Calculus of Variations and Integral Equation

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Welcome viewers to the lecture series on integral equation under NPTEL course. In last lecture, we have discussed about how the linear boundary value problems can be converted into Fredholm integral equation. In today's lecture, we are going to discuss a special type of second order boundry value problems, those are actually known as Strum Liouville boundary value problem and with the Strum Liouville boundary value problem, we can see how the concept of Green's function will comes into the picture and this Green's functions actually give us the solution for Fredholm integral equation.

Now, before going to the solution of Fredholm integral equation with the method of Green's function in this lecture, I am just going to quickly recapitulate the related theories of Strum Liouville boundary value problems and the construction of Green's function for non homogeneous second order boundary value problems with the separated boundary condition.

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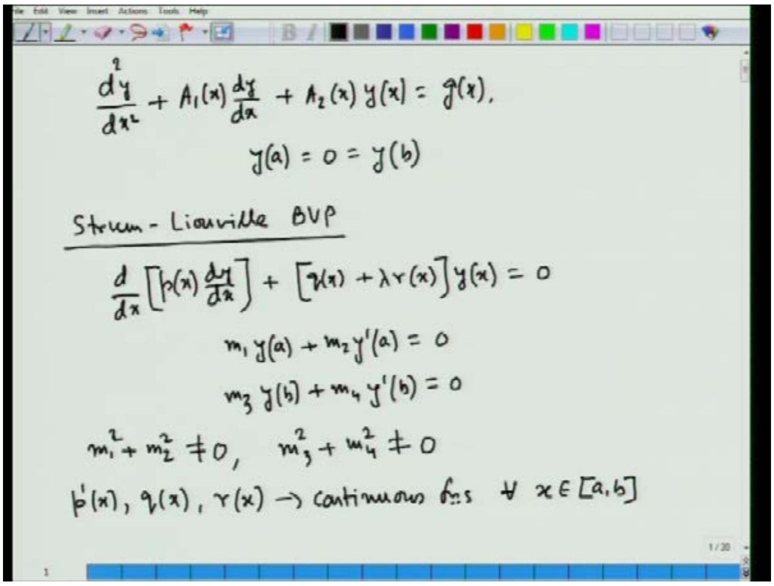

So, the equations we are going to consider those are of these type that is d 2 y d x 2 plus A 1 x d y d x plus A 2 x y x this is equal to g x with the associated boundary condition that we have discussed in the last lecture was of this particular format. And you can recall, if g x equal to 0, then this second order linear boundary value problem can be converted into Fredholm integral equation of homogeneous type. And if g x is a non zero function which is defined over the closed interval a comma b then it can be converted into a Fredholm integral equation of second kind which is a non homogeneous linear integral equation.

Now, we are going to discuss about the Strum Liouville boundary value problem Strum Liouville boundary value problem. This Strum Liouville boundary value problem actually defined as d d x of p x d y d x plus q x plus lamda r x y x this is equal to 0. For the time being, we are considering homogeneous Strum Liouville boundary value problem with the separated boundary conditions, that is m 1 y a plus m 2 y dot a this is equal to 0 and m 2 y b sorry m 3 and m 4 y dot b this is equal to 0.

So, you can see one boundary condition is defined at x equal to a, other boundary condition is defined at y equal to b. So, that why it is called separated boundary conditions and there is a restrictions on m 1, m 1 m 2, m 3 and m 4 that, for the first boundary condition m 1 and m 2 should not be simultaneously equal to 0 and m 3 and m 4 should not be simultaneously equal to 0. So, mathematically we can write that m 1 square plus m 2 square, this is not equal to 0 and m 3 square plus m 4 square, this is not equal to 0.

Now, in this particular problem p dot x q x and r x, these are all continuous functions these are all continuous functions for all x belongs to this closed interval a comma b. Lamda is a parameter and the function p x is non vanishing for all values of x belongs to a comma b.

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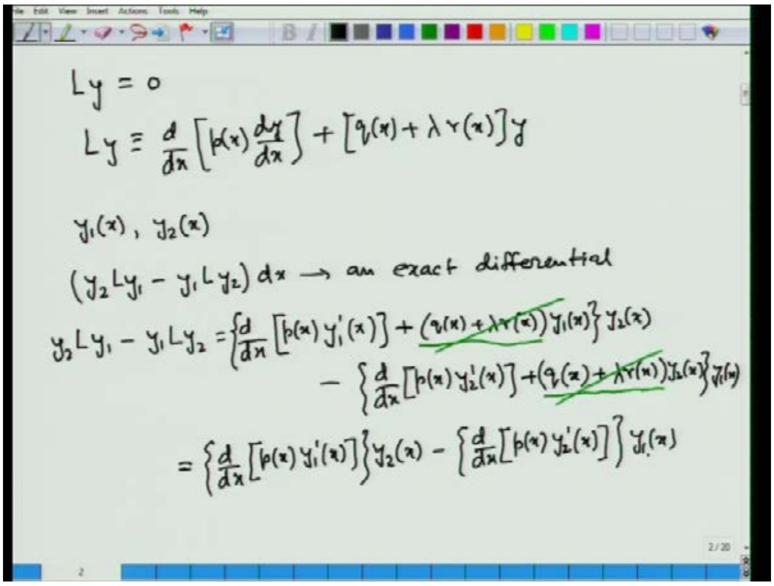

Now, the point is that this homogeneous integral equation can be simply written as in terms of an differential operator L y equal to 0, where this L y this stands for this operator d d x of p x d y d x plus q x plus lamda r x y.

This is actually definition for this operator L operated upon y. Now first of all I just like to draw your attention, this operator L is a self adjoint operator, self adjoint differential operator. This means, if we consider two functions y 1 x y 2 x which are twice continuously differentiable functions then, we can find that y 2 L y 1 minus y 1 L y 2 this d x is going to be an exact differential is going to be an exact differential.

If operator L satisfies this type of condition in case of differential operator if, y 2 L y 1 minus y 1 L y 2 d x is an exact differential. Then, we can say that operator is a self adjoint operator. Now quickly we can just try to verify this result.

So, clearly y 2 L y 1 minus y 1 L y 2 this is equal to d d x of p x for convenience, we can write here y 1 dash x. This entire expression will be multiplied by y 2, this plus q x plus lamda r x, this multiplied with y 1 x and this entire expression is multiplied with y 2 x minus d d x of p x y 2 dashed x, this plus q x plus lamda r x y 2 x, this multiplied with y 1 x. And you can see, that these two terms q x r x lamda r x q x plus lamda r x y 1 y 2 x and q x plus lamda r x y 2 y 1 x these two term cancels with each other. So that means, this will cancel with this one.

And therefore, we are left with the expressions that is d d x of d d x of p x y 1 dashed x, this quantity multiplied with y 2 x minus d d x of p x y 2 dashed x, this multiplied with y 1 x.

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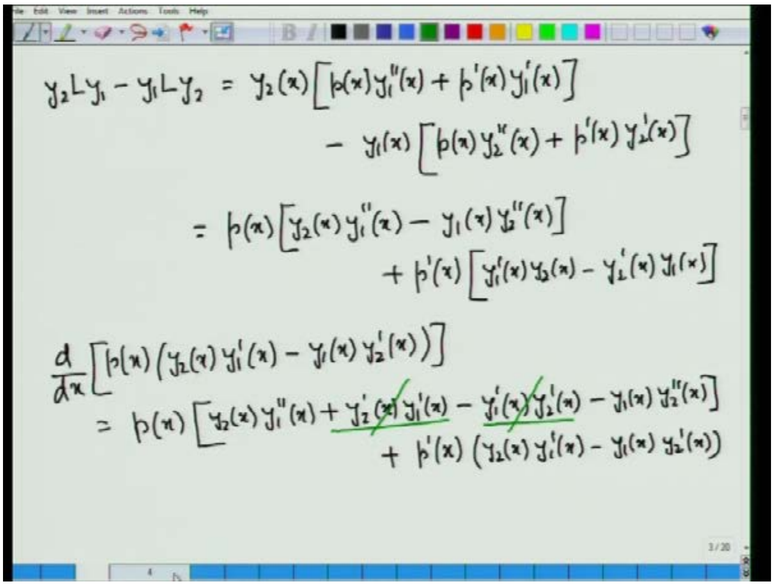

And this will be equal to, that is y 2 L y 1 minus y 1 L y 2 this will be equal to, if we differentiate this expressions so, it will be for the first part this is y 2 x d d x of p y 1 x. So, this will be this can be written as, this is y 2 x this multiplied with p x y 1 double dashed x plus p dashed x y 1 dashed x, this 1 minus y 1 x into p x y 2 doubled dashed x plus p dot x y 2 dashed x. And this will be equal to p x multiplied with y 2 x y 1 double dot x minus from here, if we take the term with p x as common. So, this will be minus y 1 x y 2 double dot x this 1 plus p dot x this multiplied with y 1 dashed x y 2 x minus y 2 dashed x y 1 x.

And now, we can easily verify that, this expression is nothing but d d x of p x multiplied with y 2 x y 1 dot x minus y 1 x y 2 dot x. Because keeping p x without differentiating here, if we differentiate this expression then we will be having y 2 x y 1 double dot x plus differentiating y 2, we can find y 2 dashed x y 1 dashed x minus differentiating y 1 will be having y 1 dashed x 2 dashed x minus y 1 x into y 2 double dot x this. And then differentiating p x and keeping rest of that from unchanged, it will be p dot x times y 2 x times y 1 dot x minus y 1 x times y 2 dot x.

And here, these two terms cancels with each other. So, you can easily verify this resulting expression is nothing, but what we have written for y 2 L 1 minus y 1 L y 2.

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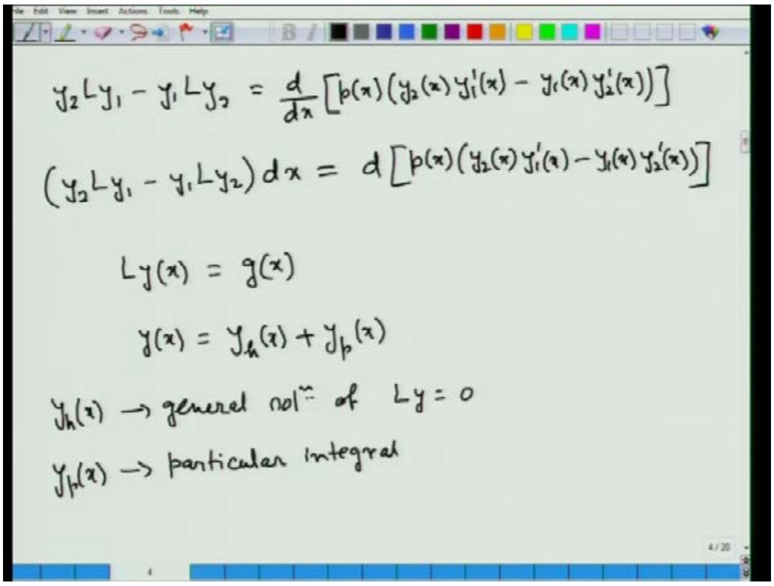

So that means, we are getting the results that y 2 L y 1 minus y 1 L y 2, this is equal to d d x of p x, then y 2 x y 1 dot x minus y 1 x y 2 dashed x. This one and hence, this expression y 2 L y 1 minus y 1 L y 2 this d x is an exact differential. That is d of p x multiplied with y 2 x y 1 dot x minus y 1 x y 2 dot x.

So, this is an exact differential for two functions y 1 and y 2 which are twice differentiable. Now, actually we have to use this result in order to construct the Green's function for the non homogeneous boundary value problem. Now before going to that part, I just like to draw your attention towards this type of problem, that L y equal to g x we have to solve this equation.

For the time being, we are not at all considering anything about the initial condition or boundary condition, but for solution of this equation. You can recall that general solution of this equation y x can be written as, superposition of two solutions y h x plus y p x, where y h x is actually solution general solution of is general solution of the homogenious equation L y this is equal to 0 and y p x, this is known as particular integral.

And since the given equation since the given equation is a linear ordinary differential equation therefore, using the superposition principle, we can find out the solution y x is equal to y h x plus y p x. And using the initial condition or boundary condition whatever provided for the given problem, we can calculate the unknown quantity C 1 and C 2 involved with y h x such that, the solution y x satisfies the given differential equation L y x equal to g x with the specified initial condition or boundary condition.

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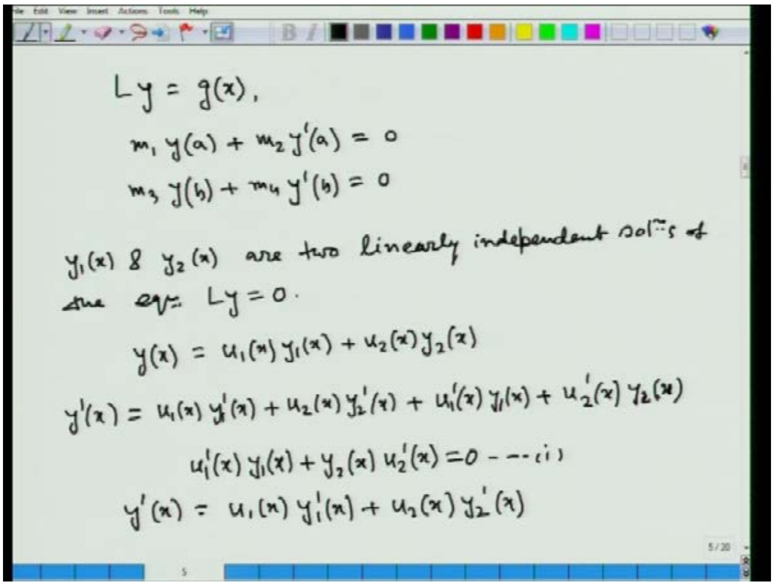

Now, here we are going to construct the Green’s function for the problem L y equal to g x with the boundary conditions, that is m 1 y a plus m 2 y dot a, this is equal to 0 and m 3 y b plus m 4 y dot b this is equal to 0. So, these are the given conditions.

Now, in order to derive the Green’s function associated with the solution of this non homogeneous boundary value problem, we are going to assume one important thing regarding the two linearly independent solution of the associated homogeneous problem. So, first of all we assume that y 1 x and y 2 x, these are two linearly independent two linearly independent solutions of the homogeneous equation of the differential equation, you can write here L y equal to 0.

So, now our assumption is that neither y 1 x nor y 2 x satisfies both the boundary condition. If, any one of them satisfies both the condition, then in order to obtain the solution for the given problem, we need some other restrictions and actually in order to derive this Green's function with help of that solution of this equation, we are not interested to address any further restriction, such that which will affect the solution of this problem.

So, without inviting any further restriction, we are going to call this problem, where our assumption is that y 1 x and y 2 x. These are two linearly independent solution of the associated homogeneous equation L y equal to 0. And if y 1 x satisfies the boundary condition on the left, then y 2 will be satisfying the boundary condition on the right or vice versa. In due course of time, I will assume which function satisfies the boundary condition at which end.

So now, we are going to solve this equation. That means, we are going to find out a solution of this equation by the method of variation of parameters. So, as for method of variation of parameter, we assume solution of the equation will be of the form y x is equal to u 1 x y 1 x plus u 2 x y 2 x, these are two unknown functions. Now just one point, I here would like to draw your attention. In this case, we are assuming that this y x equal to u 1 x y 1 x plus u 2 x y 2 x is going to be the solution of the original given non homogeneous problem. In this case, we are not dividing it into two parts that is y h x plus y p x.

In subsequent deduction, I will discuss why this assumption justifies the method of that I am going to follow here. So, assuming this y x equal to this one, now we have to find out u 1 x and u 2 x in such a way, that y x equal to this one is going to be solution of the non homogeneous problem as well as this y x will be satisfying both the boundary conditions.

Again, you have to keep in mind, y 1 x and y 2 x are two linearly independent solutions. So, we have to choose this or we have to derive u 1 x and u 2 x such that, this is a solution and boundary conditions y a that is boundary condition at a and boundary condition at b will be satisfied, but none of y 1 and y 2 is going to satisfy the boundary condition at the both end. So, first of all differentiating y x, we can find y dot x equal to u 1 x y 1 dot x plus u 2 x y 2 dot x plus u 1 dot x y 1 x plus u 2 dot x y 2 x.

Now, in order to solve a second order linear ordinary differential equation, we have introduced two unknown functions, but we are very much careful, that we are not going to introduce or going to have two more second order ordinary differential equations for u 1 and u 2. In order to avoid that difficulty, that means in order to find out this u 1 and u 2 this two unknown functions without solving a second order differential equation. We can make use of the assumption which is standard method for variation of parameters, that is u 1 dot x y 1 x plus y 2 x u 2 dot x this is equal to 0. This is going to be y 2 x 1, this is equal to 0 call it 1. And with this assumption that is y u 1 dot x y 1 x plus u 2 dot x y 2 x equal to 0, we can find y dot x, this is equal to u 1 x y 1 dot x plus u 2 x y 1 dashed x, this is the expression for y dot x. Now, we calculate L of y with the y x equal to u 1 x y 1 x plus u 2 x y 2 x and y dashed x equal to u 1 x y 1 dashed x plus u 2 x y 2 dashed x. If, we calculate L of y.

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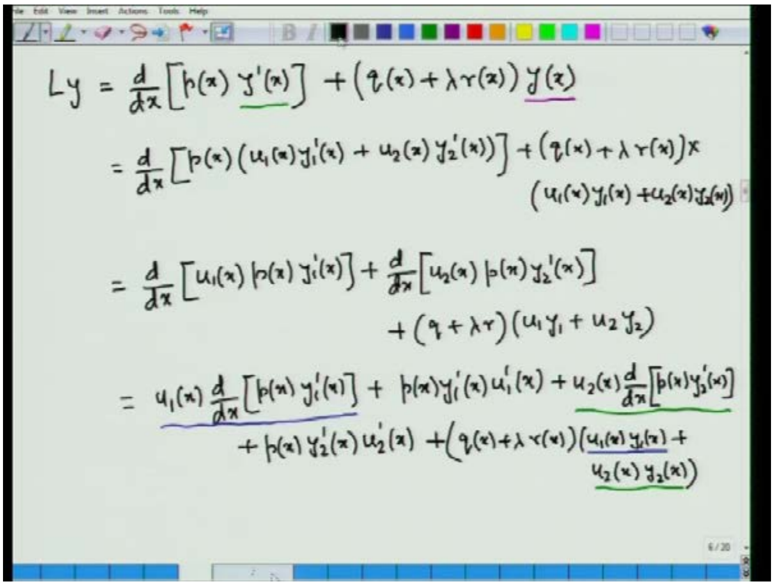

So, this will be equal to d d x of p x times y dot x plus q x plus lamda r x this into y x.

Now, we are actually going to use the result for y dashed x, this will be equal to this one. So this expression, we are going to use and for y x we are going to use the result, this one.

So, substituting these two expressions, we can find this is equal to d d x of p x with u 1 x y 1 dashed x plus u 2 x y 2 dashed x, this expression plus q x plus lamda r x this entire quantity multiplied with u 1 x y 1 x plus u 2 x y 2 x and this is equal to, we can write d d x of u 1 x, then p x y 1 dashed x plus d d x of u 2 x p x y 2 dashed x, this 1 plus rest of the expression without writing x here, that is q plus lamda r into u 1 y 1 plus u 2 y 2. I am not writing x here, as the argument of the functions.

Now, in first two differentiation, we have to be little bit careful, in order to use the result that y 1 and y 2 at the solutions of the homogeneous linear ordinary differential equation. we consider the function which we have to differentiate as product of two functions with first function as u 1 x and second function of p x times y 1 dashed x.

So, with that particular assumption, we can find this is equal to u 1 x then d d x of p x y 1 dashed x, this plus p x y 1 dashed x times u 1 dot x. So, first of all we are not differentiating p x y 1 dashed x, but we are writing u 1 x d d x of p x y 1 dot x. Similarly for the second expression d d x of this one, we can find this will be u 2 x multiplied with d d x of p x y 2 dot x plus p x y 2 dot x multiplied with u 2 dot x plus q x plus lamda r x this multiplied with u 1 x y 1 x plus u 2 x y 2 x.

Now in order to apply the result, that y 1 and y 2 are solution of the homogeneous equation, we have to rearrange this terms. First of all, we are going to take this terms together, that is this expression u 1 x this one and then q x plus lamda r x multiplied with this u 1 x y 1 x. Because with these two terms, we can take u 1 common and similarly from these expressions. That is u 2 x d d x of this and this q x plus lamda r x multiplied with u 2 x into y 2 x.

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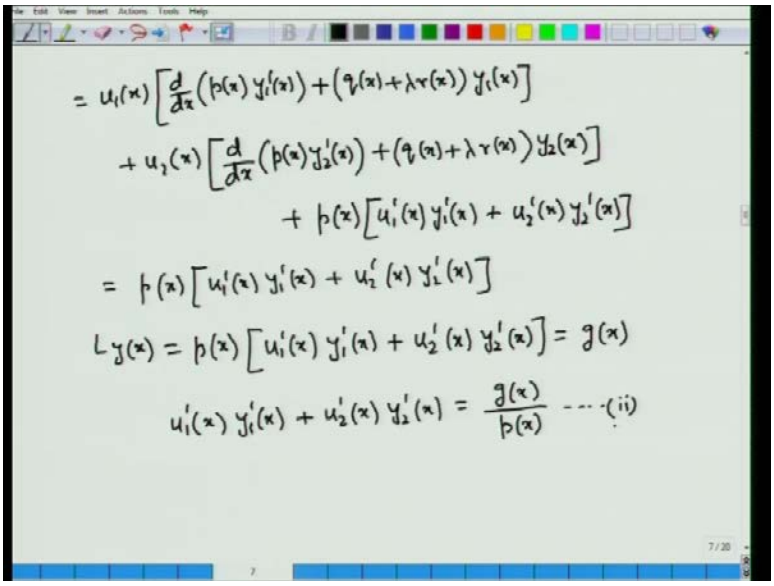

So combining, we can write this is equal to u 1 x this multiplied with d d x of p x y 1 dashed x plus q x plus lamda r x, this multiplied with y 1 x. First term, then plus u 2 x this d d x of p x multiplied with y 2 dashed x plus q x plus lamda r x, this multiplied with y 2 x and then will be left with two terms, that is p x multiplied with y 1 dashed x u 1 dashed x and p x multiplied with y 2 dot x u 2 dot x.

So, this will be plus p x multiplied with u 1 dashed x y 1 dashed x plus u 2 dashed x with y 2 dashed x. Now, these expression that is d d x of p x y 1 dot x plus q x plus lamda r x into y 1 x this entire equation is nothing but L y 1.

Now, y 1 is the solution of associated homogeneous differential equation. So, this expression is identically equal to 0, similarly d d x of p x y 2 dot x plus q x plus lamda r u 1 dot x y 1 dot x plus u 2 dot x y 2 dot x. Now, if we claim that y x equal to y 1 x into u 1 x plus u 2 x times y 2 x is going to be solution of the non homogeneous problem. So therefore, we must have this L of y x is equal to p x multiplied with u 1 dashed x y 1 dashed x plus u 2 dashed x y 2 dashed x, this should be equal to g x.

Because this is the resulting expression for L y x and if, y x equal to the assumed format u 1 y 1 plus u 2 y 2 is going to be a solution of the non homogeneous problem. Therefore, we should have this is equal to 0. And from here, we can write that u 1 dot x y 1 dot x plus u 2 dot x y 2 dot x this is equal to g x by p x and here you can see the utility of our assumption that p x is not equal to 0 for all x belongs to the close interval a comma b.

So that means, now we can solve for u 1 dot and u 2 dot from this equation, we can call this equation as two and then solving it with one we have to determine u 1 dot x and u 2 dot x .

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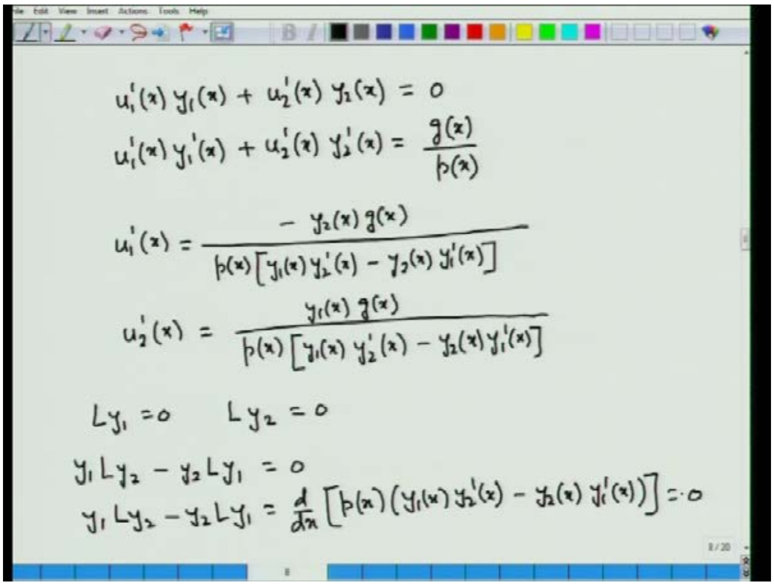

So that means, we are going to solve this two equations u 1 dot x y 1 x plus u 2 dot x y 2 x this is equal to 0 and u 1 dot x y 1 dot x plus u 2 dot x y 2 dot x. this is equal to g x by p x.

So just by cross multiplication, we can find that u 1 dot x this is equal to minus y 2 x g x this whole divided by p x into y 1 x y 2 dashed x minus y 2 x y 1 dashed x. This expression and u 2 dashed x that is equal to y 1 x g x divided by p x multiplied with y 1 x y 2 dashed x minus y 2 x y 1 dot x this 1.

Now, we have to ensure that this quantity present in the denominator for the expression of u 1 dot x u 2 dot x, these are non zero. And you can easily verify from the theory of ordinary differential equations, that y 1 x times y 2 dot x minus y 2 x times y 1 dot x is nothing, but the wronskian of two function y 1 x and y 2 x. And here, we can easily prove that these quantity is a constant quantity because L y 1 this is equal to 0 L y 2 this is equal to 0 y 1 and y 2, they are actually two linearly independent solutions of the corresponding homogeneous equation. And then from previous result, we can easily derive that y 1 L y 2 minus y 2 L y 1 this expression is equal to 0 and from the previous discussion, where we have obtained the result for y 2 L y 1 minus y 1 L y 2. So, interchanging the role of y 1 y 2, we can derive here, that y 1 L y 2 minus y 2 L y 1 this is equal to d d x of p x this multiplied with y 1 x y 2 dot x minus y 2 x y 1 dot x this is equal to 0 and as the derivative of this equal to 0. So therefore, we write p x multiplied with y 1 x y 2 dot x minus y 2 x y 1 dot x this is equal to a constant and for the time being, we can call this particular constant is alpha.

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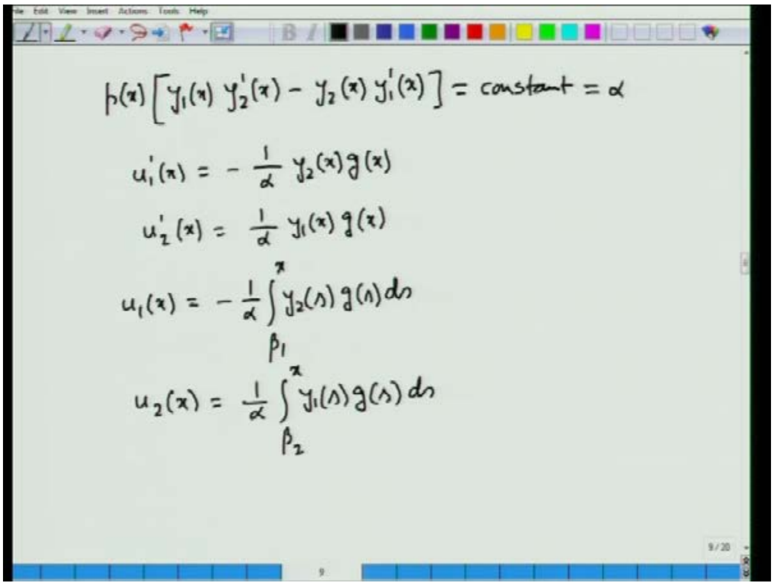

So, using this result that p x times y 1 x into y 2 dot x minus y 2 x into y 1 dot x equal to constant. We can write u 1 dashed x this is equal to minus 1 by alpha y 2 x into g x and u 2 dot x, this is equal to 1 by alpha y 1 x into g x. Because the expression for u 1 dot x was minus y 2 x g x divided by the entire expression p x times y 1 x y 2 dot x minus y 2 x y 1 dot x. And since, this is equal to constant alpha the denominator that is equal to alpha. So, we will be having this result u 1 dot x equal to this one and u 2 dot x equal to this 1.

Now, integrating this result u 1 dot x equal to this from some arbitrary constants say beta 1 2 x, we can write u 1 x this is equal to minus 1 by alpha integral beta 1 2 x y 2 s g x d s. And similarly u 2 x this is equal to 1 by alpha integral beta 2 to x y 1 s g s d x. Now, we are going to derive the constants beta 1 and beta 2 such that the boundary conditions will be satisfied by y x at the two ends, that is at x equal to at and at x equal to b.

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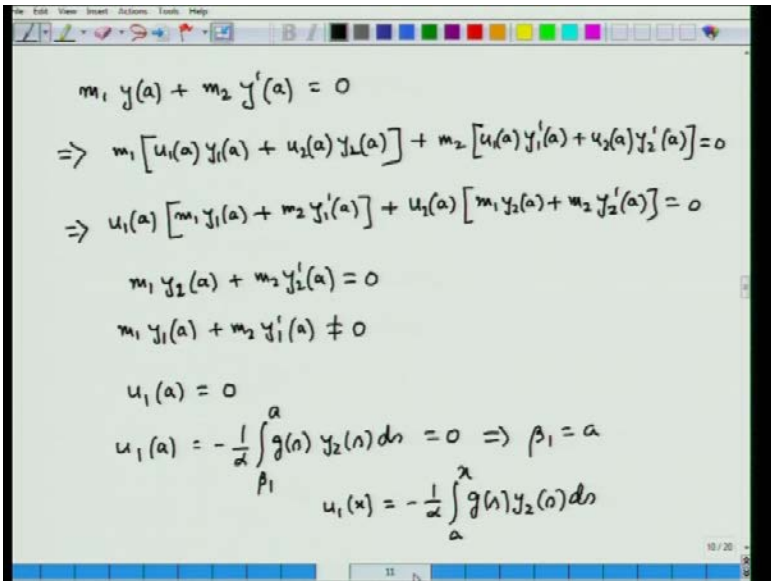
S 

So, if we write the left hand boundary condition going to satisfy by y. So that means, we must have m 1 y a plus m 2 y dot a, this is equal to 0. So now, we have the expressions called y and y dot so, this implies will be having m 1 times u 1 a y 1 a plus u 2 a y 2 a, this is the condition that is y a. And using the result for m 2 you can recall what we have used, we have derived here that is y dot x equal to u 1 x y 1 dashed x plus u 2 x y 2 dashed x. We can find this will be m 2 times u 1 a y 1 dot a plus u 2 a y 2 dot a this is equal to 0.

Now, rearranging that means taking u 1 a and u 2 a common, we can find u 1 a this multiplied with m 1 y 1 a plus m 2 y 1 dot a, this expression plus u 2 a multiplied with m 1 y 2 a plus m 2 y 2 dot a this is equal to 0.

Now, without any loss of genuinity at this point, we are assuming that y 2 satisfies the boundary condition at the left end is that x equal to 0. So that means, we are assuming that m 1 y 2 a plus m 1 y 2 dot a this is equal to 0. Already, we have discussed that this y 2 1 satisfied the boundary condition on the left. So, it should not be satisfying the boundary condition on the right and similarly y 1 a, at this moment is not satisfying the boundary condition on the left. So therefore, assuming m 1 y 1 a plus m 2 y 1 dot a, this is not equal to 0, from this expression we have u 1 a this is equal to 0.

I repeat this argument again, we are assuming that y 2 is satisfy the boundary condition on the left. So therefore, y 2 will not be satisfying the boundary condition on to the right. So therefore, right end boundary condition will be satisfied by y 1 and we have already mentioned that either of y 1 and y 2 will satisfy the boundary condition at 1 end. So therefore, y 1 is not allowed to satisfy the boundary condition on the left end point, as we have assumed that y 2 satisfies the boundary condition on the left end.

So, therefore, m 1 y 1 a plus m 2 y 1 dot a this is not equal to 0 and m 1 y 2 a plus m 2 y 2 dot a this is equal to 0. So, combine this quantity, this result m 1 y a plus m 2 y dot a equal to 0 will be satisfied whenever u 1 a equal to 0.

Now, if you recall the expression for u 1 x, what we have written earlier, then u 1 a is nothing but minus 1 by alpha integral beta 1 to a g s g s y 2 s d s g s y 2 s d s, now this will be equal to 0. Whenever, this beta 1 this is equal to a. So, with this result we can write that, y 1 x is equal to minus 1 by alpha integral a to x g s y to s d s. So, u 1 x we have obtained here.

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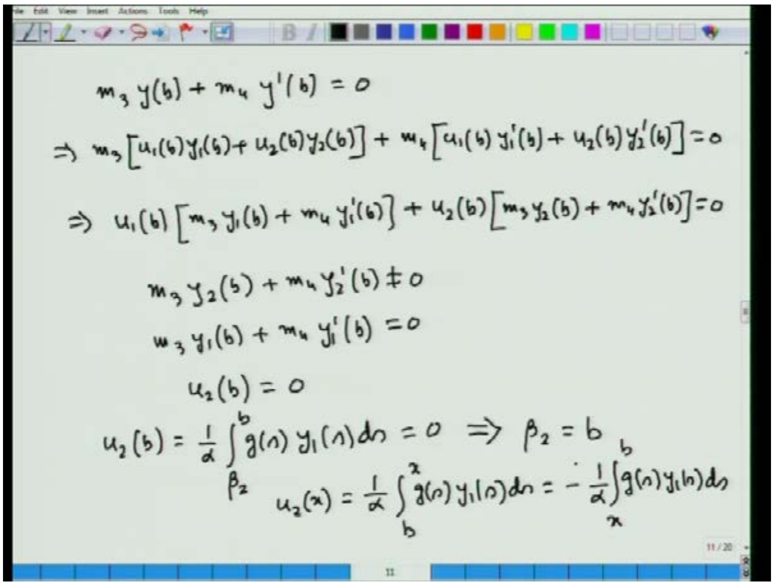

Now, we have to find out y 2 x. So, using the boundary condition on the right end, that is m 3 y b plus m 4 y dot b this is equal to 0, will be having m 3 u 1 b y 1 b plus u 2 b y 2 b, using the expression for y x equal to u 1 x y 1 x plus u 2 x y 2 x plus m 4 this will be u 1 b y 1 dot b plus u 2 b y 2 dot b this is equal to 0.

Again rearranging the terms, we can find u 1 b m 3 y 1 b plus m 4 y 1 dot b plus u 2 b ,this multiplied with m 3 y 2 b plus m 4 y 2 dot b this is equal to 0. Now in the previous step, we have assumed that y 2 satisfied the boundary condition on the left hand. So therefore, as far our assumption m 3 y 2 b plus m 4 y 2 dot b this is not equal to 0. And y 1 is going to satisfy boundary condition on the right hand. So that means, m 3 y 1 b plus m 4 y 1 dot b this is equal to 0. So, substituting these two results here, we can find that u 2 b this is equal to 0.

Now, using the expression for u 2 x, we can find that u 2 b this is equal to 1 by alpha integral beta 2 to b g s y 1 s d s this is equal to 0 and this implies beta 2 this is equal to b. And hence, we can write that u 2 x this is equal to 1 by alpha integral b to x g s y 1 s d s and that is equal to minus 1 by alpha integral x to b g s y 1 s d s. So, just remember these two expression that is u 1 x equal to minus 1 by alpha integral a to x g s y 2 s d s and u 2 x equal to minus 1 by alpha integral from x to b g s y 1 s d s, this is obtained using the condition that y x satisfies the boundary condition on the left and boundary condition o the right. And we have obtained earlier the expressions for u 1 dot x and u 2 dot x using the condition, that y equal to u 1 y 1 plus u 2 y 2 is a solution of the non homogeneous equation.

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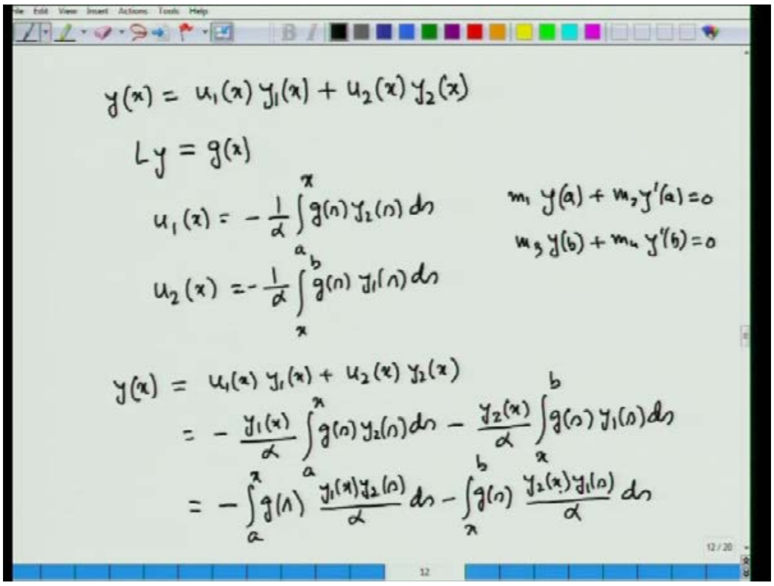

So, if we now just summarize this results so with the assumption y x is equal to u 1 x y 1 x plus u 2 x y 2 x this y x we are assume is going to be the solution of the self adjoint non homogeneous linear differential equation this one. And from here first of all, we have derived the differential equations for u 1 and u 2 and we have obtained the expression for u 1 x, this is equal to minus 1 by alpha integral a to x g s y 2 s d s this expression and u 2 x this is equal to 1 by alpha integral minus here x to b g s y 1 s d s, these two expression we have obtained by using the boundary conditions. That is y a m 1 plus m 2 y dot a is equal to 0 and m 3 y b plus m 4 y dot b this is equal to 0.

So that means combining, we can write this y x is equal to this solution u 1 x y 1 x plus u 2 x y 2 x is the solution of this non homogeneous boundary value problem. And it is given by minus y 1 x by alpha integral a to x g s y 2 s d s minus y 2 x divided by alpha integral x to b g s y 1 s d s. So, taking this expression under the integral sign, you can write minus integral a to x g s multiplied with y 1 x y 2 s by alpha, this d s minus integral x to b g s this multiplied with y 2 x y 1 s divided by alpha d x.

So now, we can define a function g x comma s that will be denoted by g x comma s which follows the definition y 1 x times y 2 s divided by alpha, when s is less then equal to x and it satisfies the definition y 1 x times y 1 x divided by alpha whenever s is greater than x.

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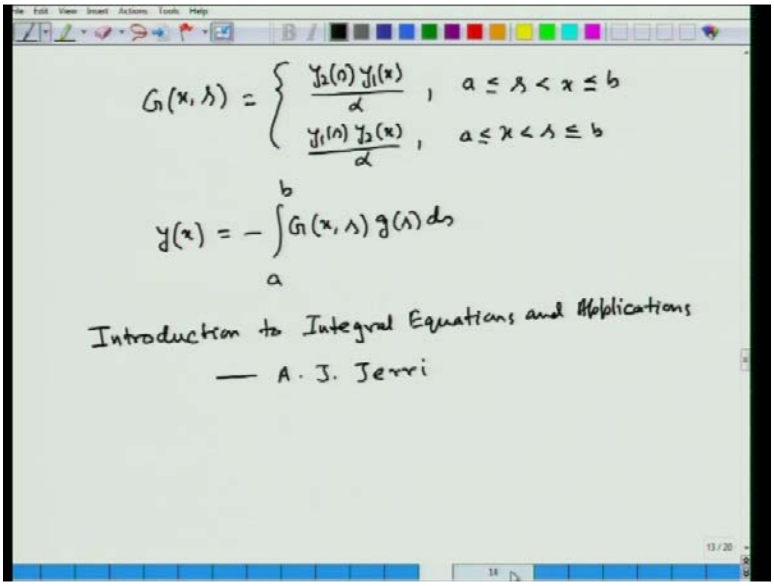

So, defining this function g x, s in this way this is equal to y 2 s y 1 x divided by alpha, this condition is satisfied for a less than equal to s less than x and here, we are writing this x is less than equal to b because our range of interest is from a to b. And then it is y 1 s y 2 x divided by alpha whenever, in general we can write this x less than s less than equal to b and here, we are introducing that this a is less than equal to x.

So, with this definition for g x comma s, we can write the solution to the boundary value problem is given by minus integral a to b g of x comma s g s d s, this is a solution for the non homogeneous boundary value problem which is a Strum Liouville boundary value problem.

Now, in some of the books, we can find that this minus sign absorb into the definition of g of x comma s and in that case, minus sign will not appear here. There are various books which have discussed the construction of the Green's function. In order to solve the non homogeneous boundary value problem but this approach that I have presented here or you can say adopted here is taken from the book by J D and name of the book is introduction to integral equations and applications by A J Jerri.

So, this is the solution of the boundary value problem, that is not homogeneous boundary value problem. So, just to recapitulate quickly what we have done today.

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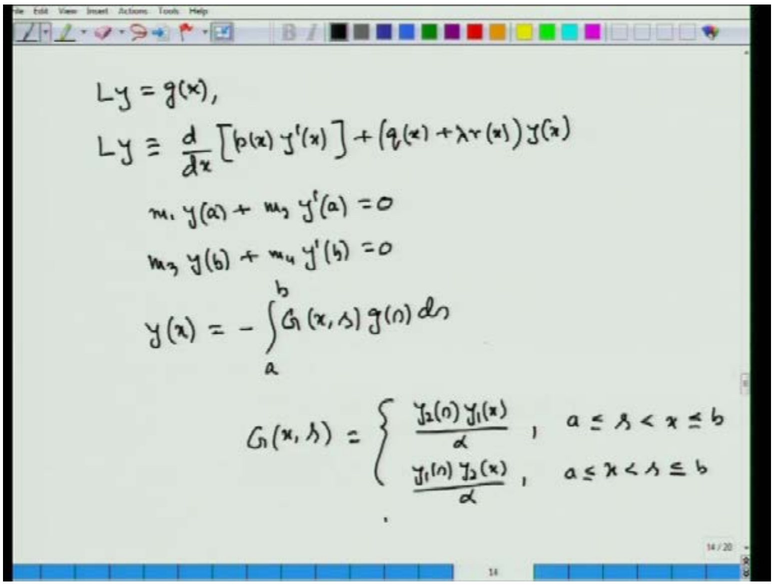

So first of all, we have considered the self adjoint operator and then we have considered the Strum Liouville boundary value problem which is defined by L y equal to g x where l y is equivalent to d d x of p x y dot x plus q x plus lamda r x this into y x. Satisfying the boundary condition, that is m 1 y a plus m 2 y dot a this is equal to 0 and m 3 y b plus m 4 y dot b this is equal to 0.

And solution of this non homogeneous boundary value problem can be written as y x is equal to minus integral a to b g of x comma s g s d s where g x comma x is the Green's function which is defined in this way g x comma s in this way, this is equal to y 2 s y 1 x divided by alpha. This condition is satisfied for a less than equal to s less than x and here we are writing this x is less than equal to b because our range of interest is from a to b. And then, it is y 1 s y 2 x divided by alpha, whenever in general we can write this x less than s less than equal to b and here, we are introducing that this a so, this the solution. And most important point that we can realize from here apart from this unknown quantity alpha, how to derive this alpha that, we will be discussing in the next lecture. That if, we know two linearly independent solutions of the associated homogeneous equation. So, we can directly write down this function g x comma s which is actually known as the Green's function and in terms of this two linearly independent solution of the homogeneous equation, you can write down the solution of the non homogeneous boundary value problem.

So, today we can stop at this. In the next lecture, we will be considering some more properties associated with this Green's function and direct application to verify that, yes this Green's function gives us solution to the non homogeneous boundary value problem. Thank you for your attention .