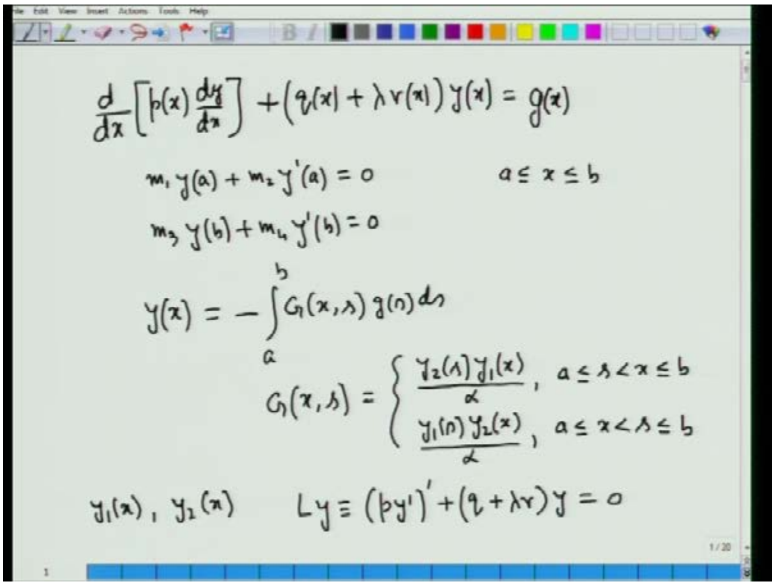
Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology Kanpur

Lecture #31

Welcome viewers once again to the lecture series on integral equation under the NPTEL courses.

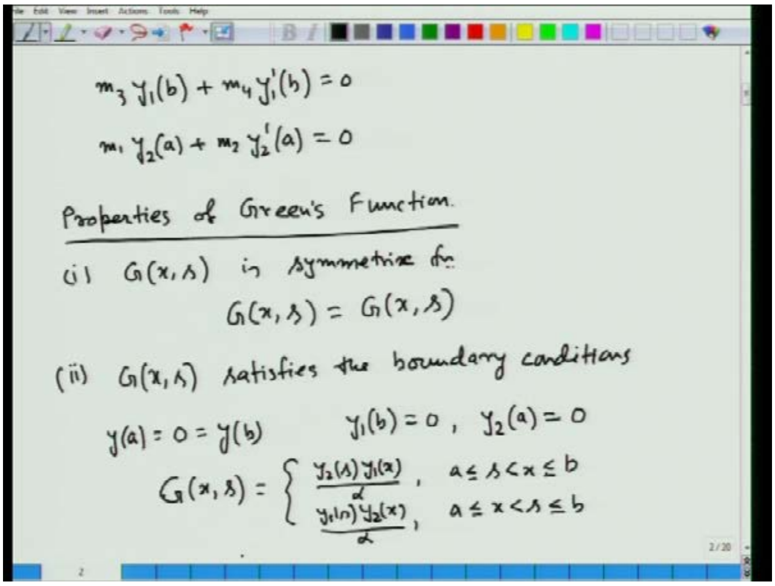
(Refer Slide Time: 00:41)



In the last lecture we were discussing about the Greens function for linear boundary value problems, just for a quick recapitulation, you can recall we were dealt with (( )) type boundary value problems, which are defined by d dx of px dy dx plus qx plus lambda rx yx. This is equal to 0 for homogenous boundary value problem and this is equal to gx for non-homogenous boundary value problem with the separated boundary conditions m 1 ya plus m 2 y dot a. This is equal to 0 m 3 yb plus m 4 y dot b, this is equal to 0. This is called separated boundary conditions where a less than equal to x less than equal to b; p dash x q x and r x, these are all continuous over the interval a to b, lambda is a parameter.

With these assumptions we arrived at the position, that solution of this equation can be written as y x, that is equal to minus integral a to b G of x, s gs ds, but this capital G x, s is the base function. And this particular greens function we defined, as well as, derived, as this is equal to y 2 s y 1 x divided by alpha for a less than equal to s less than x less than equal to b. And y 1 x y 2 x divided by alpha, where a less than equal to x less than s less than equal to b, where this y 1 x and y 2 x, these two functions were the solution of the corresponding homogenous equation L of y, that is equivalent to, in short if we can write, that is, p y dashed whole dashed plus q plus lambda r y, this is equal to 0. This was two linear independent solution of this equation.

(Refer Slide Time: 03:54)



And not only they are just linearly independent solution of this equation, there was another restriction based upon which we have constructed the Greens function. Those restrictions was, that y 1 satisfies the boundary conditions on the right hand, that is, m 3 y 1 b plus y 1 dot b, this is equal to 0 and y 2 x, that satisfies the boundary conditions on the left. That means, m 1 y 2 a plus m 2 y 2 dot a, this is equal to 0. So, based upon this we have constructed our Greens function.

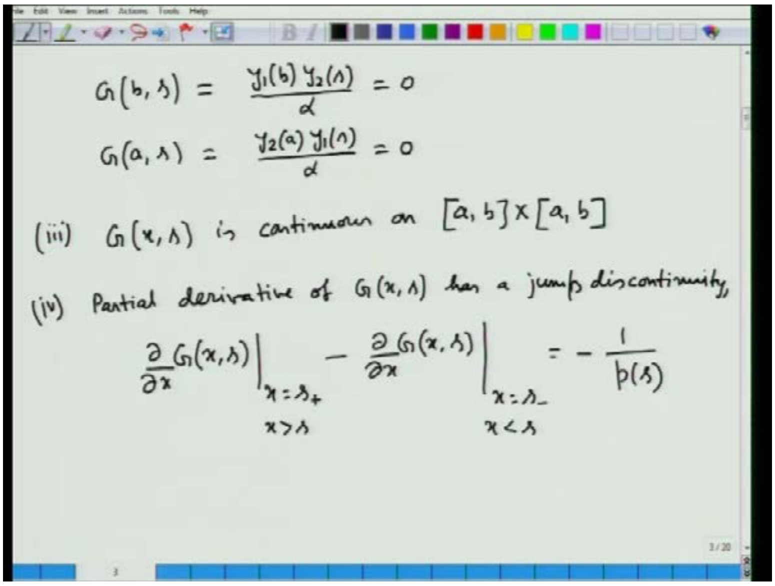
Now, in this particular formation 1 1 t, that means, 1 by alpha still remains undefined, that now we are going to define along with some discussion on properties of this Green’s functions. So, now, let us look at the properties of Green’s functions.

First of all you can easily verify, that g (x, s) is symmetric, is a symmetric function of two variables x and s, that is, G (x, s) is equal to G (x, s). So, that means, interchanging the role of x and s you can easily verify, that this condition is satisfied. Secondly, the important properties is, that G (x, s) this actually satisfies the boundary conditions; this satisfies the boundary conditions. And of course, you can verify, that d g (x, s) satisfies this type of m 1 y 2 a plus m 2 y 2 dot a, this is equal to 0.

Now, without any loss of generality if we simply assume the boundary conditions into this particular format, say y a, this is equal to 0 equal to y b. The simplest boundary condition corresponding to m 1 y a plus m 2 y dot a equal to 0 and m 3 y b plus m 4 y dot b equal to 0. So, that means, choosing m 2 and m 4 equal to 0 we can arrive at this type of boundary conditions. Now, with this a, that is, y 1 satisfies boundary conditions on the right implies y 1 p, this is equal to 0 and y 2 x satisfies the boundary conditions on the left, means y 2 a, this is equal to 0.

Now, look at the definition for the Green's function. Green's function is G (x, s), that is equal to here, y 2 s y 1 x divided by alpha. This is defined for a less than equal to s less than x less than equal to b and y 1 x y 2 x divided by alpha, where a less than equal to x less than s less than equal to b.

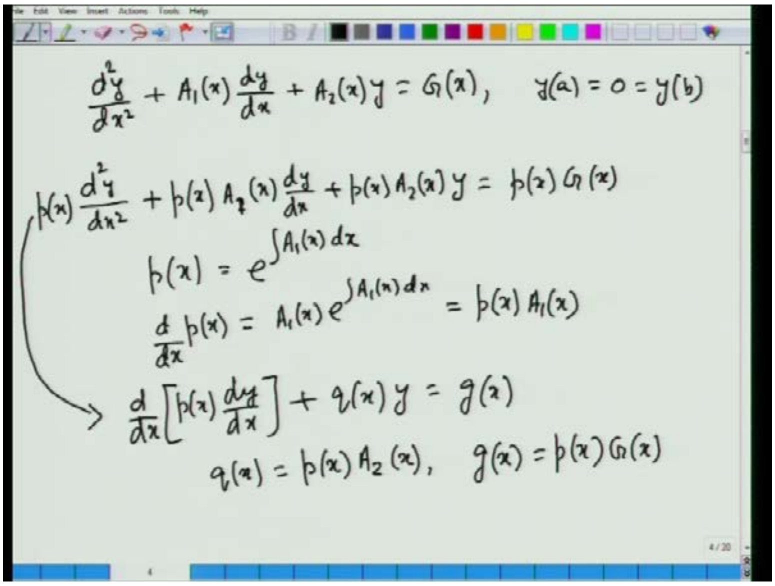
(Refer Slide Time: 08:01)



Now, look at this definition. In the first part x can be equated to b, so therefore, when you are substituting x equal to b in the Green's function, then G of (b, s), that means, you are substituting x equal to b and from this definition x equal to b is allowed for the, when G (x, s) equal to y 2 s y 1 x by alpha. So, this is, this will come out as y 1 b y 2 s divided by alpha. And already we have mentioned, that y 1 b equal to 0, so therefore, this is equal to 0.

Similarly, we can easily verify, that G a s, that is equal to y 2 a y 1 s divided by alpha, this is equal to 0. So, this, that means, the Green's function satisfies the boundary conditions on the left hand point and as well as the right hand point. And similarly, using the property, that m 3 y 1 b plus m 4 y 1 dot b, you can verify, that they also satisfy the boundary condition, when boundary condition given in general point at this G (x, s). This continues on this square domain (a, b) cross (a, b) and its partial derivative, this partial derivative has been jumped discontinuity along the line x equal to s and this is given by that partial derivative of G (x, s) has a jump discontinuity, and it is defined as del del x of G (x, s) and x equal to s plus. So, that means, we have to use the definition x greater than s minus del del x of g (x, s) at x is equal to s minus. That means, we have to use x less than s, that is equal to minus 1 by p s and from here we can easily find out what will be the value of alpha? What expectation of alpha in terms of s?

(Refer Slice Time: 10:59)



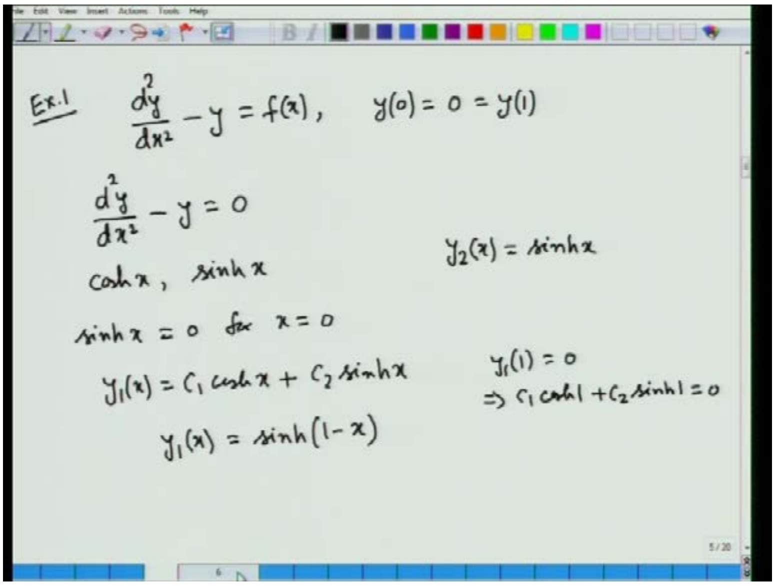
Now, before proceeding further I just want to make a remark here, that initially we started our discussion on this type of ordinary differential equation subjected to boundary conditions, that is, d 2 y d x 2 plus A 1 x dy dx plus A 2 xy equal to capital G x, purposefully I am writing here capital G x and with the condition, say ya equal to 0 equal to yb.

Now, after that we have discussed everything on (( )) boundary value problem. So, the question is whatever theory, that we have discussed on (( )) boundary value problem, that can be applied for this type of differential equation or not. So, we can put question in other way round, that whether this equation can be converted into the (( )) boundary value problem format or not, such that operator will become a self at joint operator? Answer is yes, this can be done because that d 2 y d x 2, this equation can be multiplied by the function px and which results in px d 2 y d x 2 plus px A 1 x dy dx plus px A 2 xy, that is equal to px into capital Gx.

This one now if we define px, this is equal to e to power integral A 1 x dx, then we can easily verify, that d dx of px, this is going to be A 1 x e to power integral A 1 x dx. So, that means, actually this is equal to px times A 1 x. So, if we define px in this particular way for the first two term of the last expression, that px d 2 y dx 2 plus p x times A 1 x dy dx, these can be combined into to write d dx of px into dy dx. So, that means, these equation becomes d dx of px dy dx plus qxy, this is equal to gx, where qx is px times A 2 x and small gx, this is equal to px times capital Gx. And just note, that here we have made some rearrangement of the tar mode of, you can use the transformation px equal to this one and in this process we have not disturbed anything on y.

So, that means, the boundary condition y a equal to 0 equal to y b, that remains unaltered. That means, this problem can be written as a (( )) boundary value problem. Now, next is, that we are going to consider some particular problem, that how this type of boundary value problem can be solved in term of Green's function.

(Refer Slide Time: 14:58)



So, first of all we consider this example, that is, d 2 y d x 2 minus y, this is equal to fx with the condition y0 equal to 0 equal to y1. Of course, this is a simplified version of the (( )) boundary value problems and at this moment we have no lambda here, px is 1. So, of course, this is an equation of the form (( )) boundary value problem.

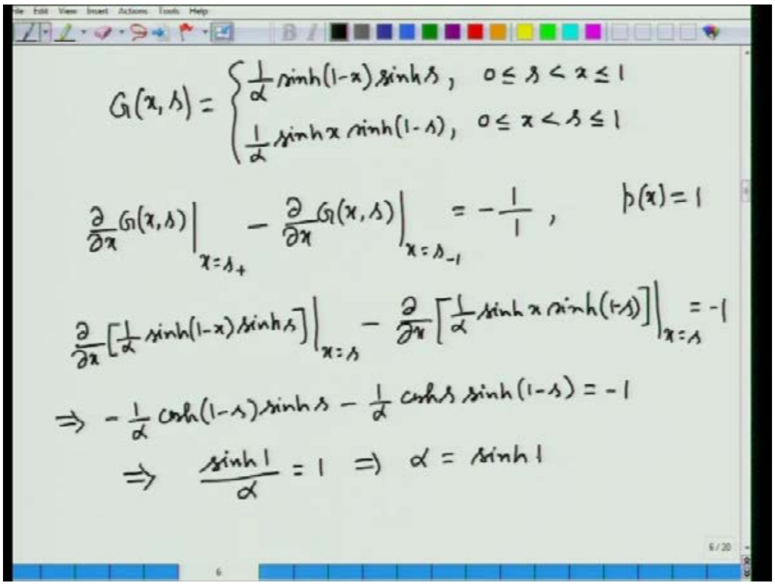
Now, first of all we have to consider the corresponding homogenous equation, that is, d 2 y d x 2 minus y, this is equal to 0. This is the homogenous ordinary differential equation associated with the given non-homogenous problem, two linearly independent solution of this particular problem are actually cos hyperbolic x and sine hyperbolic x.

Now, you can recall, in order to construct the Green's function for this problem we have to choose y 1 x and y 2 x in such a way, that y 1 x will satisfy the boundary condition on the right hand and y 2 x will satisfying the boundary condition on the left hand such that y 1 and y 2, these two functions of x should be linearly independent. Now, clearly, for all real values of x cosine hyperbolic x is always positive. So, that means, this cosine hyperbolic x does not (( )) either x is equal to 0 or at x equal to 1, but sine hyperbolic x, this is equal to 0 for x equal to 0. So, that means, the sin hyperbolic x, this is the solution of the homogenous problem and satisfying the boundary condition on the left. So, based upon this fact we can denote this sine hyperbolic x as y 2 x.

Now, we have to find out a linearly independent function, which will be satisfying the left hand boundary conditions as none of them are satisfying the boundary condition and equation is a linear equation. So, we can try to find out y 1 x into the format, that is linear combination of these two functions, that is, c 1 cosine hyperbolic x and c 2 sine hyperbolic x and using the condition, that c 1 cosine hyperbolic x plus c 2 sine hyperbolic x will be 0 at x equal to 1. You can find out c 1 c 2 and ultimately will be having these results, that is, y 1 x, this is equal to sine hyperbolic 1 minus x. This can be easily obtained because if you claim y 1 1 equal to 0, that means, y 1 1 equal to 0.

So, this implies, c 1 cosine hyperbolic 1 plus c 2 sine hyperbolic 1, this is equal to 0 and from here you can find out c 1 by sine hyperbolic 1 equal to c 2 by minus cosine hyperbolic 1 and taking this constant of proportion equal to 1, you can derive this quantity c 1 and c 2. And after substituting you can find y 1 x equal to sine hyperbolic 1 minus x. And of course, by the method of (( )) you can verify these two solutions, that is, sine hyperbolic x and sine hyperbolic 1 minus x, they are actually two linearly independent solution of the homogenous equation.

(Refer Slide Time: 19:26)

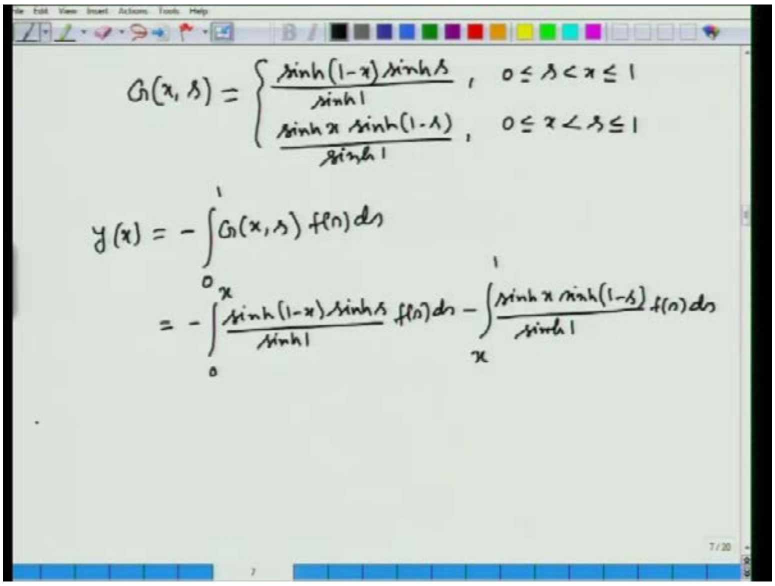


So, now, with this definition or this particular choice for y 1 x and y 2 x we can write the Green's function, G (x, s), that is equal to 1 by alpha sine hyperbolic 1 minus x times sine hyperbolic s. This is 0 less than equal to s less than x less than equal to 1 and this is equal to 1 by alpha sine hyperbolic x sine hyperbolic 1 minus s with 0 less than equal to x less than s less than equal to 1. So, this is the format.

Now, here if you applied the jump discontinuity of the derivative along the line s equal to x, then we can find del del x of G (x, s) at x equal to s plus minus del del x of G (x, s) at x equal to s minus, this is equal to minus 1 by 1 because here px, this is equal to 1. So, ps, this will be equal to 1. So, this is nothing, but minus 1 by 1 and this x equal to s plus, that means, x greater than s.

So, therefore, from this definition of Green's function we can find this will be actually del del x of 1 by alpha sine hyperbolic 1 minus x sine hyperbolic s. Now, we can substitute x equal to s here because this is actually in order to choice the proper G (x, s) and then minus integral, sorry, del del x of 1 by alpha sine hyperbolic x times sine hyperbolic 1 minus s at x equal to s. This is equal to minus 1 and this gives minus 1 by alpha cos hyperbolic 1 minus s sine hyperbolic s minus 1 by alpha cos hyperbolic s sine hyperbolic 1 minus s. This is equal to minus 1 and this implies sine hyperbolic 1 divided by alpha equal to 1 in (( )) alpha equal to sine hyperbolic 1.

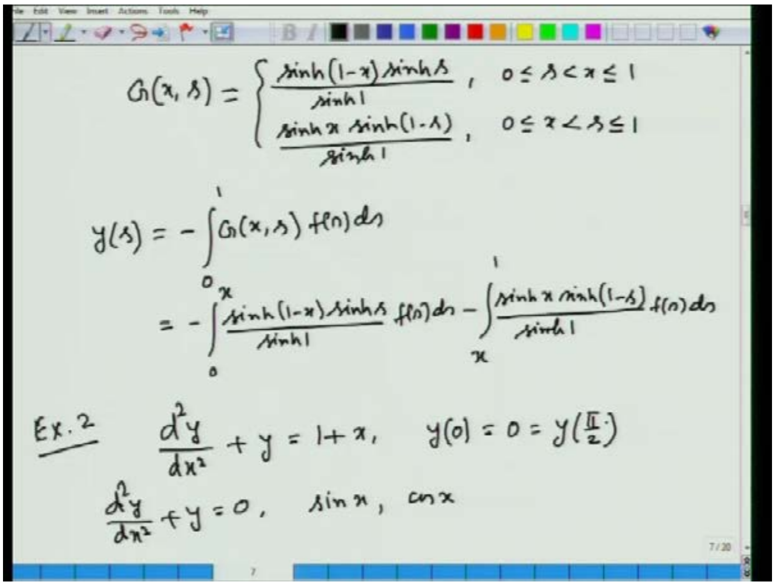
(Refer Slide Time: 22:37)



So, therefore, finally, for the given problem G (x, s), that is equal to sine hyperbolic 1 minus x sine hyperbolic s divided by sine hyperbolic 1 and sine hyperbolic x sine hyperbolic 1 minus s divided by sine hyperbolic 1. This is for 0 less than equal to s less than x less than equal to 1 and 0 less than equal to x less than s less than equal to 1 and therefore, solution to the given problem y x is equal to minus integral 0 to 1 G (x, s) than fs ds and that is equal to, you can find, that minus integral 0 to x sine hyperbolic 1 minus x sine hyperbolic s divided by sine hyperbolic 1 fs ds minus integral x to 1 sine hyperbolic x sine hyperbolic 1 minus s divided by sine hyperbolic 1 fs ds. So, this is the solution to the given problem in terms of Green's function and if we know the particular form of fs, then we can find out the complete solution of the given problem.

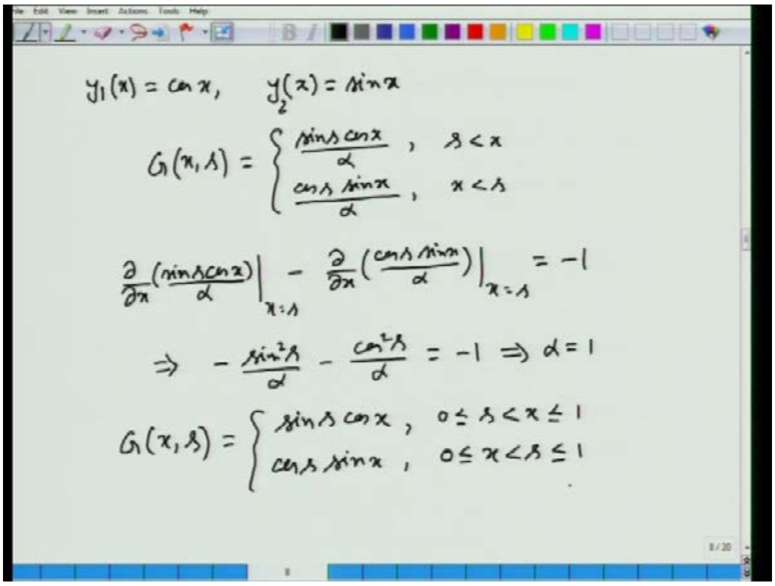
And at this moment it comes to your mind, that this course is on integral equation, but I am here discussing a solution of the (( )) boundary value problem, which is differential equation, but the point I like to make it clear here, that if we convert the given differential equation into the associated (( )) integral equation of first or second time, that will come out after the derivation, and actually affix is non-zero. So, it will be non-homogenous (( )) integral equation of the second time. Then, you can verify, for that problem this is going to be the solution of the integral equation.

(Refer Slide Time: 25:26)



Now, we consider one more example where fx is given such that you can verify the solution obtained by this method is actually satisfying the given equation. Here we consider the problem, that is, d 2 y d x 2 plus y, this is equal to 1 plus x with the condition y 0 equal to 0 equal to y pi by 2. So, now, this corresponding homogenous equation d 2 y d x 2 plus y equal to 0, it has two linearly independent solution, one is sine x, other is cos x. And clearly, this cos x satisfies the boundary condition, at x is equal to pi by 2 sine x satisfies the boundary condition at x equal to 0. So, that means, sine x satisfies the boundary condition on the left hand, cos x satisfies the boundary condition on the right hand.

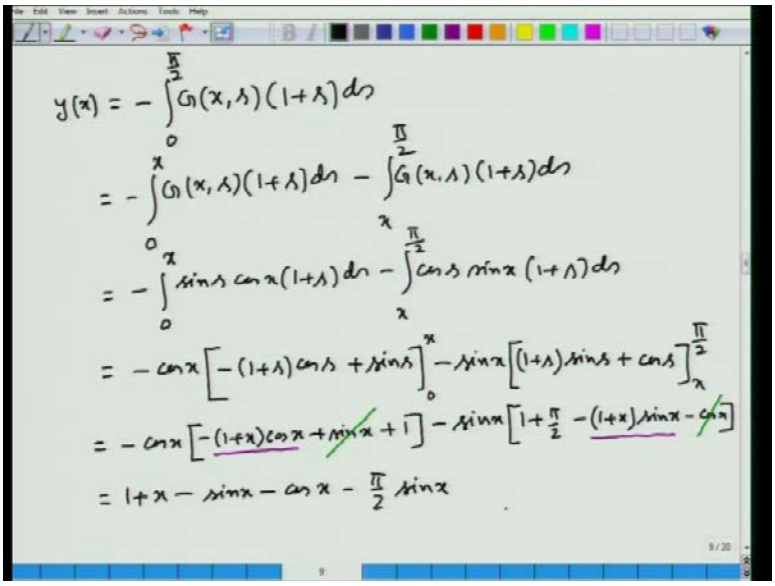
(Refer Slide Time: 26:23)



So, therefore, we can denote them as cos x as y 1 x and sine x y 2 x and therefore, this Green's function G (x, s), this will be sine s cosine x divided by alpha. This is for s less than x and cosine s sine x divided by alpha. This is for x less than s and again, using the jump discontinuity of the Green's function, that is the derivative of the Green's function we can find del del x of G (x, s) for x greater than s.

So, that means, here we have to apply on sine s cosine x, then substituting x equal to s minus del del x of minus del del x of cosine s sine x divided by alpha with x equal to s, that is equal to minus 1 because p is equal to 1 here. So, this gives minus sine square s divided by alpha minus cosine square s divided by alpha. This is equal to minus 1 implying alpha equal to 1 and therefore, this Green's function G (x, s), this is equal to simply sine s cosine x and cosine s sine x. This is valued for 0 less than equal to s less than x less than equal to 1 and this is 0 less than equal to x less than s less than equal to 1. So, these are the definitions.

(Refer Slide Time: 28:42)



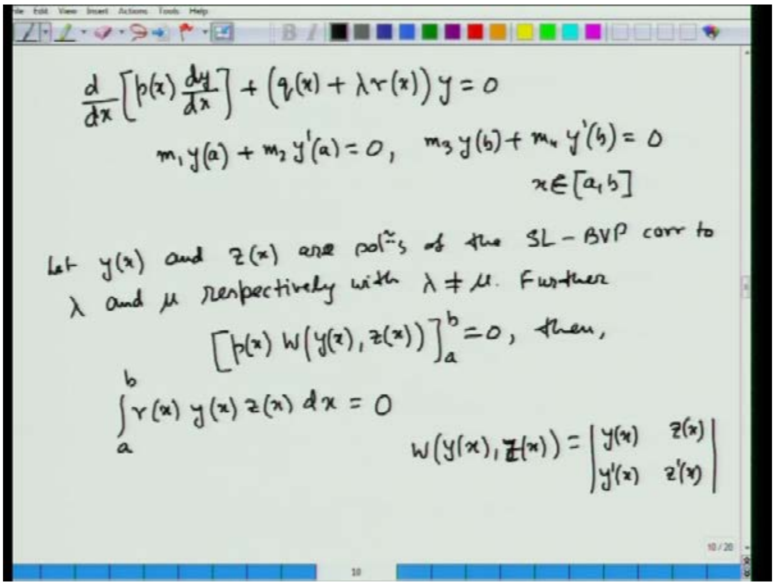
And therefore, the solution to the given problem, that is, yx is equal to minus integral 0 to pi by 2 G (x, s), this will be 1 plus s ds. So, first of all we divided into two intervals, that is, 0 to x G (x, s) 1 plus s ds minus x 2 pi by 2 G of (x, s) times 1 plus s ds and here this will be minus integral 0 to x. First integral s is less than x, so (( )) s is less than x. So, this is sine s cosine x. So, therefore, this will be sine s cosine x 1 plus s ds minus integral x 2 pi by 2 cosine s sine x 1 plus s ds. And using the formula for integration by parts, cosine x can be taken out of the integral and then we have to use (( )) formula considering 1 plus s as the first function u and sine s as the second function v.

So, we will be having this minus 1 plus s cosine s because integral of sine s minus cosine s, then (( )) will involve minus sine combined with this 1 plus 1 derivative of 1 plus s is 1, so plus cosine s. And after integration this will be plus sine s limit 0 to x and for the second integral minus sine x can be out of the integral sign, then it will be 1 plus s sine s plus cosine s limit x 2 pi by 2 and this is equal to minus cosine x at the upper limit minus 1 plus x cosine x plus sine x. At the lower limit cosine 0 is 0, so this will be plus 1 and no contribution from the sine x term. Then, minus sine x, this will be 1 plus pi by 2 into 1 sine x is 1. There is no contribution from cosine at y by 2 is 0 and then minus 1 plus x sine x minus cosine x.

Now, you just check, that these two terms can be combined together, this one and this one, they actually produce the term 1 plus x and then these two terms cancels with each other, that is, this one and this one. So, then we are left with minus sine x minus cosine x and minus pi by 2 sine x. So, this is actually solution of the non-homogenous boundary value problem, that is, d 2 y d x 2 plus y equal to 1 plus x subjected to the boundary condition, that is, y 0 equal to 0 as well as y pi by 2, that is also equal to 0.

Next, we are going to consider some properties of Eigen values and Eigen functions associated with the (( )) boundary value problem from where we can define, that set of orthogonal function and the infinite collection of set of orthogonal functions will gives us opportunity to expand any given function, which of course, satisfies certain differentiability and continuity condition such that infinite series will converge uniformly, then those functions can be expressed as an infinite series of this orthogonal functions, that means, those functions can be generated with the help of the collection of familiar orthogonal functions.

(Refer Slide Time: 33:41)

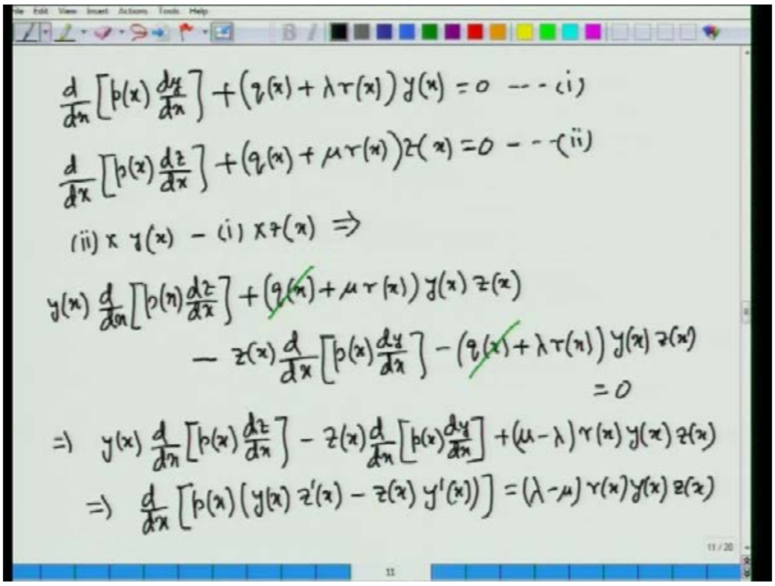


Now, first of all we consider some particular property of these problems. So, throughout the rest of the part of the discussion we will be considering this equation, that is, d dx of px dy dx plus qx plus lambda rx, this y equal to 0 with the boundary conditions, that is, separated boundary conditions m 1 ya plus m 2 y dot a, this is equal to 0 and m 3 yb plus m 4 y dot b, this is equal to 0 where this p dot x qx and rx they are continuous. And we are assuming, that p x is non-zero for all values x within the range, that is, x belongs this close interval a, b.

Now, first of all we are going to prove a result, that let yx and zx are solutions of the (( )) boundary value problem corresponding to lambda and mu respectively with lambda not equal to mu. Further, further this condition, that is, px and wronskian of yx zx, this is from a to b is equal to 0, then integral a to b rx yx zx dx. This is equal to 0.

And here, just for your quick reference, wronskian of yx and zx is nothing, but determinant yx zx y dashed x z dashed x. So, that means, for lambda y is solution for mu, z x is the solution in these conditions, that is, p x multiple by wronskian of y z from a to b. This is equal to 0, then this condition is satisfied a to b rx times yx zx dx equal to 0. And whenever this condition is satisfied, that means, integral a to b rx into yx into zx equal to 0. Then, we say, that y and z, they are orthogonal functions to each other with respect to the weight function rx depending upon the associated (( )) boundary value problem. If rx equal to 1, then condition for orthogonality will come down to simply integral a to b yx zx dx, this is equal to 0.

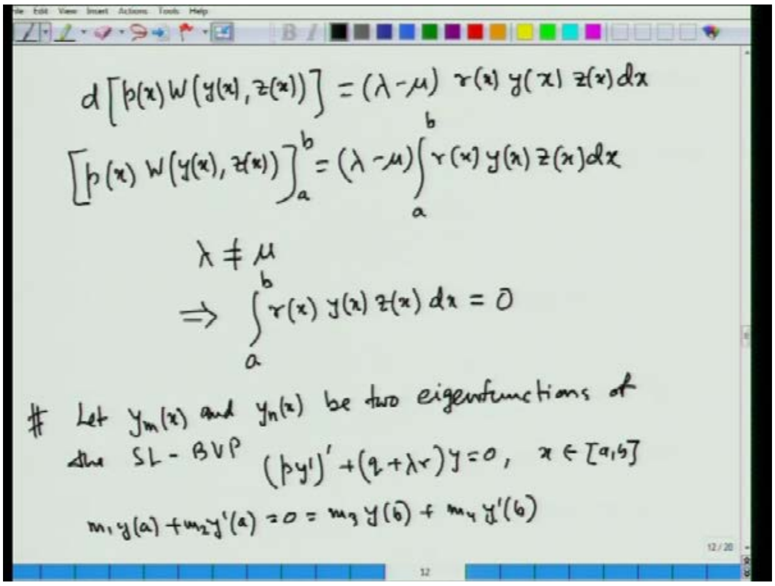
(Refer Slide Time: 37:30)



So, first of all we prove this result, y x is the solution of (( )) problem for lambda. So, that means, d dx of px dy dx plus qx plus lambda rx, this times yx, this is equal to 0, call it one. And d dx of px dz dx plus qx plus mu rx multiplied by yx, this is equal to 0, call it two. Now, if we multiply second equation by zx and first by yx and then subtract, this implies we will be having that, sorry, multiplying 2 by yx and 1 by zx, then we can find, that yx multiplied with d dx of px dz dx plus qx plus mu rx this into yx zx minus zx multiplied with d dx of px dy dx minus qx plus lambda rx times yx zx, this will be equal to 0. Here, in equation two this will be actually zx.

Now, you can see, that qx yx zx and qx yx zx cancels from, cancels with each other. So, then rest of the expression we can combine as yx with d dx of px dz dx, then minus zx times d dx of px dy dx, this expression plus mu minus lambda rx yx zx. And now, you can recall from the previous discussion, that this part, that is, yx d dx of px dz dx minus zx d dx of px dy dx, this is nothing, but the derivative of px multiplied with yx z dashed x minus zx y dashed x, this is equal to lambda minus mu rx yx zx. And this expression yz dot minus zy dot is nothing, but the wronskian of y and z.

(Refer Slide Time: 41:35)

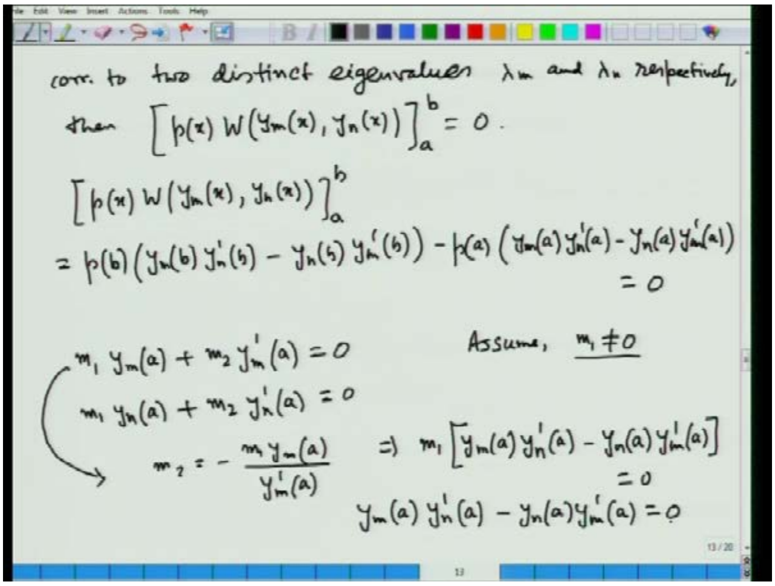


And therefore, we can write, that d of px W yx zx, this is equal to lambda minus mu rx yx zx dx. And then, integrating from a to b we can find px W yx zx, this limit a to b, that is equal to lambda minus mu integral a to b rx yx zx dx. And since this is given to be 0 and with the condition lambda not equal to mu, this implies, that integral a to b rx yx zx dx, this is actually equal to 0.

So, that means, for two (( )) boundary value problem, one with parameter lambda and other with parameter mu with same P, q, r, if you are able to find out two corresponding solution of the equations, which are denoted by yx and zx such that px wronskian yx zx evaluated at b minus the same expression evaluated at a, is equal to 0. Then, these two functions y and z are orthogonal to each other with respect to the weight function r.

Next, we are going to prove, that for a particular (( )) boundary value problem we are able to find out two non-trivial solutions, that means, Eigen values and Eigen functions and if two Eigen functions are corresponding to two distinct Eigen values, then those Eigen functions are actually orthogonal functions. So, in order to prove this, first of all we are going to prove this result, that let y m x and y n x be two Eigen functions, be two Eigen functions of the (( )) boundary value problem. In short, we can write p y dot whole dot plus q plus lambda r y equal to 0, where x belongs to (a, b) with the boundary conditions m 1 y a plus m 2 y dot a equal to 0 equal to m 3 y b plus m 4 y dot b.

(Refer Slide Time: 45:00)

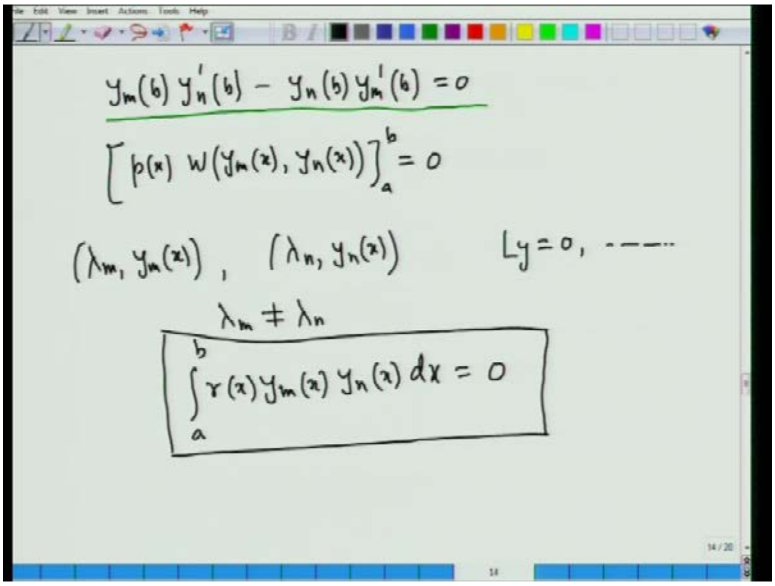


Corresponding to two distinct Eigen values, two distinct Eigen values lambda m and lambda n respectively, then px wronskian of y m x y n x a to b, this is equal to 0. This is one of the important results that we can prove. So, if we look at this expression, that is, px wronskian y m x y n x a to b, this is equal to p of b y m b y n dot b minus y n b y m dot b minus pa multiplied with y m a y n dot a minus y n a y m dot a, this is equal to 0. We have to prove this.

Now, recall, that whenever we have mentioned, that the boundary condition, then we have mentioned m 1 and m 2 are not simultaneously equal to 0. Now, y m and y n, they are solution of (( )) boundary value problem because they are Eigen functions. So, that means, y m satisfies the left hand boundary condition, that is, m 1 y m a plus m 2 y m dashed a, this is equal to 0 and m 1 y n a plus m 2 y n dot a, this is equal to 0.

Now, without any loss of generality we can assume, that m 1, this is not equal to 0 because initially we have mentioned, that m 1 and m 2 are simultaneously equal to 0. So, therefore, we are assuming m 1 not equal to 0. If m 1 not equal to 0, then from the first relation we can write, we can write m 2, this is equal to minus m 1 y m a divided by y m dashed a. So, that means, then we can substitute m 2 in this expression. That means, eliminating m 2 between these two relations we can find m 1 times y m a y n dot a minus y n a y m dot a, this is equal to 0. Now, already we have assumed m 1 not equal to 0. So, therefore, we must have y m a y n dot a minus y n a y m dot a, this is equal to 0. Now, of course, with assumption m 2 not equal to 0 you can arrive at the same result, as well as, if m 1 and m 2 both of them are not equal to 0, then in that case also you can arrive at the same result. Here, for simplicity I have proceeded with this assumption, that is, m 1 not equal to 0, so this is equal to 0.

(Refer Slide Time: 49:43)



Similarly, from the second condition, now you can easily guess, that from the second condition we will be able to derive, that y m b times y n dot b minus y n b times y m dot b this is equal to 0. So, combining these two results, that is, y m y n dot a minus y n a y m dot a equal to 0. And then, y m b y n dot b minus y n b y m dot b, this is equal to 0. You can derive, that p x wronskian of y m x y n x from a to b, this is equal to 0.

So, that means, from this result we can conclude, that for (( )) boundary value problem, if y m x and y n x are two Eigen functions corresponding to distinct Eigen values, lambda m and lambda n, then they are orthogonal to each other because in the last result we have established, if y z at two solutions such that p x w of y x comma z x from a to b, this is equal to 0. Then, integral a to b rx into yx into zx, that is equal to 0.

So, combining these two results we can say, that lambda m comma y m x is an Eigen pair; lambda n y n x, this is another Eigen pair associated with the homogenous (( )) boundary value problem L y equal to 0 with the separated boundary conditions, and lambda m not equal to lambda n. Then, integral a to b rx y m x y n x dx, this is equal to 0. So, this is the result.

So, that means, two distinct Eigen functions corresponding to different Eigen values lambda m and lambda n, they are orthogonal to each other. So, that means, today we have established this result. In the next lecture we will be proving, that under certain conditions satisfied by rx, that means, if rx maintains the same sign toward the interval a to b, then all the Eigen values are real and then we will define the important class of functions associated with the orthogonal function, that is familiar of ortho-normal functions. And from there we can find either Fourier series expansion when this orthogonal functions are trignometric functions or in general, expansion of a function in terms of infinite dimensional orthogonal functions such that every function can be expressed as a linear combination of those functions. And we will be discussing how the coefficients of those series, in terms of the orthogonal functions, can be derived using all these properties of the Eigen functions associated with a (( )) boundary value problem.

Thank you for your attention.