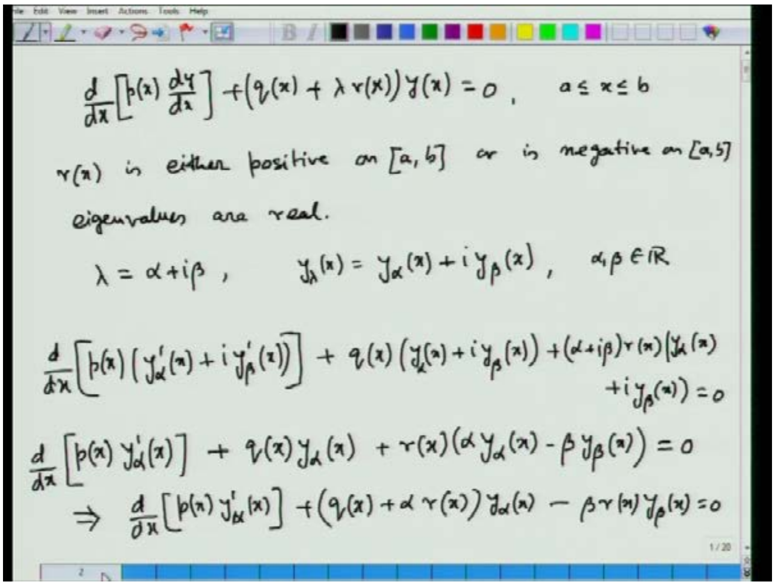
Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture No. #32

Welcome viewers once again to the lecture series on integral equation and the NPTEL courses. In the last lecture we were discussing about the Eigen values and Eigen functions of Sturm-Liouville boundary value problem. We have established certain property and one of the important properties was if y m x and y n x, these are two Eigen functions of a Sturm-Liouville boundary value problem corresponding to two distinct Eigen values lambda m and lambda n respectively, then we can prove, that these two Eigen functions are orthogonal to each other. And where the orthogonality property we have defined in terms of the integral of these two functions taken with the weight function r x integrated from a to b d x. If this is equal to 0, then we can say these two functions are orthogonal to each other.

(Refer Slide Time: 01:32)



Now in this lecture I am going to start with the proof of another theorem, that is very much interesting, that we are considering the same Sturm-Liouville boundary value problem. d dx of px dy dx plus qx plus lambda rx yx, this is equal to 0 with the specified boundary conditions and a less than equal to x less than equal to b. So, we are considering these things.

Now, apart from the continuity property of p dot x qx and rx over the interval a, b, here we are assuming, that rx is either positive on closed interval a, b or is negative on the closed interval a, b and if this happens, that means, rx maintain the same sign over the closed interval a, b. It does not process any zero within this closed interval a, b, then Eigen values, then Eigen values are real.

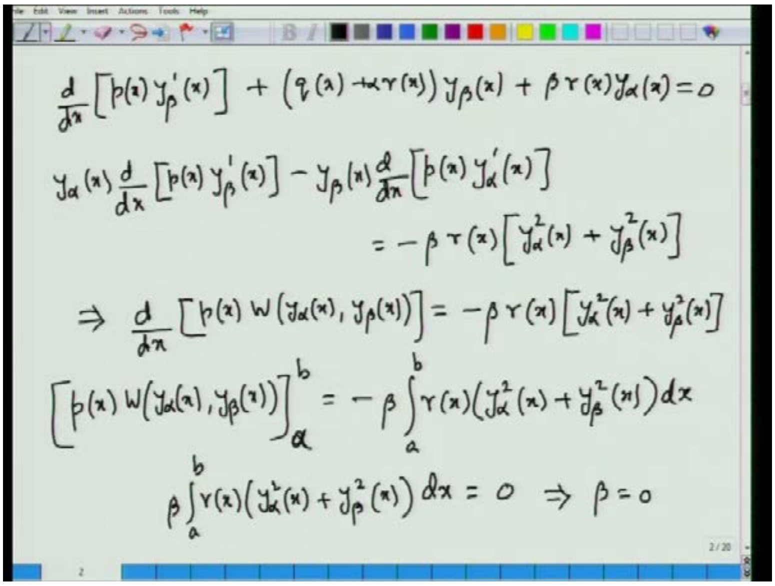
So, first of all we are going to prove this result, that whenever rx maintains the same sign over the close interval a, b, then this Eigen values are real. So, for the time being we are assuming lambda equal to alpha plus i beta be a complex Eigen value for this boundary value problem, and we are assuming the corresponding Eigen function y lambda x. This is defined by y alpha x plus i y beta x. So, where alpha and beta, these are two real numbers and the Eigen function y lambda x corresponding to lambda equal to alpha plus i beta can be separated into real and imaginary parts.

So, now our target is using the property, that rx maintains the same sign over the interval a to b, we are going to prove, that beta is identically equal to 0. So, now this y alpha x plus iy beta x is the Eigen function corresponding to the Eigen value lambda equal to alpha plus i beta for this Sturm-Liouville problem. So, therefore, it satisfies the given equation.

So, substituting into the given equation we can write d dx of px, then y alpha dot x plus i y beta dot x, this entire equation d dx of this one plus qx multiplied by y alpha x plus iy beta x. Then, for lambda we have to substitute alpha plus i beta, this multiplied with rx and then y alpha x plus iy beta x, this is equal to g, this satisfies the given equation. Now, separating the real and imaginary parts we can write, this can be actually divided into two parts, that is, d dx of px y dot alpha and plus i d dx of p x times y dot beta x. So, separating real and imaginary parts we can find that px, this with y alpha dot x, this expression with d dx. So, this is the contribution from the first term.

Then, plus qx y alpha x and from the rest of the part we can find rx multiplied by alpha times y alpha x minus beta times y beta x, this is equal to 0. And for further calculations we can rearrange this term into the form, that is, d d x of px y dot alpha x, this one plus qx plus alpha rx. Actually, we were collecting the coefficient of y alpha x, this times y alpha x minus beta rx y beta x, this is equal to 0. So, this is the expression we are getting from real part.

(Refer Slide Time: 07:14)



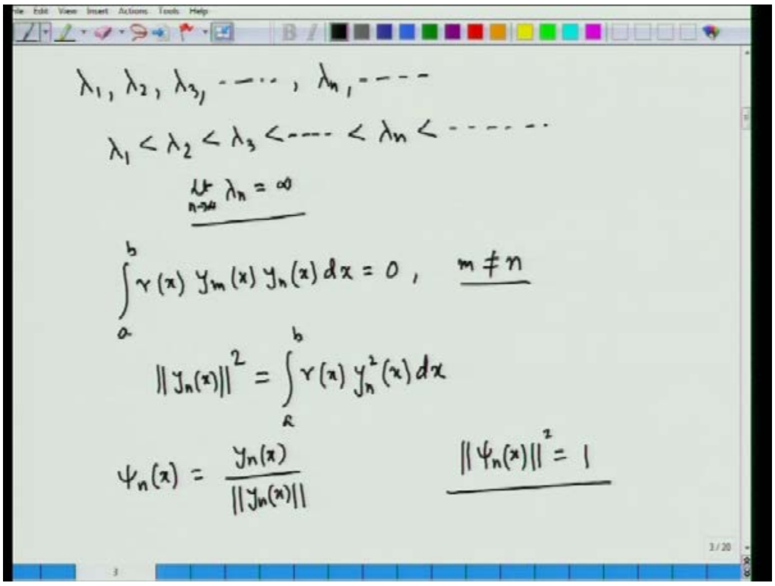
Next collecting the imaginary part we can find similarly d dx of px, this multiplied with y beta dot x and similarly, collecting the coefficient of y beta x we can find qx plus alpha rx times y beta x plus beta rx into y alpha x, this is equal to 0. So, previous expression was this one and here we can find this beta y rx y alpha x. This is the two expressions.

So, now, we can eliminate this qx plus alpha rx, this term from both the equations. So, that means, we are multiplying first equation by y alpha x y beta x and second one by y alpha x and then subtracting we can find, that y alpha x, this multiplied with d dx of px y beta dot x, this one minus beta x times d dx of px y alpha dot x, this expression. Now, this will be equal to minus beta rx this into y alpha x whole square plus y beta x whole square. This is the expression. And now you can recall the left hand side is nothing, but d dx of px times wronskian of y alpha (x, y) beta x, this one, is equal to minus beta times rx with y alpha square x plus y beta square x. Now, integrating both the sides from the limit a to b we can find, that px wronskian of y alpha x y beta x, this one from the limit a to b, this is equal to minus beta integral a to b rx multiplied with y alpha square x plus y beta square x this dx.

Now, the left hand part can be considered as, that y alpha is an Eigen function corresponding to the Eigen value alpha and y beta is the Eigen function corresponding to the Eigen value beta. So, therefore, using the previous property what we have discussed earlier here, this left hand side, is identically equal to 0. And once this is equal to 0, so therefore, we can write, that beta times integral a to b rx multiplied with y alpha square x plus y beta square x, this dx, this is equal to 0.

Now, we have assumed, that rx maintain the same sign over the interval a to b. So, whenever this rx maintain the same sign over the interval a to b, so that means, without any loss of generality if we assume r x is positive, so therefore, integrand rx into y alpha x whole square plus y beta x whole square, this integrand is positive and therefore, there is no chance, that integral a to b rx times y alpha square x plus y beta square x equal to 0. So, we are left with the only one option, that is, beta equal to 0. So, therefore, whenever this rx maintain the same sign over the interval a to b, then Eigen values are real and therefore, associated Eigen functions, those are also real and with this we can complete the all necessary proofs for the requirement, that a function can be expressed in terms of collection of orthogonal functions.

(Refer Slide Time: 12:44)



Now, before going to that discussion I like to mention one more result, that I am not going to prove, that for this kind of Sturm-Liouville boundary value problem you will be having an infinite sequence of Eigen values, actually. So, that means, if lambda 1, lambda 2, lambda 3, these are the Eigen values of a Sturm-Liouville boundary value problem with lambda n and up to infinity, so then ordering can be arranged in this particular format, that is, lambda 1, this is less than lambda 2, less than lambda 3, less than dot dot, this will be less than lambda n, less than this with the property limit n tends to infinity, lambda n, this is equal to infinity. So, I am not going to prove this result, this is quite t t s. So, you can omit this one.

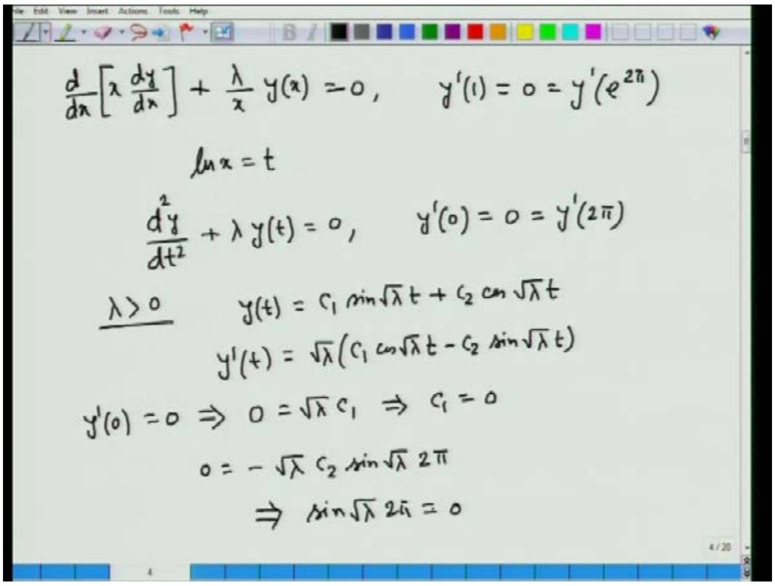
And before going to the orthogonal or orthonormal series expansion of a function f x, which is square integrable, I just like to discuss one more thing, that is, the orthogonal Eigen functions. We have already defined now what is the concept of orthonormal Eigen functions, so we are already familiar with the concept, that if a to b rx y m x y n x dx, this is equal to 0 with m not equal to n. Then, these two functions, y m x and y n x, they are orthogonal to each other.

Now, clearly you can understand, that if rx maintains the same sign and when Eigen values are real, then for m equal to n, this integral is not equal to 0. So, in that case we can use a normalizing factor such that these orthogonal Eigen functions can be converted into orthonormal Eigen functions. So, the concept is, that the norm of the Eigen function, this is defined by, actually denoted by norm of y n x and it can be obtained as this y n in norm of square is equal to integral a to b rx y n square x dx.

Now, if we define this y n x, so each y n x by its associated norm, so then we can find the family of orthonormal functions and here I am denoting this as psi n x and psi n x is nothing, but y n x divided by norm of y n x. And with this definition, that psi n x equal to y n x by norm of y n x, where each y n x at the Eigen functions of an associated Sturm-Liouville boundary value problem you can easily verify, that norm of psi n x, this whole square, this is equal to 1. So, this result you can easily obtain.

Now, before proceeding further I like to discuss one problem where you can see what are the Eigen functions, what is the collection of orthogonal Eigen functions and corresponding orthonormal functions. And in a particular manner I am going to consider the example where you can understand what is the utility of rx, that is needed to define the orthogonality property and other relevant things, that is, (( )) also norm of the function and its role to understand what is the concept of orthonormal Eigen functions.

(Refer Slide Time: 16:32)



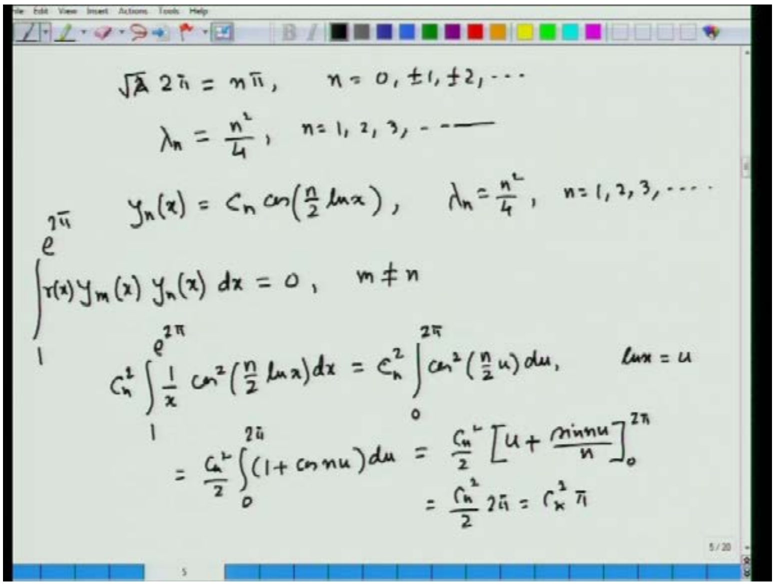
So, for this problem I am considering here the equation d dx of x dy dx, this plus lambda by x yx, this is equal to 0 with the boundary conditions y dot 1 equal to 0 equal to y dot e to the power 2 pi.

Now, we can recall this is a (( )) type second order ordinary differential equation. So, in order to solve this problem easily we can use the change of independent variable x. So, if we use the transformation of variable, that is, lnx equal to t, so then this equation, given ordinary differential equation will be converted into d 2 y dt 2 plus lambda. You have to keep in mind, this is now yt, this is equal to 0 and associated boundary conditions will be converted into y dot 0. This is equal to 0 equal to y dot 2 pi because here x equal to e to the power 2 t. So, at the limit x equal to 1, t is 0 and at the limit x equal to e to the power 2 pi, this is t equal to 2 pi. So, this is the boundary conditions.

Now, for this problem you can easily verify for lambda equal to 0 and for lambda less than 0. This boundary value problem does not possess any non-trivial solution. So, non-trivial solution exist whenever lambda greater than 0 and in that case solutions are given by yt, this is equal to C 1 sine rout over lambda t plus C 2 cosine root over lambda t, differentiating we can find y dot t, that is equal to root over lambda C 1 cosine root over lambda t minus C 2 sine root over lambda t. Now, using the condition, that is, y dot 0 equal to 0, we can find 0 equal to root over lambda times C 1, and already we have assumed lambda greater than 0. So, this implies, C 1, this is equal to 0.

Then, using the right hand boundary conditions we can find 0, this is equal to minus root over lambda C 2 sine root over lambda times 2 pi. So, lambda greater than 0, we are looking for non-trivial solution, so therefore, C 2 not equal to 0. So, therefore, we are left with only one possibility, that is, sine root over lambda 2 pi, this is equal to 0.

(Refer Slide Time: 19:58)



And finding its general solution we can find, that root over lambda times 2 pi, that is equal to n pi where n equal to 0 plus minus 1 plus minus 2 and so on. And ultimately we can get the result, that is, lambda n, this is given by n square divided by 4 and therefore, n we have to take actually 1, 2, 3 and so on. So, this type we are writing root over lambda 2 pi equal to n pi. This is for as a general solution of trigonometric equation.

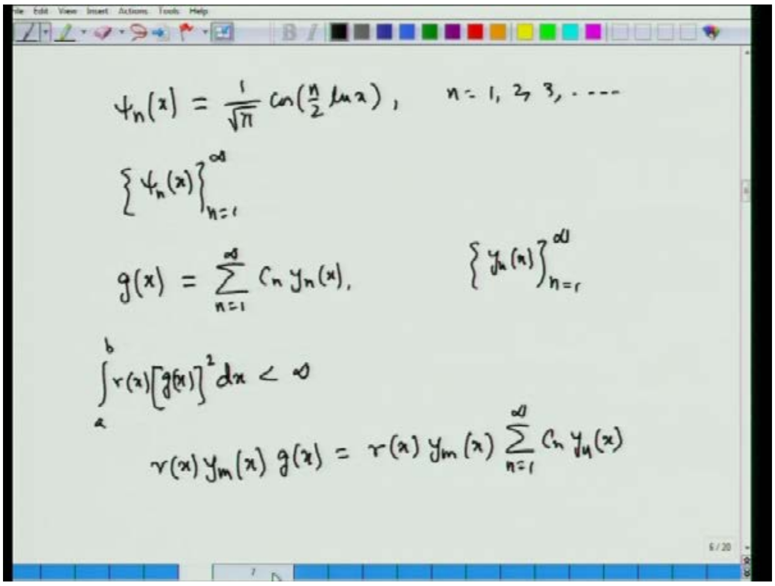
Now, for the Eigen value, Eigen function problem, lambda n is, comes down to n square by 4 and therefore, the corresponding Eigen functions is actually, we are having C 1 equal to 0, C 2 not equal to 0. So, we will be having only cosine terms. So, therefore, resulting Eigen functions are y n x, this is equal to C n cosine n by 2 lnx. This is actually Eigen functions and Eigen values are given by lambda n equal to n square by 4 for n equal to 1, 2, 3 and so on.

Now, we are actually interested to find out the corresponding orthonormal Eigen functions. So, these are orthogonal Eigen functions and here you can verify without this additive, sorry, this multiplicative constant C n, that y m x y n x with rx, this is required. Then, integral x equal to 1 to e to the power 2 pi dx, this is equal to 0 when m not equal to n. And there is no harm if you proceed with this C m and C n because integral will come out to the 0.

So, whatever may be the finite values of C m and C n, this condition will be satisfied and therefore, this norm of this particular problem you can calculate easily, that is, C n square integral 1 to e to the power 2 pi 1 by x cosine square n by 2 lnx dx. So, now, using the change of variables, that is, from lnx equal to u, you can easily convert it to the integral C n square integral 0 to 2 pi cosine square n by 2 u du. Here we are substituting lnx, this is equal to u. So, that means, the rx part, rx equal to 1 by x and then 1 by x dx will be your du and limits will be changed to from 0 to 2 pi. So, this will be equal to C n square by 2 integral 0 to 2 pi 1 plus cosine nu du. So, this will be C n square by 2 u plus sine nu divided by n, this limit from 0 to 2 pi.

Now, here you can recall the value of n is equal to 1, 2, 3 and so on. So, there is no contribution from the part sine nu by n. At 0 it is 0 and sine 2 n pi equal to 0 because n is ranging over 1, 2, 3 and so on. So, that means, we are left with C n square by 2 times 2 pi. So, this is equal to C n square pi.

(Refer Slide Time: 24:28)



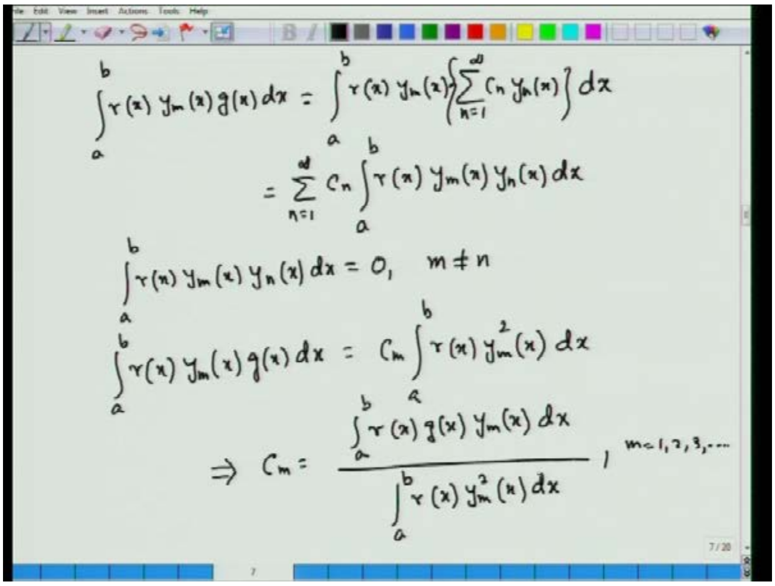
So, now if we divide this y n x equal to C n cosine n by 2 lnx by C n times root over pi, so then we will be having the corresponding orthonormal Eigen functions. These are nothing, but 1 by root over pi cosine of n by 2 lnx. So, this particular set, that is, psi n x equal to 1 by root over phi cosine n by 2 lnx square n equal to 1, 2, 3 and dot dot upto infinity. This is psi n x n from 1 to infinity, this actually a set of orthonormal Eigen functions for the associated Sturm-Liouville boundary value problem.

Now, our target is to find out the orthonormal Eigen function expansion or whatever may be the orthogonal Eigen function expansion for a particular problem, this gx. So, that means, we are interested to express gx as summation in runnings from 1 to infinity C n y n x, where this sequence y n x in runnings from 1 to infinity. This is a collection of a family of orthogonal Eigen functions associated with a Sturm-Liouville boundary value problem and where this function gx is defined over the interval a, b. And for the forthcoming discussion, that whether this expansion can be obtained or not in order to get the answer and all possible interchange of integral and the infinite summation, we are assuming the property of gx, that is, gx is square integral, that we have actually assumed.

I am not going into details of the proof of this results at where this can be done, but it can be found in any book on Fourier series and related analysis and the property is, that in terms of mathematics you can say, that when rx, that is, the function involved with the Sturm-Liouville boundary value problem, if integral a to b rx gx whole square dx, if this is finite, then we can say this gx is actually square integrable function.

So, now, we are actually going to derive the coefficient C n such that gx can be expressed as summation in running from 1 to infinity C n y n x. And in order to derive the coefficients C n we will be using the orthogonality property of the Eigen functions. So, to find out the C n we can multiply this expression on the both side by rx into y m x. So, therefore, we can write rx y m x gx, this is equal to rx y m x, then sigma in runnings from 1 to infinity C n y n x.

(Refer Slide Time: 28:01)

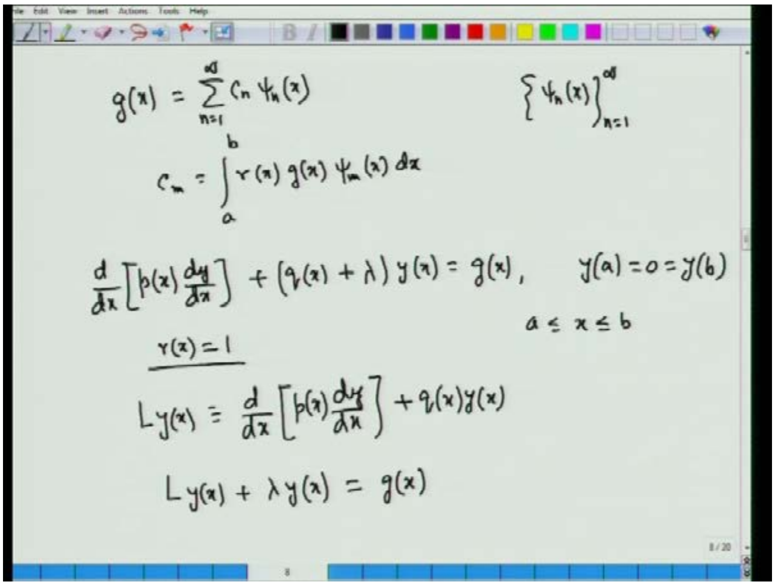


Now, integrating both sides from the limit a to b you can find integral a to b rx y m x gx dx, this is equal to integral a to b rx y m x sigma n equal to 1 to infinity C n y n x dx. Now, using the property, that gx is squared integrable and other relevant properties of uniform convergence of this series and then interchanging the integral and summation we can find this will be equal to sigma in running from 1 to infinity C n integral a to b rx multiplied with y m x y n x dx.

Now, you can recall, that this integrals, integral a to b rx y m x and y n x, since they are taken from collection of orthogonal functions, this is equal to 0 for all m not equal to n and this quantity is non-zero whenever this m equal to n. So, ultimately you will be having integral a to b rx y m x gx dx, this is equal to C m integral a to b rx y m x, this square dx, this is actually C m, the unknown coefficients. And therefore, each unknown coefficients C m can be obtained as integral a to b rx gx y m x dx divided by integral a to b rx y m square x dx. So, this result is valid for m equal to 1, 2, 3 and so on. So, this is the coefficient for C m.

So, in this way if we calculate the coefficient C m, then we can find the expansion of a function gx in terms of orthogonal Eigen functions y m x or it can be said as the Fourier series for the y m x, where this gx, that is, gx square integrable over the close interval a, b, that condition is satisfied. And if instead of orthogonal Eigen functions, if we consider the orthonormal Eigen functions, then this y m and y n will be replaced throughout by psi m and psi n. And since the norm of psi n, this is equal to 1, so this expression appearing in the denominator for C n will be identically called 2 1.

(Refer Slide Time: 31:32)



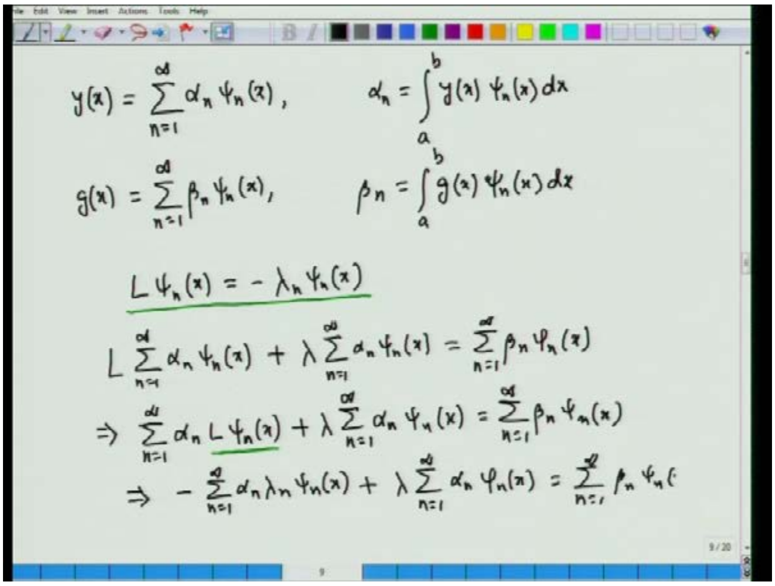
So, that means, just for your note, if we consider gx is equal to sigma in runnings from 1 to infinity C n psi n x, where psi collection of psi n x, these are actually orthonormal Eigen functions n runnings from 1 to infinity, then you will be having this coefficients C m. This is equal to integral a to b rx gx psi m x dx only. So, no terms will be appearing in the denominator. So, this is actually expression for, that is, orthonormal Eigen function expansion of a function gx, which is defined over the closed interval a, b.

Now, we are going to apply this concept of expanding a function defined over the closed interval a to b in terms of orthonormal Eigen functions to construct the Greens' function for a non-homogeneous Sturm-Liouville boundary value problem. So, that means, our target equation is d dx of px dy dx, this one plus qx plus lambda yx equal to gx with the associated boundary conditions, say y a equal to 0 equal to y b. Of course, you can use the general separated boundary condition, but for your simplicity I am considering here the simple zero boundary condition at the both end for y, that is, y i equal to 0 and y b equal to 0.

And one more important property, that you have to notice here, for simplicity I am assuming rx equal to 1, so without assuming rx equal to 1 same result can be derived, but for simplicity of forthcoming mathematical calculations I am assuming rx equal to 1.

This is one thing and one more thing, here I can denote this Lyx, this is equivalent to d dx of px dy dx, this plus qx yx only. So, this is just a difference in notation for Lyx. In the previous lectures you can find, that Ly actually involves this lambda rx yx terms also. So, with these definitions if we denote this expression that is, d dx of px dy dx plus qx yx as Lyx, then the given equation can be written as L of yx plus lambda yx, this is equal to gx.

(Refer Slide Time: 35:04)

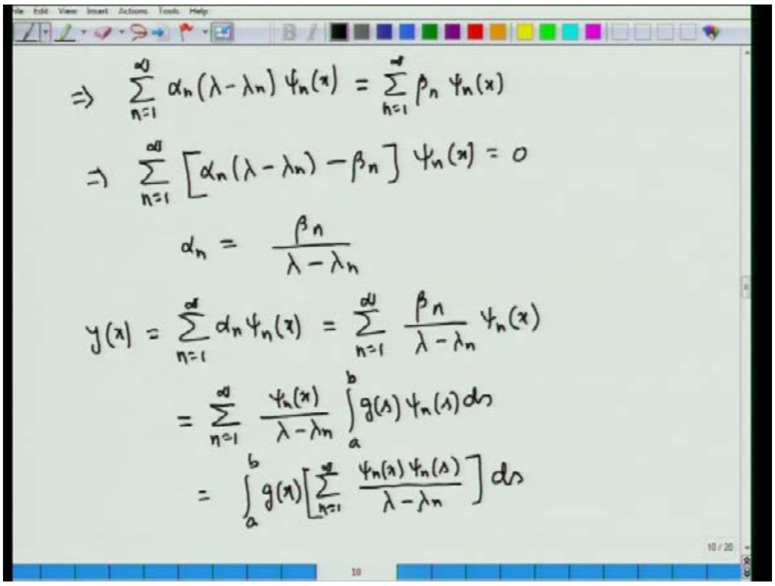


Now, in this case yx is the unknown function and gx is a given function, so we are assuming the expression for yx in terms of orthonormal Eigen functions can be written as yx equal to summation in running from 1 to infinity alpha n psi n x, where alpha n is equal to integral a to b yx psi n x dx, rx will not be coming here because we have assumed rx equal to 1. And then gx, this will be equal to summation in runnings from 1 to infinity beta n psi n x, where beta n is equal to pntegral a to b gx psi n x dx. And one more property, that psi n x, this satisfies this relation, that is, L of psi n x, that is equal to minus lambda n psi n x because this psi n x, they are actually Eigen functions corresponding to the Eigen values lambda n of the given problem.

Now, if we substitute these two expressions y and g into the given equation, then we can find, that L of sigma n runnings from 1 to infinity alpha n psi n x plus lambda integral, sorry, summation in runnings from 1 to infinity alpha n psi n x, that is equal to sigma n runnings from 1 to infinity beta n psi n x. And here we can write this is equal to sigma n runnings from 1 to infinity alpha n L of psi n x plus lambda sigma n runnings from 1 to infinity alpha n psi n x. This is equal to summation n runnings from 1 to infinity beta n psi n x and here we can use this result, that is, L of psi n x equal to minus lambda n psi n here for each term at this particular position.

So, using this result we can write this will be equal to minus summation in runnings from 1 to infinity alpha n lambda n psi n x plus lambda sigma n runnings from 1 to infinity alpha n psi n x. This is equal to sigma n runnings from 1 to infinity beta n psi n x.

(Refer Slide Time: 38:24)



And at the next step we can write, this implies sigma n runnings from 1 to infinity, this will be alpha n times lambda minus lambda n psi n x. This is equal to sigma n runnings from 1 to infinity beta n psi n x. And at the next step you can write sigma n runnings from 1 to infinity, this alpha n multiplied with lambda minus lambda n minus beta n psi n x, this is equal to 0.

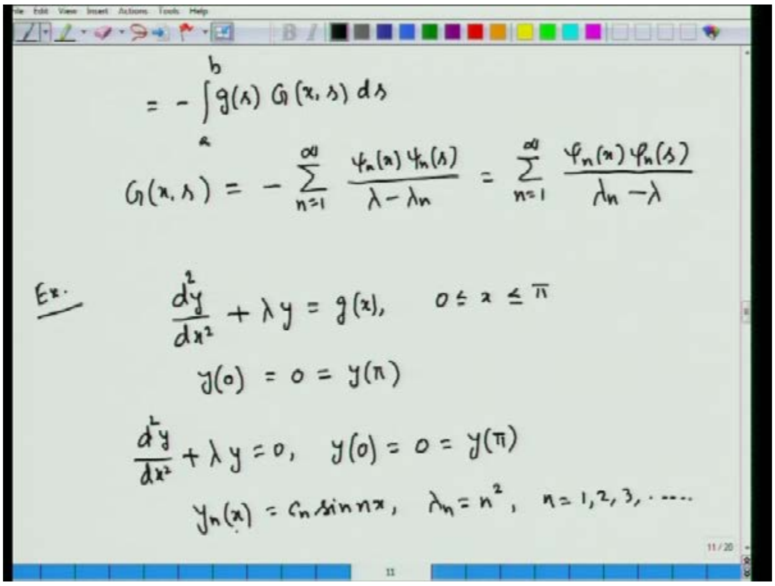
Now, this is an expression of the form, you can try to understand, that C 1 psi 1 x plus C 2 psi 2 x plus C 3 psi n x plus dot dot up to infinity, that is equal to 0. And we can easily prove, that if this happens, then every coefficients of psi n x will be identically equal to 0 because the family of orthonormal Eigen functions, they are set of linearly independent functions. So, using the set of linearly independent property or you can apply the orthogonal property in order to derive all these coefficients, are exactly equal to 0. So, from here we can write that alpha n, this is equal to beta n divided by lambda minus lambda n. So, this is the derivation of alpha n equal to beta n by lambda minus lambda n.

Now, this was actually our main goal because for the given problem y x is unknown, gx is a given function, so once we are able to solve the Eigen value Eigen function problem associated with the given Sturm-Liouville boundary value problem, so immediately you will be having lambda. Hence these are known psi n x, those are also known and once you know the lambda n and psi n, so using the known function you can find out beta n because beta n at the coefficients involved in the orthonormal Eigen function expansion for g x. So, therefore, beta n is known.

So, from this relation you can clearly understand, that lambda is a parameter, beta n are known, lambda n are known, so each unknown quantity alpha n, which are actually involved with the unknown function yx, is now determined. So, therefore, using the definition of yx we can write yx, this is equal to sigma n running from 1 to infinity alpha n psi n x. So, this is equal to sigma n runnings from 1 to infinity beta n divided by lambda minus lambda n psi n x.

And here we can recall the definition for beta n. So, therefore, this is equal to sigma n runnings from 1 to infinity psi n x divided by lambda minus lambda n integral a to b. This beta n is nothing, but gs psi n s ds because x is already involved here. So, we are writing this definition for b n in terms of the variable s and then interchanging the summation and integral sign, that is allowed with the assumption, that g x is square integrable. We can write this is equal to integral a to b, integral a to b, gx then sigma n runnings from 1 to infinity psi n x psi n s divided by lambda minus lambda n, this ds.

(Refer Slide Time: 43:02)



And if we write this expression as equal to minus integral a to b gs capital G x, s ds, then G x, s, this is equal to minus sigma n runnings from 1 to infinity psi n x psi n s divided by lambda minus lambda n. So, this is equal to simply minus lambda n. So, this is equal to simply sigma n runnings from 1 to infinity psi n x psi n s divided by lambda n minus lambda. So, this is the way by which for a given Sturm-Liouville boundary value problem if we are able to find out the family of orthonormal Eigen functions and associated Eigen values lambda n, then Greens function can be also expressed as series of orhtonormal Eigen function. So, that means, this is actually the expression for Greens function in terms of orthonormal Eigen function. You can say this formula is involved with the orthonormal Eigen function expansion of a Greens function associated with a Sturm-Liouville boundary value problem.

Now, we consider a specific example. So, this example is that we are considering this given equation d 2 y dx 2 plus lambda y, this is equal to gx, which is defined over the interval x less equal to 0 less than equal to pi; this is the interval. And where the given boundary condition y 0, this is equal to 0 equal to y pi. So, first of all we have to find out the Eigen values and Eigen function for the associated homogenous problem.

So, our target problem is d 2 y dx 2 plus lambda y, this is equal to 0 with associated boundary conditions y 0 equal to 0 equal to y pi. If you proceed in a usual manner you can find the Eigen functions y n x, this will be C n sine n x and Eigen values lambda n.

These are given by n square where n equal to 1, 2, 3, and so on. So, with these definitions of y n x if we calculate the norm of this function y n and divide it by its norm, so then we can obtain the corresponding family of orthonormal Eigen functions y n x, this is equal to root over 2 by pi sine of n x, where n equal to 1, 2, 3, and so on.

And just for your verification you can check, that here the given problem is d 2 y dx 2 plus lambda y equal to 0. So, if you compare with the standard format of the Sturm-Liouville boundary value problem. So, rx equal to 1, that is positive, which is positive throughout, the interval 0 to pi and therefore, all the Eigen values are real, each Eigen values are real and further, these Eigen values lambda n equal to n square for n equal to 1, 2, 3, and so on.

So, therefore, you can easily verify, that the ordering, that I have mentioned earlier, it also satisfy also the increasing order, that is, lambda 1 is less than lambda 2 less than lambda 3 and so on. And easily you can verify limit n tends to infinity lambda n, that is also equal to infinity and then using the normalization condition you can find out the family of orthonormal Eigen functions psi n x equal to this one. And therefore, using the formula just we have derived, that is, G x, s is equal to sigma n runnings from 1 to infinity psi n x psi n s divided by lambda n minus lambda. So, we can write for this problem Greens function G x, s is nothing, but 2 buy pi summation in running from 1 to infinity psi n x psi n s divided by n square minus lambda. This will be n square (( )) lambda square, this will be n square minus lambda.

So, this is the Greens function for the given problem and then solution to the given problem y x, this will be equal to 2 by pi sigma n runnings from 1 to infinity sine n x divided by lambda minus n square integral 0 to pi g s sine ns d s. Here, this 1 by n square minus lambda, this is changed to 1 by lambda minus n square because the solution y x is equal to minus integral a to b, where a equal to 0, b equal to pi minus integral a to b g s, then capital G x, s ds, so that minus sign will be observed here. So, results in y x equal to 2 by pi integral in running from 1 to infinity sine n x divided by lambda minus n square integral 0 to phi Gs sine n s d s.

So, of course, with some known G s you can evaluate this integrals, that what will be the result for integral 0 to pi Gs sine n s d s. So, this is actually the solution of the Sturm-Liouville boundary value problem in terms of orthonormal Eigen functions. And if you try to understand, that this y x will be solution of the associated Fredholm integral equation, that means, if the given ordinary differential equation is converted into a Fredholm linear integral equation, then this expression is also a solution to the given problem.

Now, these are the necessary results what we will be required to discuss further, the Fredholm alternative and Hilbert theory associated with Fredholm integral equation. So, these are all necessary tools in order to find out the solution of the Fredholm integral equation. So, today this lecture I can stop at here. In forthcoming lectures I will be considering similar approach for we have adopted for the Volterra integral equation. Some of them can be applied in order to find out solution of the Fredholm integral equation and then you will be considering the Fredholm three theorem, first theorem, second theorem, first theorem and then Hilbert theory to obtain the solutions for Fredholm integral equations.

So, thank you for your attention.