**Calculus of Variations and Integral Equation**

**Prof. Dhirendra Bahuguna**

**Prof. Malay Banerjee**

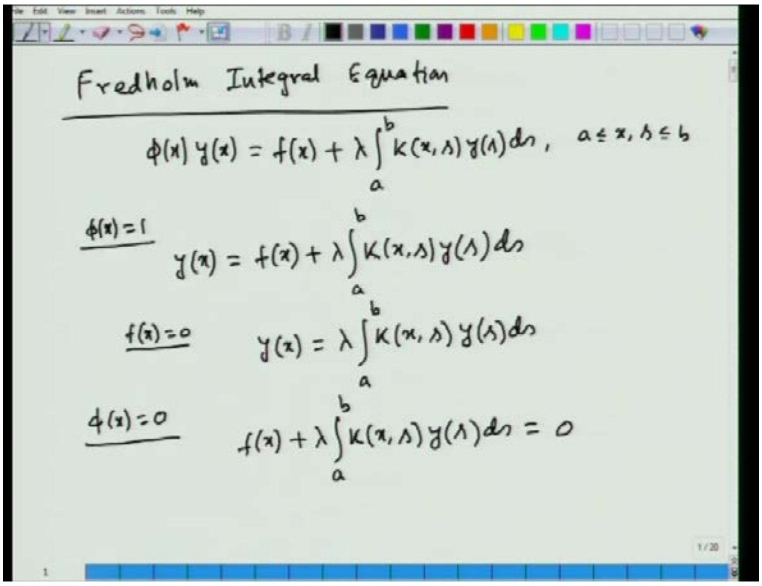
**Department of Mathematics and Statistics**

**Indian Institute of Technology, Kanpur**

**Lecture. #29**

Welcome viewers to the lecture series on integral equation under NPTEL course. From this lecture and onwards, we are going to discuss on Fredholm integral equation

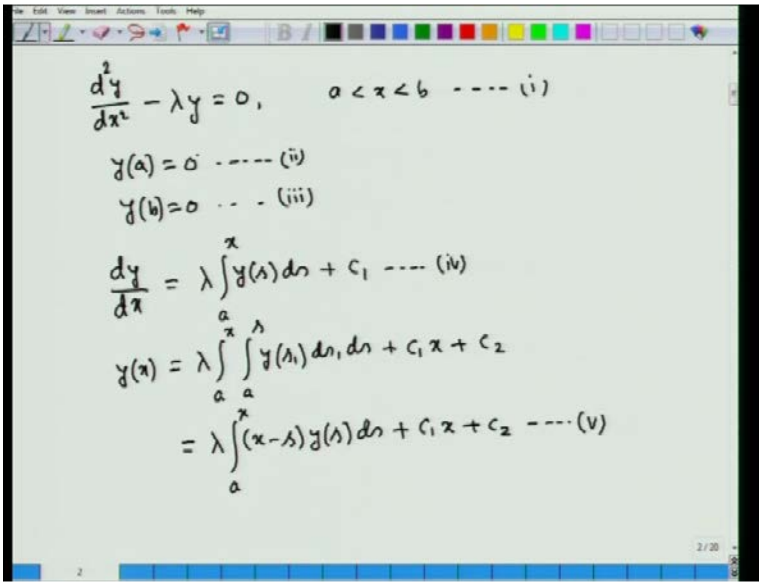
(Refer Slide Time: 00:37)




You can recall the general forum of Fredholm integral equation is given by phi (x) y(x) equal to f(x) plus lambda integral a to b k of x,s y(s) ds, where a less than equal to x,s less than equal to b; and this particular phi (x) actually responsible to determine the kind of the integral equation. If we consider phi (x) equal to 1, then we will be having y(x) equal to f(x) plus lambda times integral a to b k of x,s y(s) ds. This equation is actually Fredholm integral equation of second kind, and also it is a non-homogenous Fredholm integral equation. And in particular, if we consider here f (x) equal to 0, then you can find this equation f(x) equal to lambda times integral a to b k of x,s y(s) ds. This is actually Fredholm integral equation of second time, and it is homogenous Fredholm integral equation.

And if we choose phi (x), this is equal to 0; then f(x) plus lambda times integral a to b k (x,s) y(s) ds, this is equal to 0, this last equation is actually Fredholm integral equation of first kind. Now, in case of Volterra integral equation, the Volterra integral equation can be generated from linear ordinary differential equation, which was of the form of initial value problem. And this Fredholm integral equation are results of converting the linear ordinary differential equations of boundary value problem type problems to an integral equation leads us to Fredholm integral equation. And as we are going first work on linear ordinary differential equation and boundary value problem, so will be landed at Fredholm integral equations, those are linear Fredholm integral equation.

(Refer Slide time: 03:52)

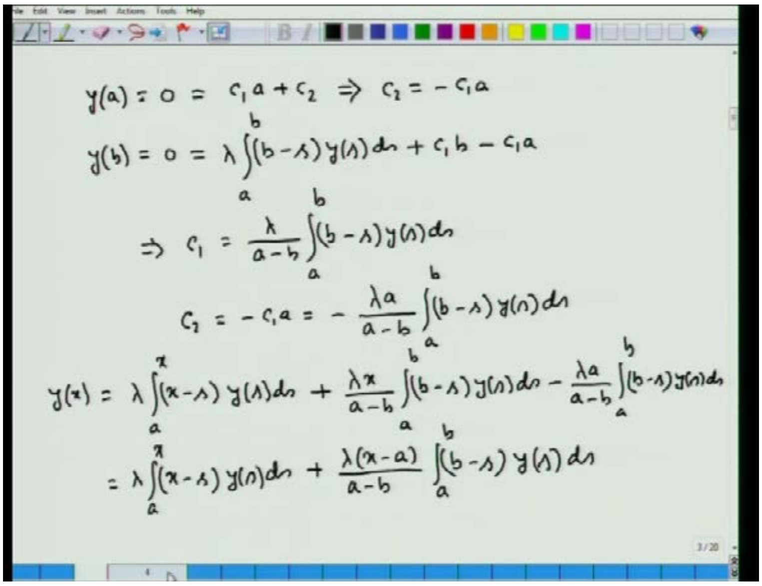



So first of all, we consider one formulation that is construction of Fredholm integral equation from a given boundary value problem. Consider this bounded value problem d 2 y d x 2 minus lambda y this is equal to 0, where less than a less than x less than b and given initial conditions are y(a) equal to 0 and y(b), this is equal to 0. Consider this as the first equations; y(a) equal to 0 first boundary condition and y(b) equal to 0 second boundary condition. Actually this type of boundary condition, when specified at the two ends that is at x equal to a and x equal to b, these are actually called separated boundary conditions. And we will be considering the Greens functions approach, then we define the general formulation of second order boundary value problem with separated boundary conditions.

Now, from this given equation, if we integrate, then we can find d y d x, this is equal to lambda times integral a to x y(s) ds plus c 1. On the right hand side, you can see I am integrating from a to x, and since x is at the upper limit, so variable considered here as s. And I am writing here c 1, because for the given problem, there is no information of d y d x at x equal to a. So, we will determine this c 1 at a later stage. So, call this equation as 4. If we integrate again, then we can write y(x), this is equal to lambda. Now this a to x y (s) ds, this we have to integrate within the limit a to x. So, on the upper limit x, it will be replaced by s, and variable s under integral sign will be change to s 1. So ultimately, we will be having integral a to x, then integral a to s y(s 1) ds 1 ds plus c 1 x plus c 2.

Now at this point, you can note that c 2 I am writing here, this c 2 takes care of minus c 1 a, and all though the value of y at a equal to 0 is given, still I am writing c 2 here, because using these two boundary conditions, we will be able to evaluate this c 1 and c 2. And now this repeated integral can be converted into a single integral, you can recall the using formula for generalized replacement lemma. So, using the generalized replacement lemma or you can interchange the order of integration, you will be getting this result integral a to x x minus s y(s) ds plus c 1 x plus c 2 call it 5. So, this last expression, we are getting from the previous one by applying generalized replacement lemma.

(Refer Slide Time: 07:46)



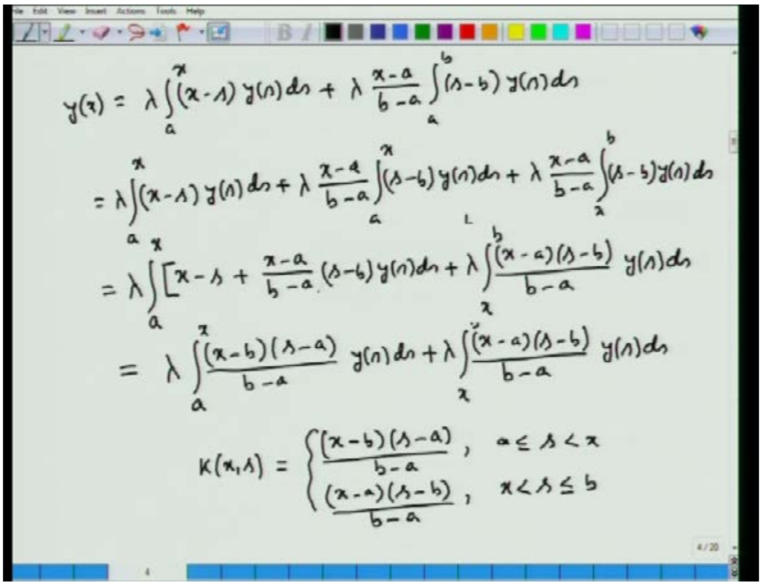
Now, if we use the condition 2 - y(a) equal to 0 in 5, then we can find that y(a), this is equal to 0, equal to contribution from this integral will be exactly equal to 0, because we are integrating from the range a to a, because x will be replace by a. So, a to a a minus s y(s) ds this is equal to 0. So, we are only left with c 1 a plus c 2 and this implies c 2, this is equal to minus c 1 a. So that means, the same thing you can obtain from here by writing y(a) equal to c 1 x minus a; and at y(a) equal to 0, so then c 1 x minus a if it appears here, then c 2 will not comes into the picture.

And using the second condition that is y(b), this is again equal to 0. We can find this is equal to lambda times integral a to b b minus s y (s) ds plus c 1 b; and then for c 2, we can write minus c 1 a. And therefore, from this expression we can find that c 1 equal to lambda divided by a minus b integral a to b b minus s y(s) ds; and then c 2 equal to minus c 1 a, so that is equal to minus lambda by a divided by a minus b times integral a to b b minus s y(s) ds. So, these are two expressions for c 1 and c 2.

Now you can see that for the given problem c 1 and c 2 are all evaluated in terms of lambda a, b, and this integral involving the unknown quantity y(s). If we substitute this expression into expression for 5, then we can find y(x) this is equal to lambda times integral a to x x minus s y(s) ds, this is the first part; then for c 1 x, we will be having lambda x divided by a minus b integral a to b b minus s y(s) ds and then minus lambda a divided by a minus b integral a to b b minus s y(s) ds. So, just look at this expression, y x equal to lambda times integral a to x x minus s y(s) ds plus c 1 x plus c 2. We have obtained c 1 and c 2, so substituting here, we are getting this egression y(x) equal to lambda times integral a to x x minus s y(s) d s plus this expression.

Now, we have to make some rearrangement. So, this is equal to lambda times integral a to x x minus s y(s) ds. And in last two integrals, the integrant and limit of integrals are all same. So, we can combine them to write lambda times x minus a divided by a minus b integral a to b b minus s y(s) ds. Now for simplicity, we can observe the minus sign involved with the second integral in the denominator that is we can change a minus b to b minus a, and then it will be s minus b.

(Refer Slide Time: 12:21)

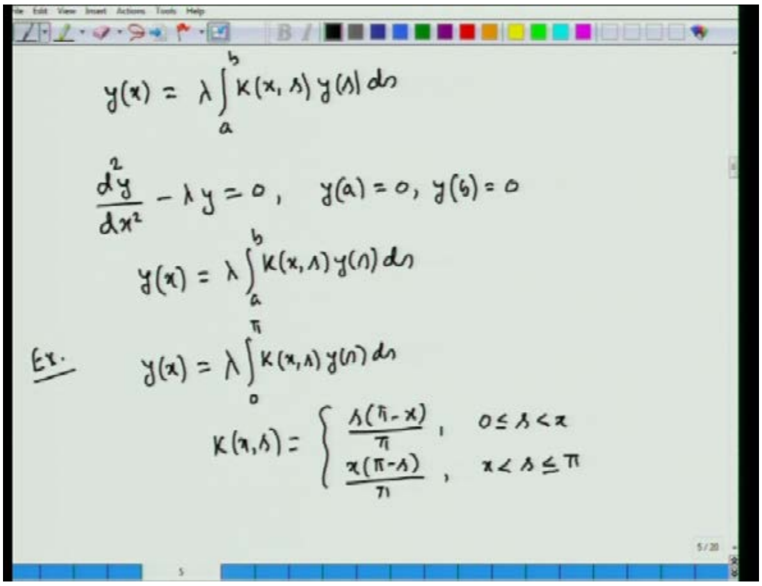


So, making this rearrangement, we can write that y (x), this is equal to lambda times integral a to x x minus s y(s) ds plus lambda x minus a divided by b minus a integral a to b s minus b y(s) ds. Now our target is to convert this problem into a Fredholm integral equation. Now in the first integral, you can see we have the limit from a to x, but in the second integral we have to, the limit a to b. Our target will be to convert this entire expression to an integral involving limit from a to b. And we, now we have to do some manipulation; this range of integration for the second integral can be divided into two parts, we can introduce one intermediate point x here, so integral from a to b will be converted into a to x plus x to b.

So, introducing that particular sub division of the range of integration, you can find lambda integral a to x x minus s y(s) ds plus lambda times x minus a by b minus a integral a to x s minus b y(s) ds plus lambda x minus a by b minus a integral x to b s minus b y(s) ds, this expression. Now, combining these two integrals, we can write this is lambda times integral a to x, then x minus s plus x minus a divided by b minus a multiplied with s minus b y(s) ds plus lambda times integral x to b without any loss of generality, we can take that term x minus a by b minus a under the integral sign, we can write this is x minus a times s minus b divided by b minus a y(s) ds; and after some simple algebra, you can able to derive that this will be equal to lambda times integral a to x x minus b times s minus a whole divided by b minus a y(s) ds plus lambda times integral x to b x minus a, then multiplied by s minus b whole divided by b minus a y(s) ds.

So, now, you can see we can write this integral as lambda integral a to b some kernel k (x,s) y (s) ds whenever we will be able to define y(x) into two parts that is one part of the y k sorry k(x,s) is x minus b times x minus a by b minus a, when s is ranging between a to x; and then this kernel is equal to x minus a times s minus b divided by b minus a, whenever s is grater then x. So that means, defining this kernel k(x,s), this is equal to it is define by x minus b times s minus a divided by b minus a, this definition is valid for a less than equal to s less than x; and this is equal to x minus a times s minus b divided by b minus a this one, where x less then s less then equal to b.

(Refer Slide Time: 17:18)



So, if we define the kernel in this particular format, then we can write this expression y (x) equal to some of these two integrals as y(x) is equal to lambda times integral a to b k of x,s y(s) ds. Now, at this stage, if we try to summaries what we have done just now. We have started with the differential equation d 2 y d x 2 minus lambda y equal to 0, subjected to the boundary conditions y a equal to 0, and y be equal to 0. We have converted this boundary equal problem of ordinary differential equation into an integral equation y (x) equal to lambda integral a to b k(x,s) y(s) ds, where this k(x s) is given by this expression that means, this is the kernel.

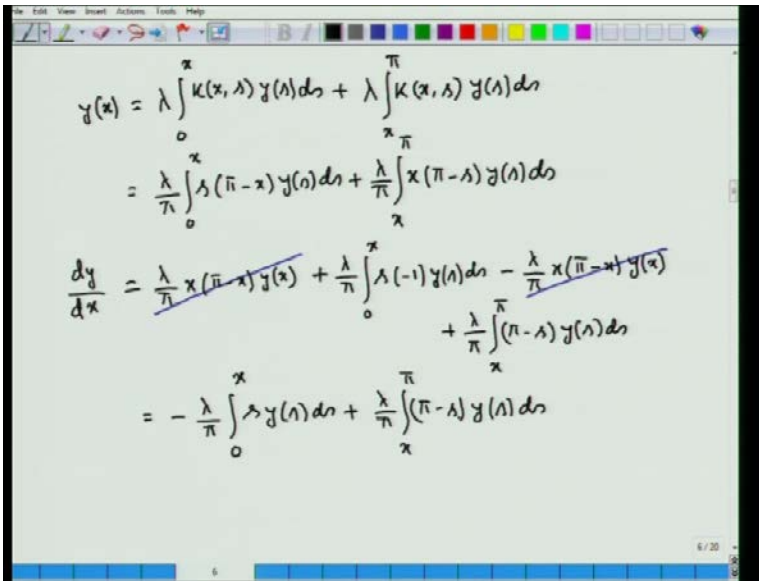
So, the point is that once we start form a linear homogenous ordinary differential equation, which is a boundary equal problem with separated boundary condition. We arrived at a Fredholm integral equation, this is of second kind, and this equation that integral equation is also a homogenous equation. So this is an important point, you can take note of this that linear homogenous ordinary differential equations with boundary condition that is separated boundary conditions can be converted into a Fredholm integral equation of second kind, which is again a homogenous linear Fredholm integral equation.

Now, the point is that if we are able to find out a solution of the given differentials equation, that particular solution of the differential equation will satisfy this integral equation and conversely. So sometimes, we can use this integral equation, and try to solve this integral equation in order to find out solution of this differential equation and vice versa, depending upon the availability of mathematical tools as well as applicability of the format involve with the problem that is the nature of kernel and other conditions given for the particular problem.

Now at this point, we can try to look at this particular situation that suppose this integral equation is given, so whether will it be possible to get back the original differential equation or differential equation corresponding to the given integral equation. And now I am going to show you and example where a given integral equation can be converted into an ordinary differential equation, and we will be able to solve that ordinary differential equation. And after looking at the solution of the ordinary differential equation, we can try to find out some implication, and also we will be able to make some comments regarding the solution of Fredholm integral equation.

So, for this purpose we consider the example that y (x) is equal to lambda times integral 0 to pi k of x,s y(s) ds, where the kernel k(x,s) this is given by s times pi minus x whole divided by pi, and x times pi minus s whole divided by pi. First definition is for 0 less than equal to s less than x; and second one for x less than s less than equal to pi. So, this is our integral equation. Now, first of all, we will be substituting this expression for the kernel into the given integral equation. And after substituting, you can find that the limit from 0 to pi have to be divided into two parts; one integrals of with limit 0 to x, other integral with limit x to pi, and then we have to apply the Leibniz formula in order to differentiate y(x) and its derivative.

(Refer Slide Time: 22:13)

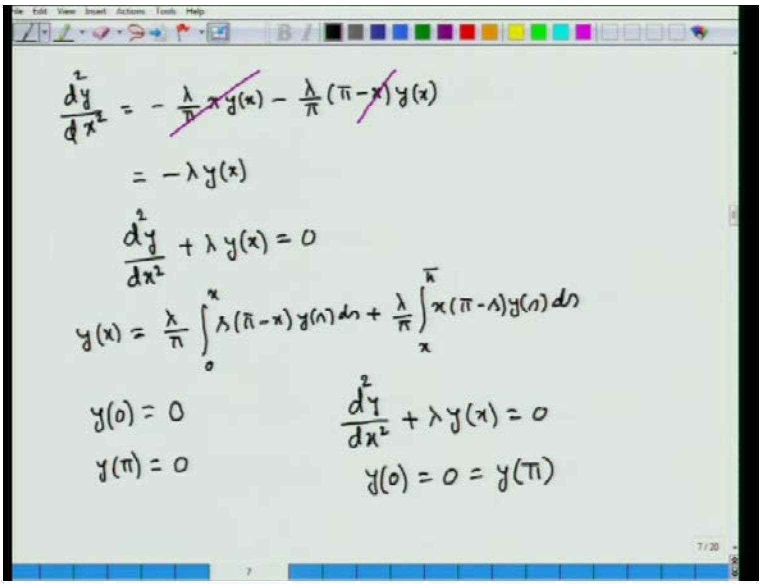



So, we can write this y (x) is equal to lambda times integral 0 to x k of x,s y(s) ds plus lambda times integral x to pi k of x,s y(s) ds. So, using the definition for kernel, this is will be equal to lambda by pi integral 0 to x s into pi minus x y(s) ds plus lambda divided by pi integral x to pi x times pi minus s y(s) ds. So, here we are just substituting the expressions for k (x,s), this is the definition that is s by pi into pi minus x for s less than x and x by pi times pi minus s this is for s greater than x.

Now, we apply the Leibniz formula to get dy dx, this is equal to lambda divided by pi times x into pi minus x y(x). So, here we are just substituting s equal to x into the integrant, and derivative of x with respect to x is 1. There will be no contribution from the lower limit, because it is 0. And then differentiating the integrant with respect to x, we can find plus lambda divided by pi integral 0 to x s into minus 1 y(s) ds, this is the expression. Then substituting the lower limit into the integrand of the second integral, we can find minus lambda divided by pi x into pi minus x y(x); and then differentiating the integrand under integral sign with respect to x partially we can find x to pi times pi minus s y(s) ds.

And now you can see these two terms cancels with each other that is lambda by pi x times pi minus x y(x) and minus lambda by pi, this one, this cancels with each other. So, this is equal to minus lambda divided by pi integral 0 to x s y(s) ds, this one, plus lambda divided by pi integral x to pi pi minus s y(s) d s. On this result, we are again going to differentiate with respect to x; and on the right hand side, we have to use the Leibniz formula once again.

(Refer Slide Time: 25:42)

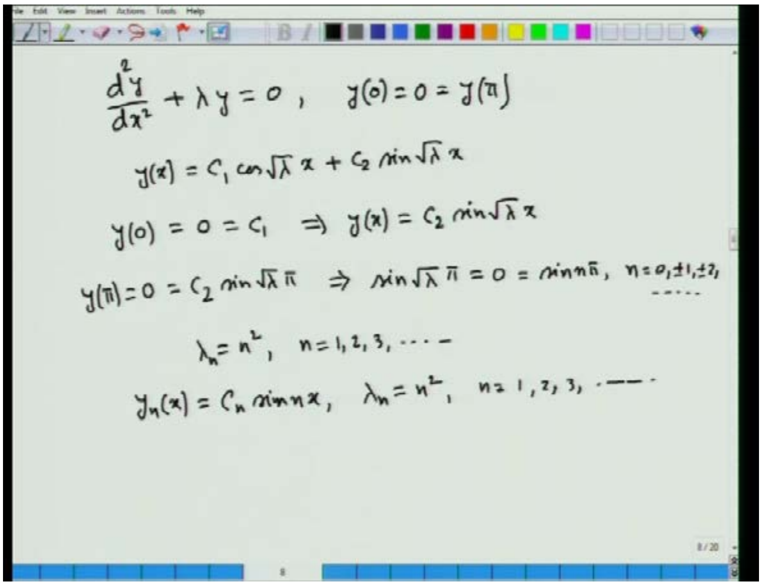


So, using that formula, you can find dy dx, this is equal to minus lambda divided by pi x y(x) minus lambda divided by pi times pi minus x y(x) no term with integral sign will come up, because in both of these two integrals, integrant are of the variables s only. So, if we differentiate the integrant with respect to x in both the cases, we will be having the result 0. And clearly, this minus lambda by pi x y(x) cancels with plus lambda by pi x y(x); so that means, this term cancels with this one. So, then we are left with the expression minus lambda y(x) this one, this will be d 2 y d x 2. So, finally, we are having d 2 y d x 2 plus lambda y(x) this is equal to 0.

Now, in order to find out of the boundary conditions, we have to use the given problem; after substituting the kernels, so that means, we have to again recall the formula that is yx) equal to lambda by pi this one, y(x) equal to lambda by pi integral a to x s times pi minus x y(s) ds plus lambda by pi integral x to pi x into pi minus s y(s) ds; here lower limit will be 0. And then if we substitute y(0) here, so then you can see that integral will be from 0 to 0; so this contributes to be 0. And here in the integrant, you have x. So, substituting x equal to 0, this integrant comes out to be 0. So ultimately, we will be having y(0) equal to 0.

And similarly if you substitute x equal to pi here, so first integral vanishes, because integrant is identically equal to 0 as involves pi minus x term. So, substituting x equal to pi this integrand will be 0, so no contribution from the first integral. And in the second integral, limit is from pi to pi. So, this is also equal to 0. So, will be having y(pi) equal to 0. So that means, starting from the given integral equation, we have arrived at the ordinary differential equation d 2 y d x 2 plus lambda y(x) this is equal to 0, with the boundary conditions y(0) equal to 0 equal to y(pi).

(Refer Slide Time: 29:16)



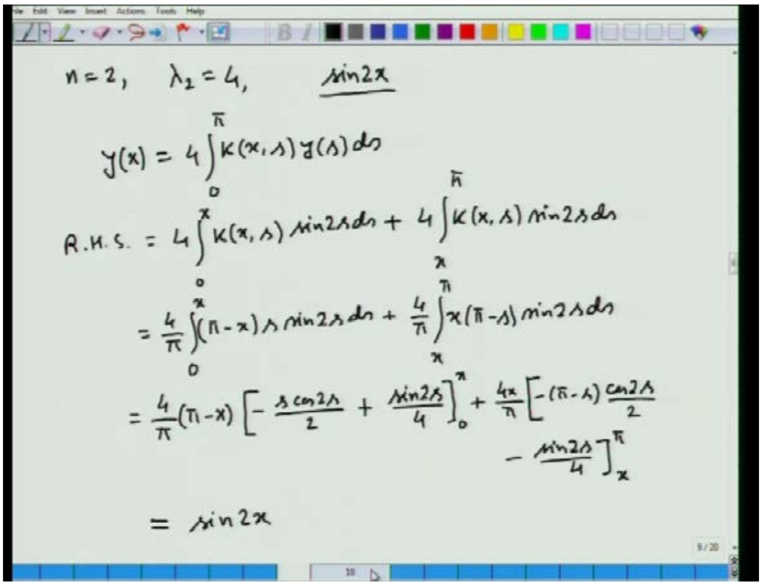

Now, this particular ordinary differential equation is well known for the problems involving Eigen values and Eigen functions. So, first of all we solve this equation quickly, because I hope you are familiar with this solution. So, you can verify for lambda less than 0 or lambda equal to 0, there is no solution other than the trivial solution. So that means, they does not exists any non-trivial solution for lambda less then equal to 0. And for lambda greater than 0, we will be having this y(x) that is the general solution, it is given by c 1 cosine root of lambda x plus c 2 sine root of lambda x. And substituting y(0) equal to 0, we can find this is equal to c 1, because cosine 0 is 1 and sine 0 equal to 0, so c 1 identically equal to 0; and this implies y(x) is equal c 2 sine root of lambda x.

And then using the second condition that is y of pi equal to 0, so this will be equal to c 2 sine root over lambda pi, and as we are interested for non trivial solution, so that means, assuming c 2 not equal to 0, we can find sine root over lambda pi equal to 0. So, this is equal to sine of n pi, where n is equal to 0 plus minus 1 plus minus 2 and so on. And writing general solution, we can find lambda equal to n square these are actually Eigen values; and in this case, n equal to 1, 2, 3 and so on; because here minus 1, minus 2, minus 3, all these values will be clubed with 1 to 3 as lambda n equal to n square; and we are not writing n equal to 0 here, because for n equal to 0, lambda n identically equal to 0; and we have already explain that this particular integral equation as no non trivial solution other than the trivial solution y equal to 0 for lambda equal to 0. So, these are the Eigen values.

And corresponding Eigen functions y n (x) these are given by c n sine n x, and Eigen values are lambda n equal to n square, where n equal to 1, 2, 3 and so on. This c n are actually normalizing factor. In later lecture, I will be discussing little about c n; how this will work for different values of n. So, what we are getting that this differential equation d 2 y d 2 x plus lambda y equal to 0 with the boundary condition y(0) equal to 0 equal to y(pi) posses non trivial solution for a particular set of values for lambda. If we choose any values of lambda, then solution does not exists; and also solution of the problem is surely linked with lambda.

So that means, if we consider the corresponding Fredholm integral equation associated with this boundary equal problem, then we have to keep in mind that when we will be having the solution of this problem, we should have this same set of solution that means, solution can be obtained whenever lambda is equal to same n square, n is taking the values of 1, 2, 3 and so on; and for particular choice of lambda, we will be having corresponding solution of the form sine n x. And actually here the solution is a linearly dependent with the, that is a scalar multiple of sine n x.

(Refer Slide Time: 33:57)



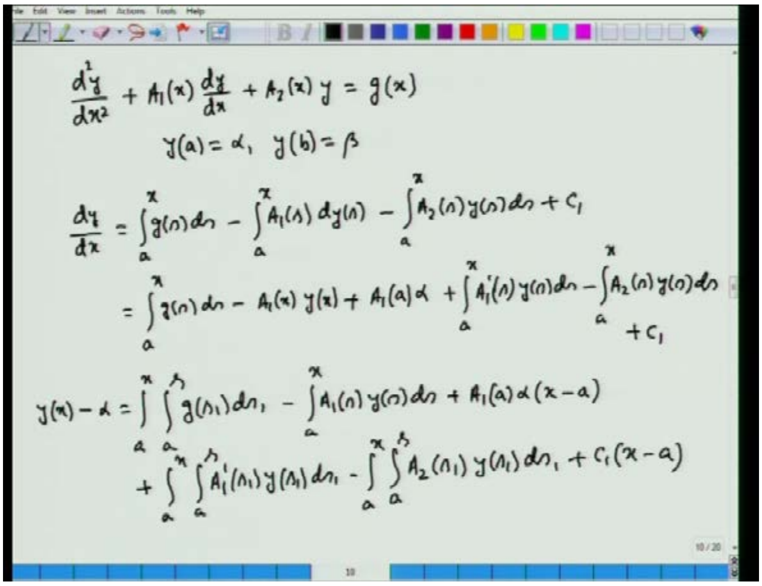

And now we just verify whether this type of solution satisfies the given integral equation or not, for a particular value of lambda. So for verification, we can choose n equal to 2, then lambda 2 is equal to 4, and we are intended to check whether this sine 2 x is a solution to the given problem or not. So, then integral equation becomes y(x) equal to four integral 0 to pi k of x,s y(s) ds, and we are going to check whether sine to x satisfies this integral equation or not. So, if we consider the right hand side, so this is equal to 4 times integral 0 to x k (x,s) sine 2 s d s plus 4 integral x to pi k of x,s sine of 2 s ds. And this is equal to 4 by pi integral 0 to x pi minus x s sine 2 s ds plus 4 divided by pi integral x to pi x times pi minus s sine 2 s ds. Of course, this pi minus x can be take outside of integral sine, and x can be take out from the second integral sine.

And then using the formula for integrating by parts, we can find 4 by pi multiplied with pi minus x; and then integrating, we can find s minus cost 2 x divided by 2; we are considering this s as u and sine to s as v; so this is u integral v d x; so cos 2 s by s. Then for Bipher's formula, we will be having 1 minus and integral of sine produces 1 minus, so this will be plus; and then it will be integral cos 2 s by 2. So, again integrating, we will be having sine 2 s divided by 4 and limit will be from 0 to x. And for the second part, we will be having 4 x by pi outside the integral sine, and then we have minus pi minus s cosine 2 s divided by 2; and then it will be minus sine 2 s divided by 4 limit from x to pi; this minus sine coming here, because for first integration that is integral v ds, we will be having 1 minus; for Bipher's formula, we will be having another minus, and derivate of pi minus s with respect to s will produce another minus 1. So, ultimately sine will be minus, because integral of cosine does not affect the sine just appearing before this integral.

And after substituting this limit, and after some simplification, we will be able to verify that this is comes out to be simply sine 2x. So that means, the function y(x) equal to sine to x satisfies this integral equation; and therefore, the solution of the given problem is surely linked with the Eigen values; and there is no non trivial solution, whenever lambda takes the values other than 1, 4, 9, 16, 25 and onwards that is square of the positive integers.

So, this point we have to keep in mind that most of the time, we will be having this type of solution. And we will be discussing these things with the light of sturm liouville boundary value problem in details. Now before going to that part, now I am going to consider that a general second order linear ordinary differential equation, which is a non homogenous ordinary differential equation with the specified separated boundary condition, can be converted into a Fredholm integral equation. And you can see that when you convert it, then we will be having a Fredholm integral equation of second kind, which is non homogenous.

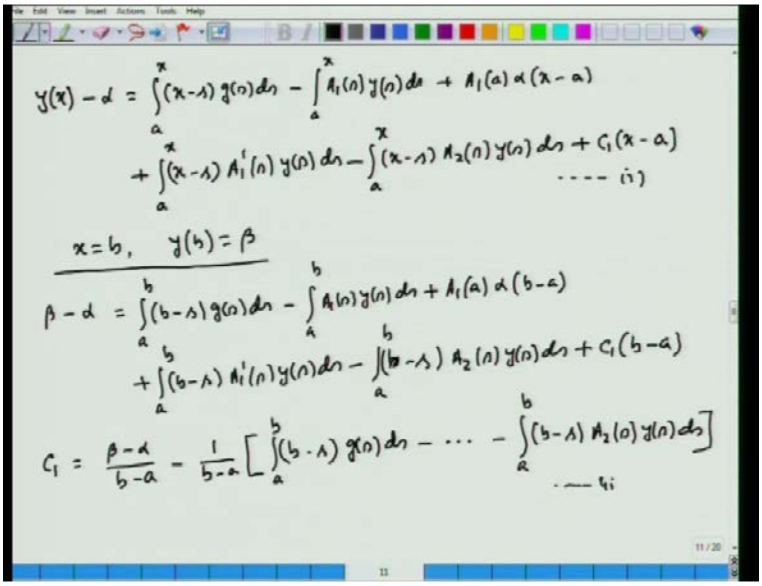
(Refer Slide Time: 38:50)




So for this purpose, we consider the equation d 2 y d x 2 plus A 1 x dy dx plus A 2 (x) y, this is equal to g(x), where given conditions are y(a) equal to alpha, y(b) equal to beta, these are the boundary conditions. So, first of all integrating the given differential equation, we can find dy dx that is equal to integral a to x g(s) ds minus integral a to x A 1 s d of y(s) minus integral a to x A 2 s y(s) d s plus c 1, because dy dx at x equal to a is not known, so we are writing c 1 here. And in order to write the middle integral, we have used the notion that dy ds, operated with ds produces dy(s). And here after using the Bipher's formula, we can get this is equal to a x g(s) ds minus A 1 (x) y(x) this one plus A 1 (a) multiplied with alpha, because y(a) equal to alpha here minus integral plus integral a to x A 1 dashed (s) y(s) ds minus integral a to x A 2 s y(s) ds plus c 1. So, this is the expression for dy dx.

If we integrate it again, then we will be having y(x) minus alpha is equal to integral a to x, then integral a to s g of s 1 ds 1 minus integral a to x A 1 s y(s) ds plus A 1(a) alpha times x minus a plus integral a to x, then integral a to s A 1 dashed (s 1) y(s 1) ds 1 minus integral a to x integral a to s A 2 (s 1) y(s 1) ds 1 plus c 1 times x minus a. Here for little bit simplicity, I am substituting limit here without going to introduce A 1 x plus c 2 here.

(Refer Slide Time: 42:40)



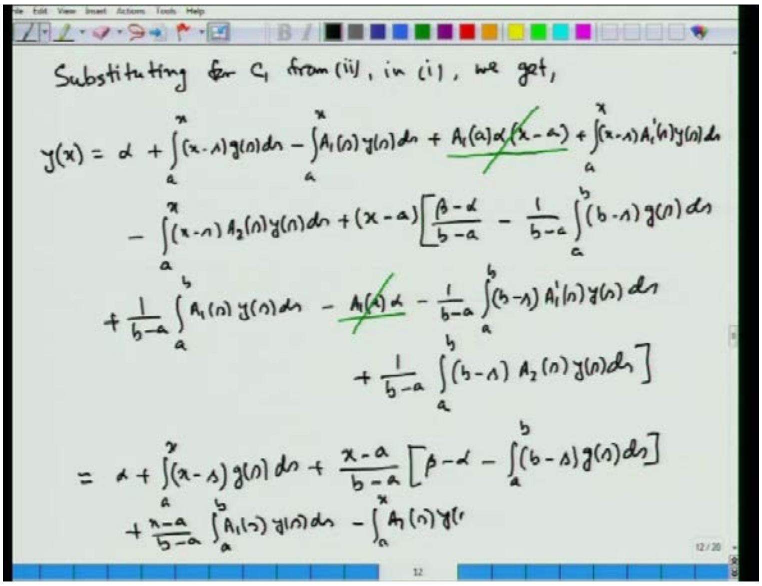
And again here, we can apply the formula that is fundamental detection formula to successive integration formula to convert this double integral into second integral, and will be having y(x) minus alpha is equal to integral a to x x minus s g(s) ds minus integral a to x, there is no need to apply that formula here at A 1 (s) ys ds, then plus A 1

(a) alpha times x minus a remains unaltered, and last two integral will be a to x, then x minus s A 1 dash s y(s) ds minus a to x x minus d A 2(s) ys ds, and last term will remain unaltered c 1 times x minus a.

Now, in this expression, only unknown quantity is c 1 that we have to evaluate. And in order to find out c 1, we are substituting x equal to b, and we can utilize the condition y(b) equal to beta. So, using this substitution into this integral, we will be having beta minus alpha, this is equal to integral a to b then b minus s g(s) ds minus integral a to b A 1 (s) y(s) ds plus A 1 a alpha times b mines a plus integral a to b b minus s A 1 dashed (s) y(s) ds minus integrals a to b b minus s A2 (s) y(s) ds plus c 1 times b minus a. So ultimately, we will be having this c 1 is equal to beta minus alpha divided by b minus a, that is beta minus alpha divided by b minus a.

And then this entire quantity we have to sit on to the left hand side, and then interchanging the left hand and right hand all this expression will be having a minus 1 factor at the beginning. So, then it will be minus 1 by b minus a with the expression that is a to b b minus s g(s) ds starting from this 1 minus dot dot up to the last expression integral a to b b minus s A 2 (s) y(s) ds. So now, we have to substitute this expression for c 1 into here that is y (x) minus alpha is given by this expression. So, if we call this as 1, and this expression for c 1 this as 2.

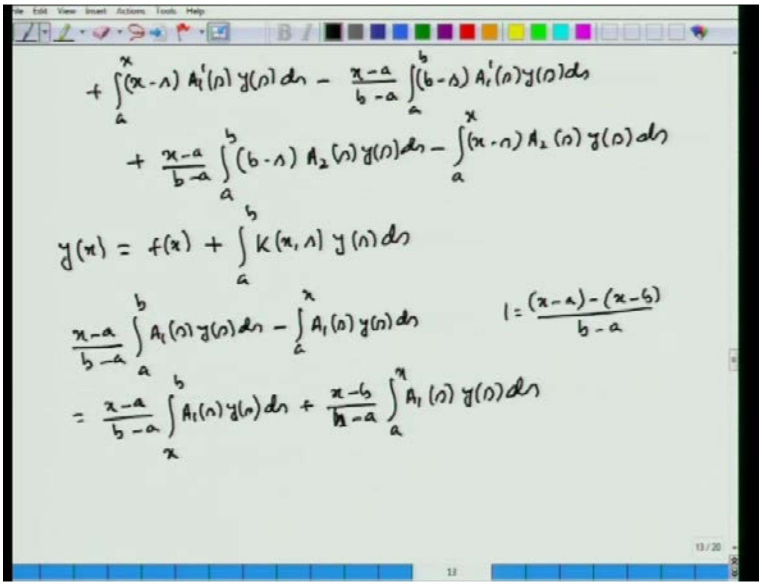
(Refer Slide Time: 46:19)



So, we can write that substituting for c 1 from 2 in 1, we get. Now, this is going to be gigantic expression, but step by step we can try to write it down y (x) equal to alpha plus. So first of all, we are transferring this alpha on to the right hand side, then we have to write this entire expression, and then plus x minus a will be multiplied with the expression for c 1. So, writing this terms, you can find this will be integral a to x x minus s g(s) ds, then minus integral a to x A 1 (s) y(s) ds plus A 1(a) alpha times x minus a plus integral a to x x minus s A 1 dashed s y(s) ds minus integral a to x x minus s A 2 (s) y(s) ds plus x minus a multiplied with the formula for c 1 that is the expression for c 1 not formula, that is beta minus alpha by b minus a minus 1 by b minus a integral a to b b minus s g(s) ds plus 1 by b minus a integral a to b A 1 (s) y(s) ds minus A 1 (a) times alpha minus 1 by b minus a integral a to b b minus s A 1 dashed (s) y(s) ds; and finally, plus 1 b minus a integral a to b minus s A 2 (s) y(s) ds this one.

Now, this x minus a times minus A 1 alpha a will cancels with this one. So that means, this 2 terms, this one and this one cancels with each other. And now our target will be to separate the integral involving y(s) term and the terms free from y(s). So, if we do that, then will be having one expression that is alpha plus integral a to x x minus s g(s) ds plus x minus a divided by b minus a, then beta minus alpha minus integral a to b b minus s g(s) ds this one, and then plus will be having six terms that is x minus a divided by b minus a integral a to b A 1 (s) y(s) ds. And now, I am writing the similar terms side by side, then will be having this term so, that is this is this term, and then minus integral a to x A 1 (s) y(s) ds.

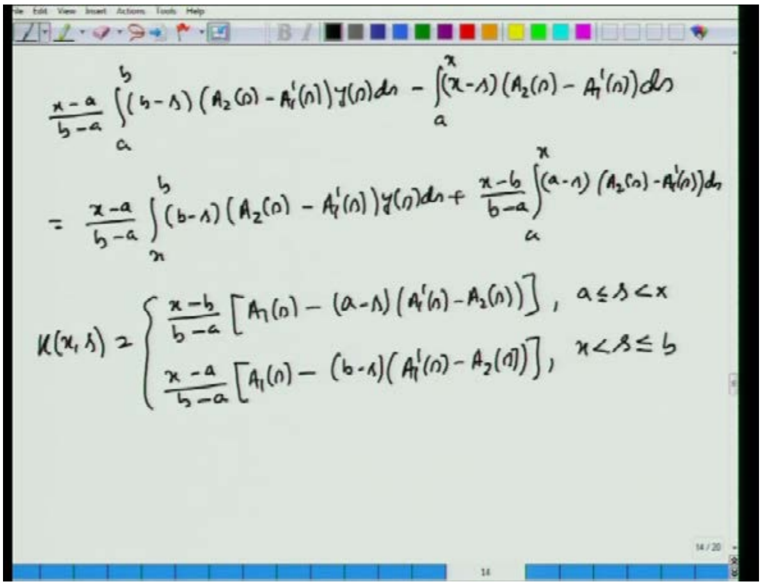
(Refer Slide Time: 50:24)




Then other four terms plus integral a to x x minus s A 1 dashed s y(s) ds minus x minus a divided by b minus a integral a to b b minus s A 1 dashed (s) y(s) ds plus x minus a by b minus a integral a to b minus s A 2 (s) y(s) ds minus a to x x minus s A 2 (s) y(s) ds. Now, this particular term that is already we have written here, if we marked with this some particular color that is alpha plus this integral this; these particular expression constitutes our f (x). So, then y (x) will be equal to f (x) plus we can put it into the form a to b k (x,s) y(s) ds, where expression for f(x) is given earlier. Now we have look at this, how we can find out this k (s, x,s).

An the idea is that as we have done earlier, every integral of a to b have to be divided into 2 parts that is a to x and x to b, then after rearranging the terms, we can get this results that is x minus a divided by b minus a integral a to b A 1 (s) y(s) ds minus integral a to x A 1 (s) y(s) ds; and if we use this result that is 1 equal to x minus a minus x minus b divided by b minus a, then we can find this will be equal to x minus a divided by b minus a integral x to b A 1 (s) y (s) ds plus x minus b divided by b minus a integral a to x A 1 (s) y(s) ds; and for rest of the parts we can take this A 1 dashed (s) minus a to s as common term.

(Refer Slide Time: 53:19)



Then, rest of the part can be written as x minus a divided by b minus a integral a to b b minus s times a to s minus A 1 dashed (s) y(s) ds this minus integral a to x x minus s times a to s minus A 1 dashed (s) ds, this can be written as equal to x minus a divided by b minus a integral x to b b minus s a to s minus A 1 dashed (s) y(s) ds plus x minus b divided by b minus a times integral a to x a minus s times A 2 (s) minus A 1 dashed (s) d s, so this one. So, we will be able to arrive at this stage. And now combining this result, we can define that the kernel of the integral equation k (x,s) this is given by x minus b divided by b minus a times A 1 (s) minus a minus s times A 1 dashed s minus a to s, this one, this is valid for a less then equal s less than x, and then x minus a divided by b minus a times A 1 (s) minus b minus s times A 1 dashed (s) minus a to s this one for x less than s less than equal to b.

And you can check from the previous integral, we will behaving for a to x range x minus b by b minus a A 1 (s) that is appearing here; and then from this part that is a to x it will be x minus b by b minus a multiplied with minus a minus s times A 1 dashed (s) minus A 2 s. So, this is the expression for s less than x. And the next on that is x minus a by b minus a equal to A 1 (s) minus b minus s this one. So, this is the kernel, so with this kernel, we can convert a general second order linear non homogenous boundary value problem to a Fredholm integral equation linear integral equation of second kind.

And of course, this is a non homogenous Fredholm integral equation. So, this lecture I can end at this point. In next few lectures, I will be considering the concept of Greens function that is very much important to obtain the solution for non homogenous linear boundary value problem and sturm liouville theory, which will be required to understand the solution methods for Fredholm integral equation. Thank you