Calculus of Variations and Integral Equations

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Module No. # 01
Lecture No. # 03

Welcome viewers to the NPTEL lectures on the calculus of variations. This is the third lecture of the series. Here, we continue with our discussions on preliminaries which will be required subsequently in our analysis of various problems of the calculus of variations. We started in the last lecture with the certain concepts on continuity, differentiability, and piecewise continuity, piecewise differentiability and representation of various curves in three dimension as their parametric representation.

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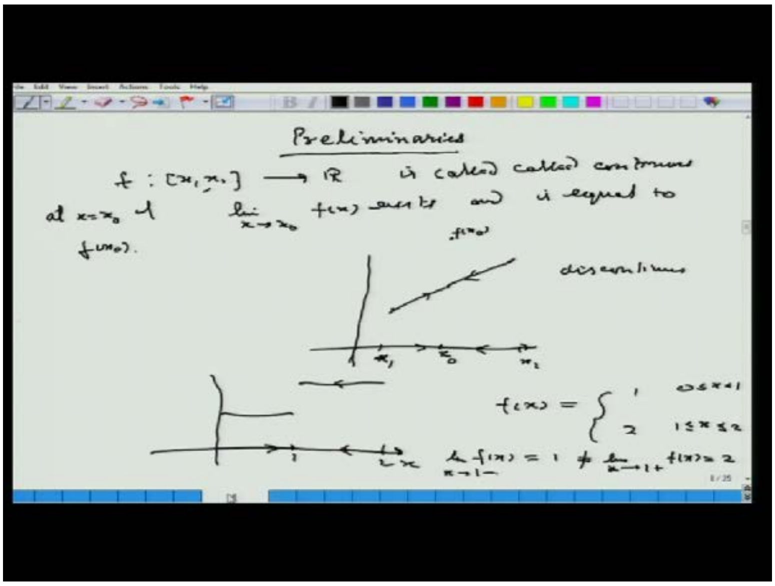

Here, we had started with various concepts of our piecewise continuity where we have in a finite interval in a length of in an interval of finite length there can be only finitely many points of discontinuities and at those points of discontinuities left and right derivatives exist, and they may differ by certain amount which is called the jump of the discontinuity at that point of discontinuity.

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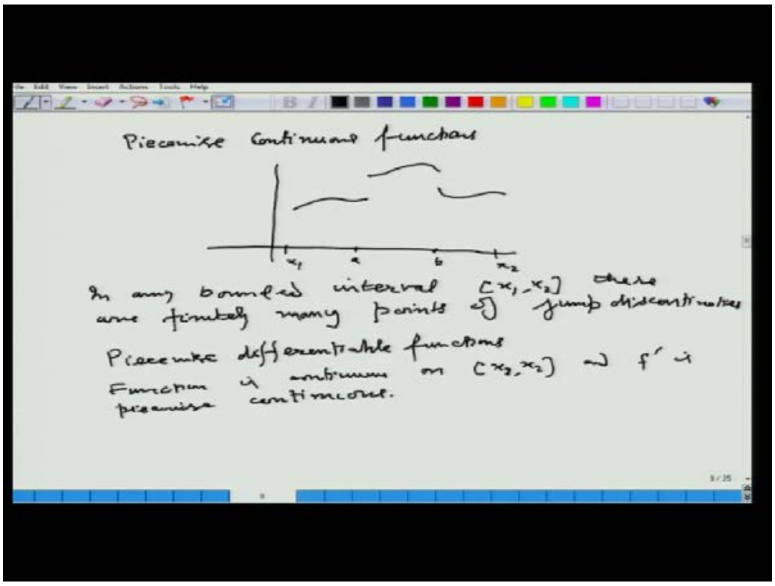

Similarly, here we… So, piecewise continuous function can look like this, where there are for example, in this case there are two points of discontinuity, and here left limit at this point a of the function will exist. Similarly, right limit when you approach to this point from the right, the right limit will exist and this will be this point, the value of the function at that point from the right. Similarly, here the left limit will be this and right limit will be this and so there is a gap here and that is what is the jump of the discontinuity at the points of discontinuity.

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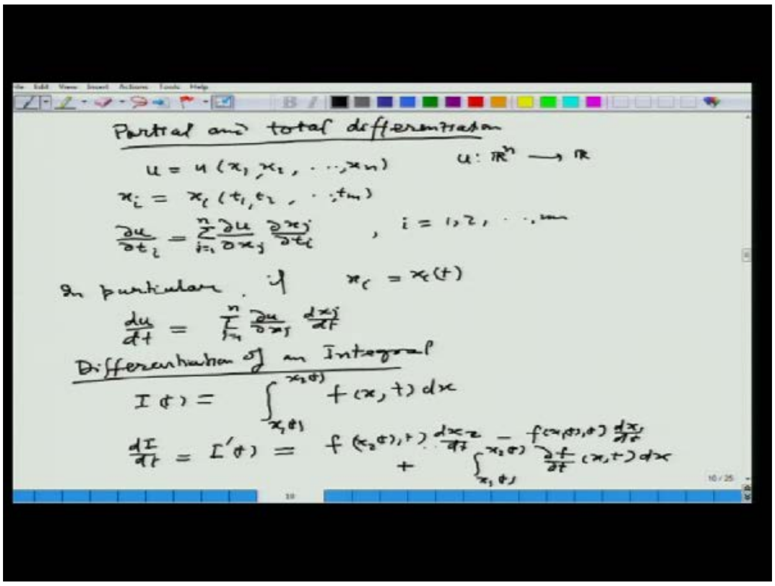

We then define the partial derivatives of function of several variables where the variables itself are then functions of some m variables t 1, t 2, t m, and the partial derivative was defined in this manner. And when the these variables are functions of single variable than the this partial derivative reduces to be ordinal derivative and it is given by this sum, then we had the notion of differentiation of an integral - definite integral where the limits are functions of a variable say t and then the integrant is function of several variables like here there are 2 variables, x is the variable of integration and t is the variable, here which is appearing as a parameter and so this integral will be function of this parameter t.

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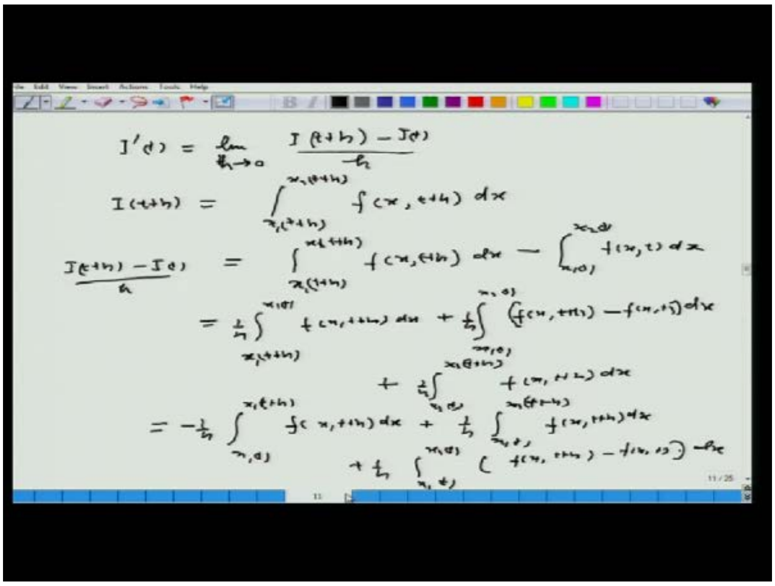

And so we can consider the differentiation of this integral I (t) with respect to the variable t and which is given by this Lebanese rule which was proved in the last lecture in this manner.

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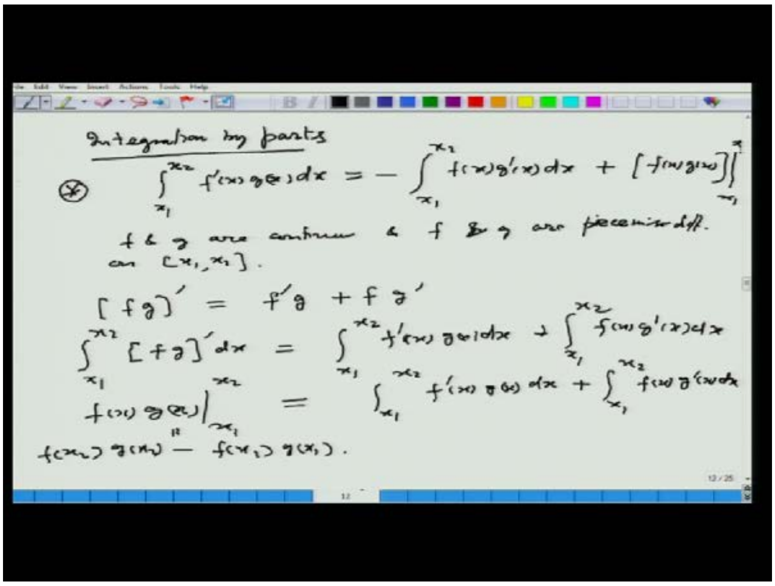

And then another concept of integration by parts were discussed where we use the this fact that f into g prime is a prime g plus f g prime and then integrating it over the interval x 1 to x 2 gives us this integration by parts formula which will be very useful in our access.

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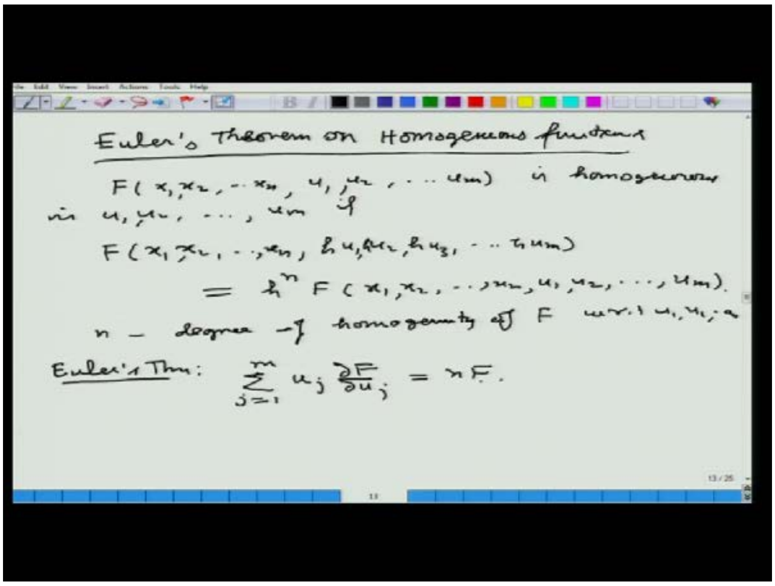

And we considered the function of several variables where function F is there are n variables x 1, x 2, x n, and u 1, u 2, u m, and it is homogeneous of degree m in u 1, u 2, u m if this condition is satisfied, when you replace the variables u 1, u 2, u m, by h u 1, h u

2, h u m, then that h to the power n comes out and then that is multiplied by F of x 1, x 2, x n, and u 1, u 2, u m. This n is called the degree of homogeneity and Euler's theorem states that a summation over j equal to 1 to m u j del F over del u j is n times F.

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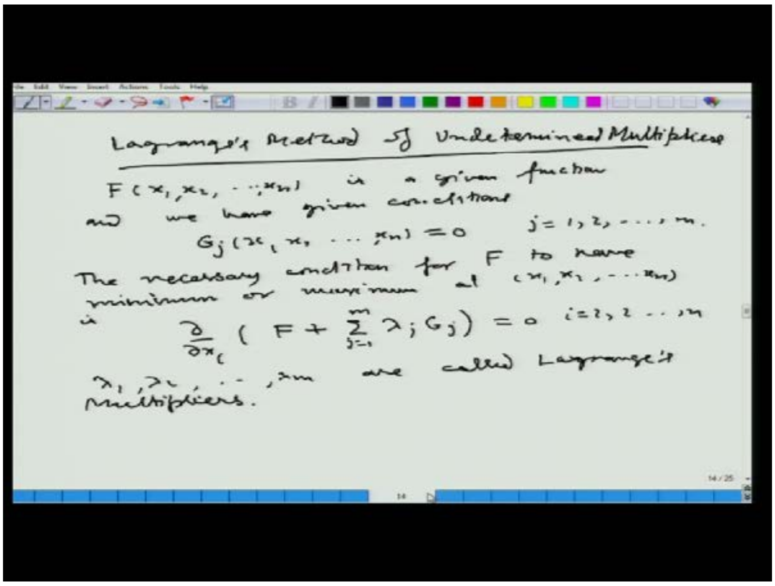

Then we consider the Lagrange's method of undetermined multipliers lambda 1, lambda 2, lambda m. Here there is a function given of n variables which is to be optimized at the points where x 1, x 2, x n, can they will also be found out in the process and these numbers lambda 1, lambda 2, lambda m will also be obtained in the same process. So, these lambda js j equal to 1 to m are actually unknown here, and the points at which x 1, x 2, x n where this F can attain minimum or maximum is also to be found out. So, this is a process of Lagrange's multipliers method. These these multipliers lambda 1, lambda 2, lambda m are called Lagrange's multipliers and they are also unknown in this process and they are to be obtained. So, that is why we call it undetermined multipliers.

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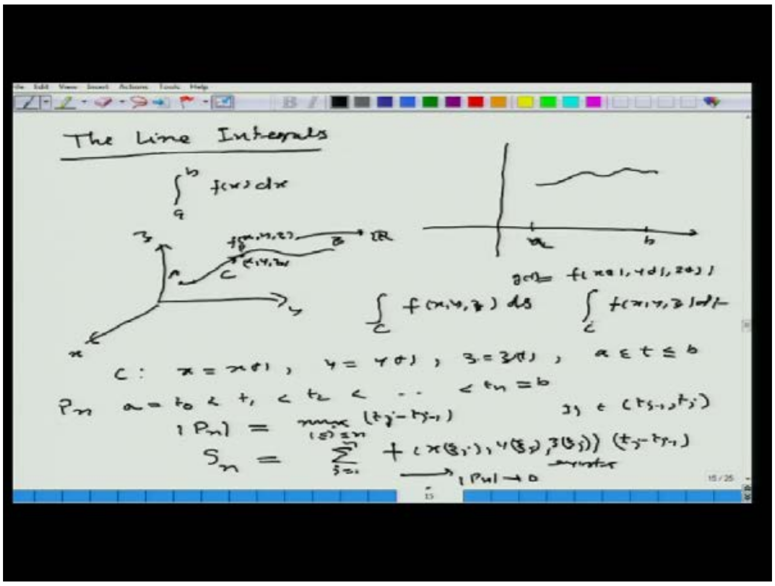

Then we went on to discuss the line integrals. It is a generalization of the integral o integration over an interval it to be f(x) dx. Here we take in a general R n, here n equal to 3 in particular will be R 3 three-dimensional (x,y,z) space, and here we consider a curve c which is a piecewise smooth means that it is a continuous and here it derivative is actually having jump discontinuities of first time.

So, this curve c can be represented parametrically like this x of x(t), y y(t) and z equal to z(t) where t ranges between a to b. Then how we define this line integral, integral f(x,y) f(x,y,z) ds where s here is arc length. So, in general, we will take a parameter t like this f(x,y,z) dt. This line integral in particular will reduce to this, when we take t equal to s, s s is the arc length which is measured from a fixed point of the distance along this curve of the point p measured from a fixed point on the curve.

So, here in general we will take parameter t and when we want to take s as the parameter we replaced t by s here. So, this can be defined like this, we partition this interval a to b in n sub intervals, and the magnitude are the norm of this partition is defined mod P n maximum of mod t j minus t j minus 1, j running from 1 to n. So, when this maximum length goes to 0 if this limit limit of this sum S n which is defined f of x psi j y psi j z psi j where psi j is a point in the sub interval t j minus 1 to t j multiplied by here by the length t j minus t j minus 1, if this some is having a limit as the number of partitions increase and this it is independent of the partition P n considered here it does not matter which way we take these points on this, as long as we see that the maximum length of these sub intervals go to 0. So, it should not happen, because why we want this maximum length, so that we should not partition only certain intervals and leaving the others, we should partition all the sub intervals and so that the maximum length goes to 0. Under that condition if the limit of the sum exist we say that this line integral exist, and denoted by f (x,y) x, y, z are functions of t here and this f(x,y,z) dt is defined as the limit of the sum.

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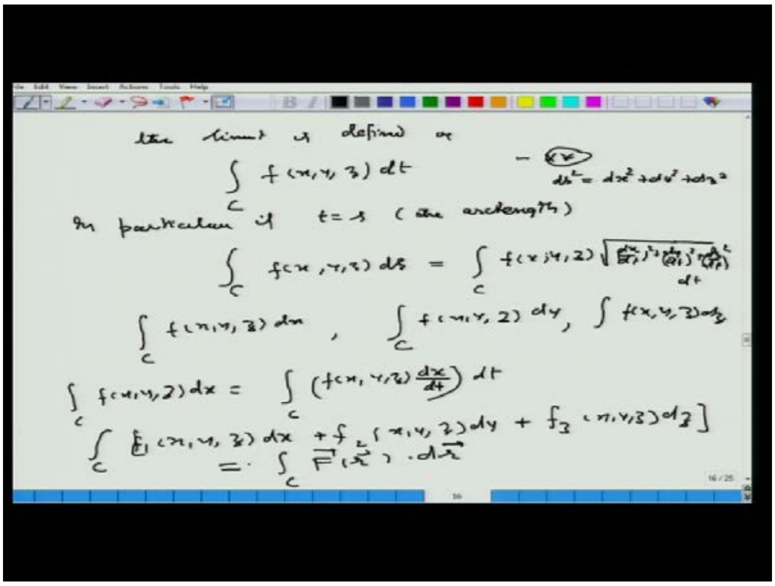

So, in particular, if we take s here then parameter t as s then we can consider this integral f(x,y,z) here then x, y, z will be functions of s, and here ds we know that this ds arc length is given by see ds square is dx square plus dy square plus dz square. So, here when we take x as a function of t, y as a function of t, z as a function of t, then we can have ds over dt whole square is dx over dt whole square dy by dt whole square plus dz by whole square and so ds is can be saying that this is equal to a square root of this quantity.

So, there are other kinds of line integrals also can be considered like when we take x itself as parameter or y itself as parameter or z itself as the parameter then we take here this integral f(x,y,z) dx, similarly f(x,y,z) dy, f(x,y,z) dz, and these can be seen that can be written like this f(x,y,z) dx by dt times dt here, and so this is like some other function here which can be considered as g(x,y,z) dt. So, this is a particular case of what we have already considered. So, all these line integrals are particular cases of the one which we have already considered. More general form of the line integral is considered in this way that we take this sum of all these. Here F is replaced by f 1, f 2, f 3 respectively and then we sum these terms up and it can be seen that this is actually the vector form of the line integral given by integral over c F r dr here f is vector valued function given by this which has components f 1, f 2, f 3; i, j, ks are the unit vectors along the coordinate axis and a radius the position vector r is given by which is actually this directed vector op and so, this position vector of p is given by x i y j plus z k where x, y, z are the coordinates of the point p.

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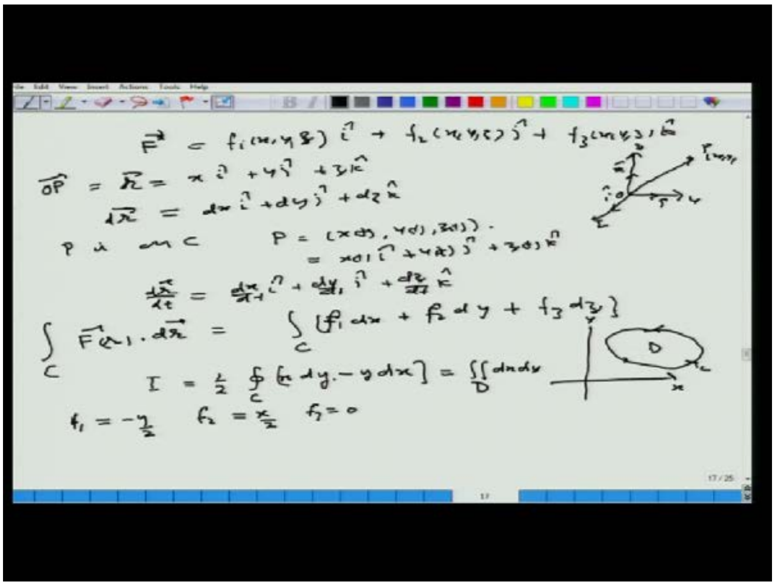

So, this dr the element length - vector element can be given by dx i dy j plus dz k, and so F r dot dr can be seen that this is actually the same thing here f 1 dx plus f 2 dy plus f 3 dz which is the same thing which we had here on the left side left hand side of this equation. In particular, if we take here this line integral i where we take f 1 equal to minus y by 2, and f 2 equal to x by 2, and f 3 equal to 0. You see that this line integral triple s star reduces to this I equal to half of integral over here c is a closed curve, and we are taking the direction that the positive direction as anticlockwise direction here in this way, and so the integral when we have a close curve we write to distinguish it from the other integrals, we write o here in this to denote that this integration is over a closed curve.

And so, we have in this particular case f 1 equal to minus y by 2. So, this is written here 1 by 2 minus y dx and f 2 is x by 2, so 1 by 2 x dy is written here and f 3 is 0. And we will see that this is actually gives us a convenient way of finding areas of various domains in two-dimensional plane which is actually given by dx dy integration over D which is the area of D, here this can be seen from the Green's theorem which is established here.

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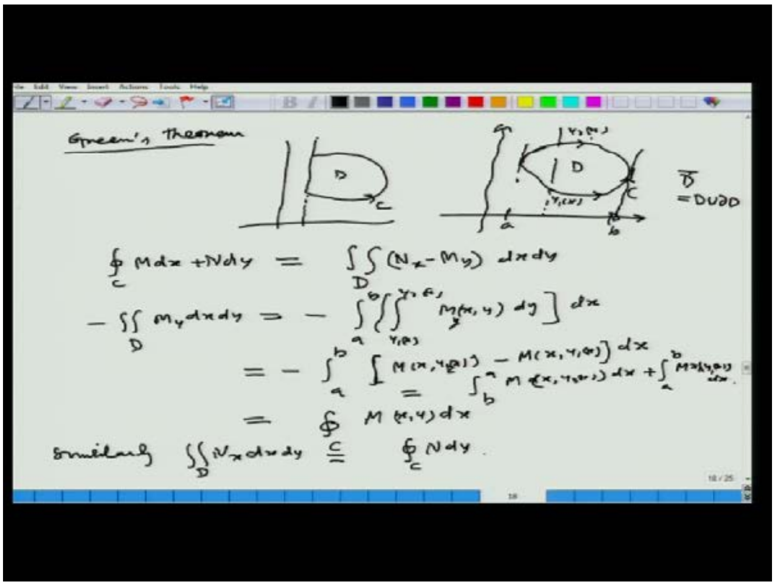

Here, Green's theorem a first we established in a very specified the special region and then extend it to other complicated regions. Here, we assumed that any vertical line or horizontal line, the lines parallel to the coordinate axis across the boundary of the area at most at two points. So like here, this line cuts this area only at this point, similarly here from D the vertical line cuts this c boundary of this area at one point and in any interior here, it cuts at these two points this boundary c. So, there are at most two points common with a vertical line or horizontal line with the boundary of this D that is the c curve c.

So, for this particular case, we have seen that it can be the Green's can be stated here and proved like this M dx plus N dy integration over the close curve c is actually N x minus M y dx dy. Here it is… So, if you take this term M y dx dy integration over D can be seen by iterated integrals like this and so this differentiation and integration cancel each other and so we get the values of M at the boundary curves like this, and adjusting this minus sign with the direction of the integration. So, this minus a to b gives you b to a of

M dx; similarly, the second term gives you minus minus becomes plus here so it gives you a to b M dx, and collectively if we see the directions then we get the integration over the whole curve c of M dx.

Similarly, we can see this second term N x dx dy will come out to be N dy integration over the close curve c. So that establishes the Green's theorem for this particular region and then we extend it to more complicated division like if you have the whole line segment common with vertical line or horizontal line than it can be extended, because here dx becomes zero. So, it does not matter, here this contribution along this curve integration will be zero. So, it actually reduces to the earlier case in these two particular cases.

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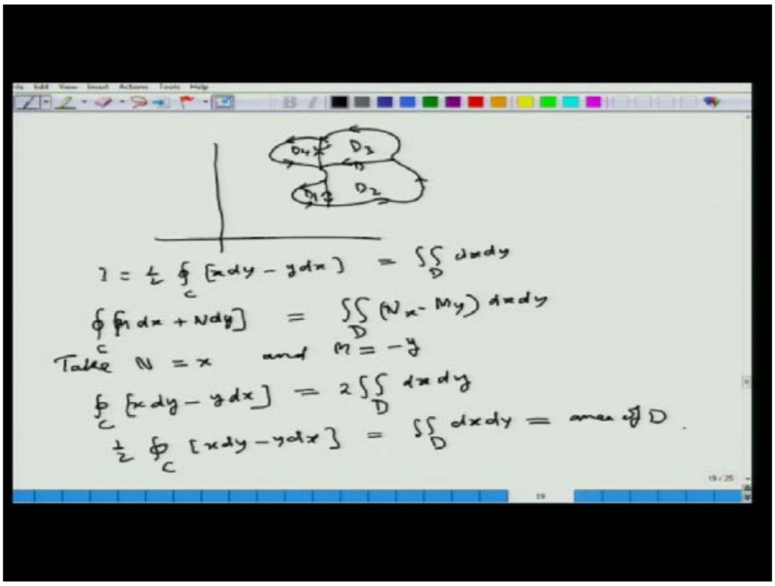

And then you extend it for more complicated region here like this and subdivide this region into simple regions like the one we have earlier and then add up. And then the artificial boundaries introduced here, the integration over the those boundaries will be cancelling when you take like, here you go like this anticlockwise, when you go in D 3 anticlockwise like this, then here integration once you get in this direction and then next time the direction is reversed, so they cancel each other. Similarly, along any interior boundary which we introduced artificially integration, so cancel and so we get the Green's theorem extended to more complicated regions.

So, now to see that our result that it is… This is gives you the area over this D dx dy. Here we take N x, so recall that what we have is from Green's theorem; here we take N x equal to… So, if we write this integral over close curve c M dx plus N dy equal to double integral over D this N x minus M y dx d y. So, here we can take N x equal to x and so take N x equal to x and M sorry N equal to n equal to x and M equal to minus y. So, then we see that the right hand side or left hand side reduces to x dy, so N is x, so x dy minus y dx rather we take… So, this will be equal to double integral over D and so you get N x becomes 1 here so you get twice dx dy. So, taking this 2 here we can see that half of this x dy minus y dx is actually integral over dx dy which is which gives us the area of D. So that formula will be very convenient here for the problem of optimization for the problem of the calculus of variations where we have a double integral, we can reduce by this to integration over a curve which is actually the boundary of that region.

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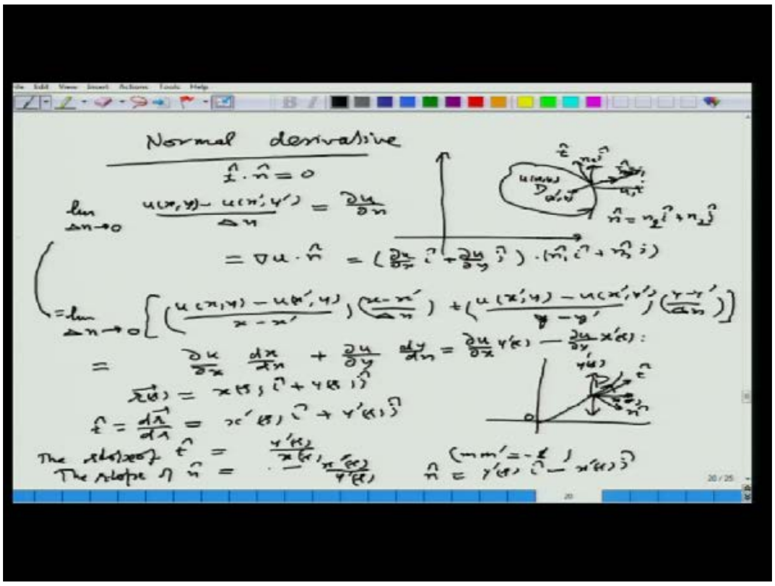

So, next we will be considering the concept of directional derivative. So, here in let us say first we do it for two dimension. So, here we have certain region D here and your function u is defined. So, u(x,y) is defined here and we want to see that. So, normal here will be like this, outward normal n cap. So, n cap will be having like this, n cap equal to n 1 i plus n 2 j. So, this n has 2 components, n 1 in this direction n 1 i and this n 2 j. So, these are the components of this n like this unit normal. So, in the x, y direction, we have these components n 1 and n 2 of this normal. So, here this actually like if we take this direction as the positive direction it counterclockwise, we know that then the tangent will be like this at this point. So, tangent can also be here, can be seen it is a components in x, y directions like this, like this and so this if we denote t cap as the unit tangent, then we know that t cap dot n cap is 0, because they are orthogonal to each other.

So, we want to find at the derivative of this u(x,y) in this direction n which is known as the derivative of u in the normal direction. So that is what we are going to consider the it is called normal derivative rather than directional derivative in the normal direction, so will write it rather normal derivative instead of. We can consider differentiation in any directions. So, in particular, we will take normal derivative, the derivative in the direction of normal - outward normal here so similar. Here you can have the inward normal like this. So, if n is the outward normal, minus n is the inward normal. So, we take only the outward normal here. So, then we this u need not be defined outside. So, how we defined this along this, actually we take a point (x dash,y dash) inside and then consider this limit - limit delta n, delta n is the distance along this normal, limit delta n tending to 0 of u(x,y) minus u(x dash,y dash) upon delta n and it is denoted as delta u by delta n. So, this is the definition of the normal derivative.

Here it is at that point (x,y) actually, so (x dash,y dash) is going to… So, (x dash,y dash) is a point here on the normal inside in the region, because outside we cannot take point, since u is not defined there. Usually you will take when you are taking along this derivative along this will take a point here and then pass it to the point (x,y). Since we cannot take points outside, we will take points inside because u is defined only on D. So, we have to take points only inside D. So, here we take (x dash,y dash) on this normal, extend it inside and we consider this difference. So, that is why we have taken (x,y) u at (x,y) minus u(x dash,y dash) instead of u at (x dash,y dash) minus u(x,y).

So, if you take u(x dash,y dash) minus u(x,y) over delta n and delta n tends to 0 that will give you the derivative along this direction. So, minus of that will give it derivative along the outward normal. So that is what we have done here. Now, this can be seen this will be actually equal to gradient of u dot n cap where n cap is this outward unit normal, here gradient is actually del u by del x i plus del u by del y j, and you have this normal and 1 cap i plus n 2 cap j.

So, this gives a convenient way of finding this normal derivative instead of using the definition, we just take the gradient of u and then take the dot product with the outward normal, the unit normal and so that is what gives you the normal derivative along the outward normal. How do we see this? Because you can do it like this that limit delta, so this thing can be seen that this is equal to limit delta n tending to 0 u(x,y) minus u(x dash,y) over x minus x dash, and then x minus x dash over delta n plus u(x dash,y) minus u(x dash,y dash) over x over y minus y dash to y minus y dash over delta n.

Now here, when we take this del u or del x that is actually x dash y minus u(x,y) and x dash minus x. So, you can multiplied by minus up and down. So, it is just the same thing. So and here this will give you. So, this will be simply when it passed to the limit you get del u or del x at (x,y) and you get this. This is the normal derivative, so this will be like dx by dn and plus del u by del y, so this in the limit you get del u del y and then dy by dn.

Now, here how to see that these are actually n 1, n 2, here we see that the tangent at any point, here if you have the curve going like this is the positive direction. So, at this point tangent t cap is this. So, if you take s as the arc length then this t cap… So here, if you take first this position vector r, function of s. So, you have x (s) i plus y (s) j. So, this is the position vector like this, and when the p moves is here it is actually a function of s and so this op can be seen like this that is the position vector here. So, these components x and y are functions of the arc length and so we know that this t cap that is dr by ds the unit tangent vector given like this, x dash (s) i plus y dash (s) j and so the this is… So, if you take… So, this is x (s) and this is y (s) component here x dashes and y dashes here.

So, these are the comments of the tangent vector. And so here we know that t dot n cap is… If the slope of t… So, slope of t is y dash as upon x dash as and so slope of normal, the slope of n cap will be like plus minus x dash as over rather minus of this, minus x, just reverse of this with a negative sign y dash, because you know that m, m dash equal to minus 1 for orthogonal lines, if m is a slope of line and orthogonal to this slope has m dash then we know that mm dash equal to minus 1. So, using that we can see that the slope of this n cap is minus x dash y dash so minus x dash over y dash. So, here these components will change with the one of them becoming minus. So, which one to take minus will be seen that see this normal is along this direction now, and so is x component is in the same direction which is actually positive, here both x and x dash and y dash both are along positive directions.

So, normal only y component will change its sign and for this normal n cap, here we can see that this for the tangent it is up, so for normal it is going down. Whereas, x component remains in the same direction. So, we can see that n cap than can be written as here x component becomes y component of t, so y dash (s) I, and then this y component goes in the other direction, so this becomes minus x dash (s) j. So, that is what is the outward normal, and so we can see that this actually comes out to be… So, here we can see that this will be now del u by del x. So, x component here is y dash. So, at y dash (s) minus del u del y and y component is minus x because of that we get minus here x dash (s). So, this gives you a convenient way of finding the directional derivative.

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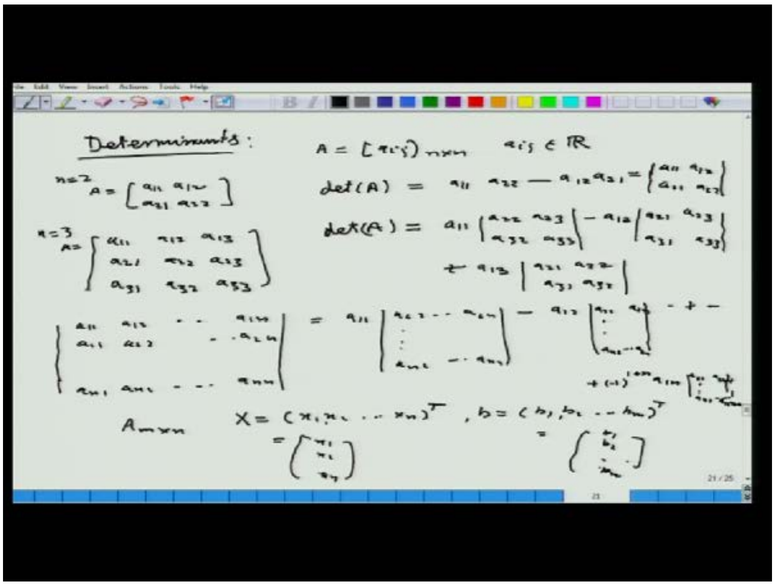

Next we consider the concept regarding the matrices and determinants. So, here… (No audio from 30:49 to 30:57) And in terms of determinants certain jacobians will also be considered. These jacobians will be involving certain transformations of variables and so they will be the determinant determinants of certain certain matrices which get we get from considering various transformations from one place to another. So, here we know that this determinant is defined like this. Supposing you have A matrix of n by n size like this, a i j n by n, these a i js are real numbers. We are considering only real matrices here, and here for n equal to 2 for n equal to 2 we have A as a 11 a 12 a 21 a 22 and so determinant we denote here, it does not mean it is a absolute value actually or let us use the other notation might confuse with the absolute value.

So, let us use d e t - det here determinant of A is given as a 11 a 22 minus a 12 a 21. So, it is actually cross multiplied like this a 11 into a 22 minus a 12 into a 21 like this. So, for n equal to 3 we extend it like this, if you have a like this a 11 a 12 a 13 a 21 a 22 a 23 a 31 a 32 a 33, so so determinant of A will be given like this a 11 times here determinant of the lower order 1. So, here this also another notation here like this a 11 a 12 a 21 a 22 like this to between two parallel lines like this. When we take this brackets than in matrix, when we take these bars between this then these notes are detriment, so here you get a 22 a 23 and a 32 a 33, and then here minus. So, the indices are 11. So, i plus j is even, when i plus j is odd we get minus sign, so minus a 12 and then crossing out this row and this column. So, we are left with those entries put those entries here a 21 a 23 and a 31 a 33 then plus because now 3 plus 1 is 4. So, we get plus sign here a 13 and then leaving crossing out this row and this column. So, we have a 21 a 22 a 31 a 32. So, like this we extend. So, any n th order determinant will be like this.

So, if you have general a 11 a 12 and so on a 1n a 21 a 22 a 2n and then a n1 a n2 so on a nn, so like this we will have. Here if you… See we can open along any row and any column using this rule that you take plus sign when i plus j is 1, i plus j is even and minus sign when i plus j is odd and then crossing out that. So, a i j if the your are all have that entry, than it remove i th row and j th column, remaining one the lower order determinant you take. So that is what we can do here a 11 and then this lower one a 22 a so on a 2n and a 12 so on here a nn, and then you get minus sign a 12 like that; removing this and this, removing the first row and second column like that the remaining ones will put here like a this is 21, so a 21 a 2n so on and a n1 then leaving this one and you get a nn so on we get and then plus minus plus minus so on. And then here you will get minus 1 to the power 1 plus n a 1n and then the determinant of these entries put here, a 21 and then a 2n minus 1, and then so on a n1 and then up to a nn minus 1. So, that is how you define the determinants of various square matrices.

Now, these are used in order to solve the linear equations like this if a if a is m by n matrix and then you take x as the n-dimensional vector x 1, x 2, x n transpose taking as row vector and then b as b 1, b 2, b m transpose, transfer means you take the row vector as column vector. So, you x transpose will then be written in column x 1 x 2 x n. This will be actually written like this x 1 x 2 x n. Same way this transpose will be written as b 1 b 2 b m.

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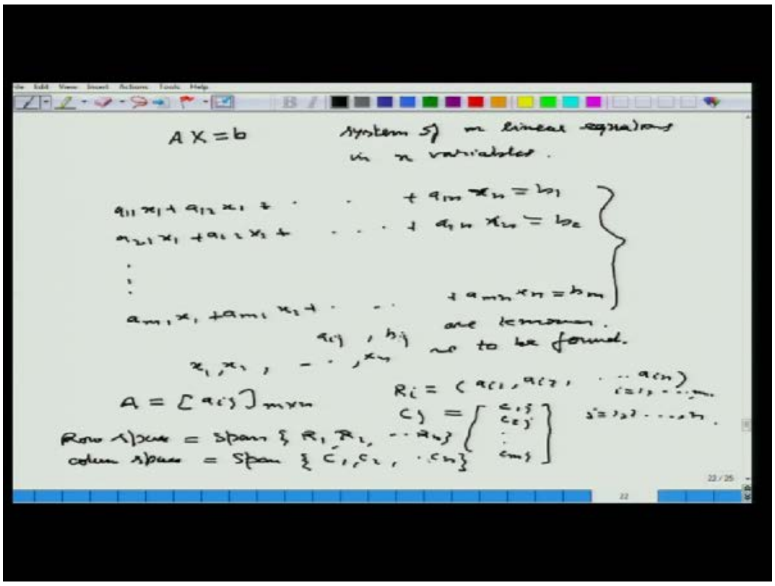

Then, one can consider this system A X equal to b. So, given b you find X such that this system of linear equations. This is system of m linear equations in n variables. So, in expanded form it is like this a 11 x 1 plus a 12 x 2 so on plus a 1n x n equal to b 1, and then a 21 x 1 plus a 22 x 2 plus a 2n x n equal to b 2, and then this is m th a m1 x 1 plus a m2 x 2 so on plus a mn x n equal to b m. So, this is system of m linear equations in n unknowns x 1 x 2 x n where the right hand side b 1 b 2 b m is given vector. So, a i js are known a i js are known a i j and b j are known. We are to find this x 1, x 2 and x n are to be found.

Now, here we know that if we consider this matrix A this is actually a i j m by n. So, we have concept of rank of the matrix that means these rows so R j R i row is a i1, a i2, a in, this is i th row, and similarly you have C j column that is c 1j c 2j and c mj. So i equal to 1, 2 to m and j equal to 1, 2 to n. So, these are m rows and n columns here, this row r i this m rows span the row space. So, this span row space is called the span of R 1 R 2 R m and column space span of C 1 C 2 C n.

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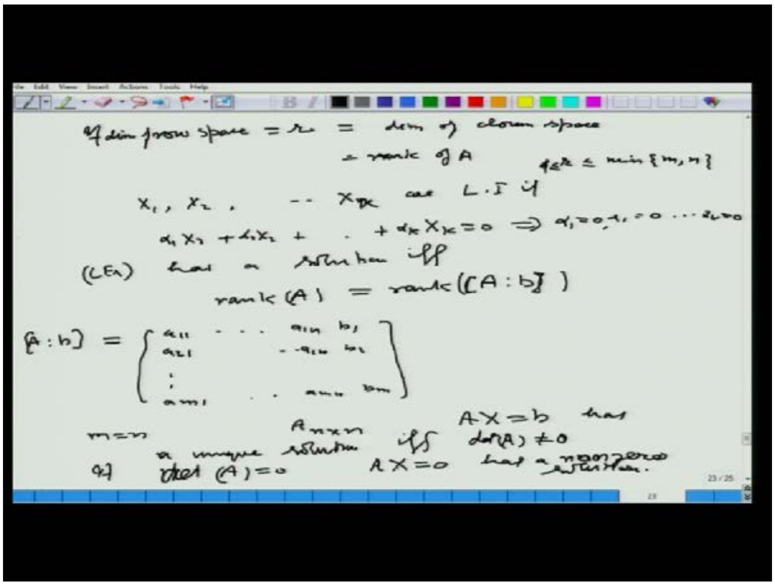

Now, you see that here these two spaces are different spaces, but they have the same dimension that the row space, if there are R linearly independent vectors then if row space if the dimension of row space is R then we have R 1 certain R number of row vectors as linearly independent vectors and we see that these out of these C 1 C 2 C n there will be then only R number of… So, this is also equal to the dimension of column space. So, these two spaces which are actually generated by taking the linear combinations of rows rows of this matrix and same time taking the column space that is a space generated by the columns C 1 C 2 C n, it happens that it has the these two spaces have the same dimension R and it is that is what is called the rank of rank of A.

So, rank is uniquely defined, it is the number of linearly independent row vectors which is also same, this number is same as the number of column in linearly independent columns; independence means here see these vectors x 1, x 2, x n or let say x k are linearly independent if this alpha 1 x 1 plus alpha 2 x 2, this is the linear combination, alpha k x k 0 implies alpha 1 equal to 0, alpha 2 equal to 0 and so on alpha k equal to 0. If you can find alpha 1, alpha 2, alpha k such that at least one of them is non-zero and this linear combination is 0, then we say that x 1, x 2, x k are linearly dependent. So, either the set is linearly dependent or linearly independent.

So, when you cannot find non-zero alphas such that this linear combination is equal to zero. That means if you consider any combination like this then it implies that these alphas are 0, then we say that these x 1, x 2, x k are linearly independent. And so here by this, this is the dimension of row spaces R, so the we will have R linearly independent rows of the matrix A, similarly will have R linearly independent columns out of those and columns, here out of those m rows, certainly R has to be less than equal to… If A matrix A is non-zero, certainly this is to be greater than equal to 1 and less than equal to minimum of mn, because there are m rows so dimension cannot be greater than m and there are n rows of dimension cannot be greater than n. So, it will be minimum it will be less than the minimum of those mn.

And if your matrix A is non-zero then at least there is one non-zero row and therefore, the dimension is at least one. So, by this we see that this system system of linear equation, so we will say this as LE - linear equations, LE is at other. So, this LEs has a solution if and only if the rank of rank of A is same thing as rank of the augmented matrix A: b. So, here this is the augmented matrix, we put it bracket for the augmented matrix, so that means here you have augmented matrix means you have A here that a 11 so on a 1n and then put this b 1 here and a 21 so on a 2n and b 2 a n m1 a mn and b n. So, that is the augmented matrix. So, since again… Here it cannot be the rank of the augmented matrix cannot be larger than m, but it can if r is smaller than m then it might be that the rank might increase. So, if the rank increases obviously, then will not have any solution because it this is if and only if. So, if the rank of A is same thing as rank of the augmented matrix. That means, if you adjoin this column, it rank does not increase then here it will be having a solution.

And here when we have m equal to n then we have a square matrix size of n by n, and then in that case we can have this, than A X equal to b then b also the size of b is also n, and in this case this has a unique solution if and only if the determinant of this. Now, we can consider determinant versus square matrix and therefore, if the determinant of the this is non-zero then rather we are using non determinant. Determinant of A is not 0 then this is as unique solution. And if determinant if determinant of A is 0 then this homogenous equation A X equal 0 has a nontrivial solution has a non-zero solution. So, that is the (()) here. So, these two are associated AX equal to b and b is not 0 when we take b equal to 0, this is the associated homogeneous system for this non homogenous system.

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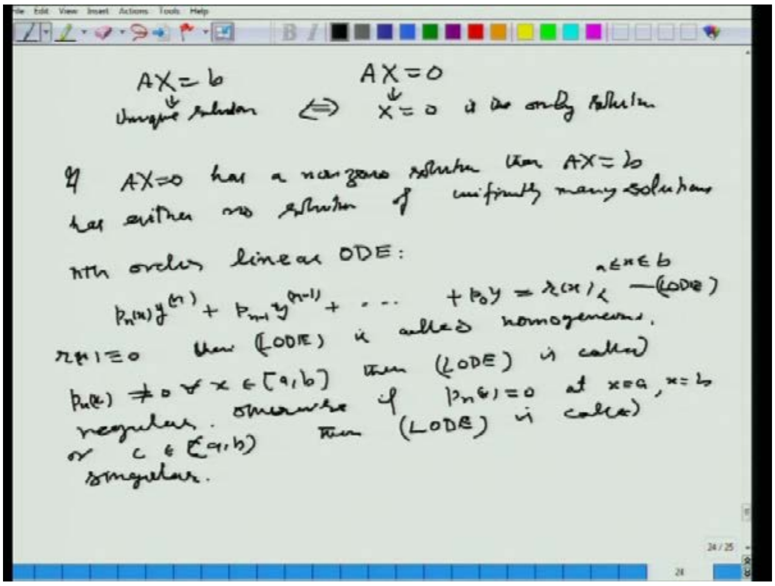

And so if this has only 0 solution than this AX equal to b, we can summarize like this that we have AX equal to b and then the associated homogeneous system is AX equal to this 0. So, if unique solution solution if and only if this as this as unique solution if and if this homogenous system has x equal to 0, x equal to 0 is anyway solution is the only solution, and if this if a if this has a non trivial solution then if x 0 has a non-zero solution, then AX equal to b has either no solution or infinitely many solution.

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So that is the case here, in this case, when we have AX equal to 0 if it has a only trivial solution then the non homogenous has a unique solution, and when this has a nontrivial solution then it can this non homogenous one cannot have a solution in general or if it has a solution it will have infinite remaining solutions.

Now, here we consider certain transformations supposing we consider the n th ordered partial n th order ordinary differential equation ODE linear. So, you have we will consider the highest derivative coefficient of highest order derivative as 1, if it is something here that say p n x plus p n minus 1 y n minus 1 and so on plus p 0 y equal to r x. So, this is a linear non homogeneous equation. If r x equal to 0 then we have homogenous equation, if r x is identically 0 we get this this ODE - linear ODE like this then is called homogeneous.

Now, if this p n x, so this to be considered on certain interval like this a less than equal to x less then b, now if p x is not equal to 0 for all x in (a,b), then LODE - linear nth order ODE given here is called regular. If otherwise, if p n x is 0 at x equal to a, x equal to b or for any c in open interval (a,b) then LODE is called singular like you have Lysander equation or Bessel equation. So, we will consider these kind of questions and their solutions next time, and so, thank you very much for viewing this lecture attentively.