 1 y x d x is our y 1, so this y 1 is equal to after integration it will be x e to the power x minus x limit from 0 to 1, then minus x square by 2 limit 0 to 1 plus y 1 x square by 2 limit 0 to 1.

So, from here we will be having y 1 this is equal to e minus e, this two things are coming from the upper limit, then minus x e to the power x at x equal to 0 is 0 and then from here, we will be having this is equal to minus e plus 1 sorry, this will be actually the result of integration will be x e to the power x minus e to the power x (Refer Slide Time: 50:49). So, therefore, e minus e plus 1 then from here you will be having minus half plus half y 1, so this e cancels with e this is half, this will goes on the right hand side, so ultimately you will be having y 1 is equal to 1.

So, with y 1 equal to 1 if you substitute on the first line, then we can find y x that is y x equal to x e to the power x minus x plus x into 1, so this is equal to x e to the power x, so by calculating this y 1 we have obtained y x equal to x e to the power x as a solution.

So, that means, what we have discussed today, that in Fredholm integral equation which are of non homogeneous type, non homogeneous Fredholm integral equation with separable kernel that can be converted into a system of linear equations. And here we have considered a simple example, where we have obtained a unique solution and for a specific hallow of lambda, now in case of this separable kernel this integral equation can be converted into a problem of finding solution for a system of linear equation.

And depending upon uniqueness of the solution of the system of linear equation, which is actually in turns depending up on the magnitude of lambda this there may be unique solution, may be no solution, may be infinite number of solution will be having corresponding conclusion for the solution of the Fredholm integral equation.

And in next few lectures, we will try to relate these idea with the concept of resolvent kernel, where this resolvent kernel can be obtained in a unique fashion or not and those theories are actually Fredholm theory for solving integral equation which are known as actually Fredholm integral equation.

And where we will be discussing, three particular theorems of Fredholm and after discussing some other problems of these type where the integral equation of Fredholm integral type with separable kernel can be converted into linear system of linear equation; and by solving those equation will be discussing the rest of the theory for Fredholm integral equation, that is Fredholm theorem one, Fredholm theorem two, Fredholm theorem three and there is actually one one correspondence between existence, if unique solution and non existence of the solutions.

So, today I can stop at this point in the next lecture we will be considering few more example of these type, and with help of a particular example, we can try to understand how this type of situation comes into the picture that these may have unique solution, may not have unique solution, and in case of this problem does not possess unique solution, what will be the to our solution for the Fredholm integral equation, so thank you for your attention.