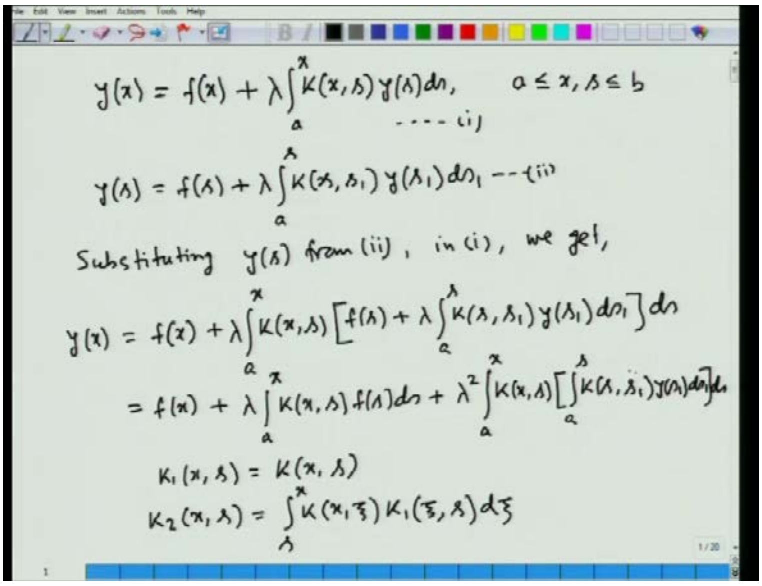
Calculus of Variations and Integral Equation

Prof. Dhirendra Bahuguna
Prof. Malay Banerjee
Department of Mathematics and Statistics
Indian Institute of Technology, Kanpur

Lecture #28

Welcome viewers once again to the lecture series on integral equation under NPTEL program. In the last lecture, we have discussed about the successive substitution method, which leads to the resolvent kernel for non-homogeneous Volterra integral equation of the second kind. In this lecture, we are going to discuss about the uniform convergence of the resolvent kernel; and then using the resolvent kernel, the result we have obtained for integral equation, that result is also uniformly convergent that we are going to prove. Now, before going to the proof of this result, we just recall what we have discussed for this method.

(Refer Slide Time: 01:08)



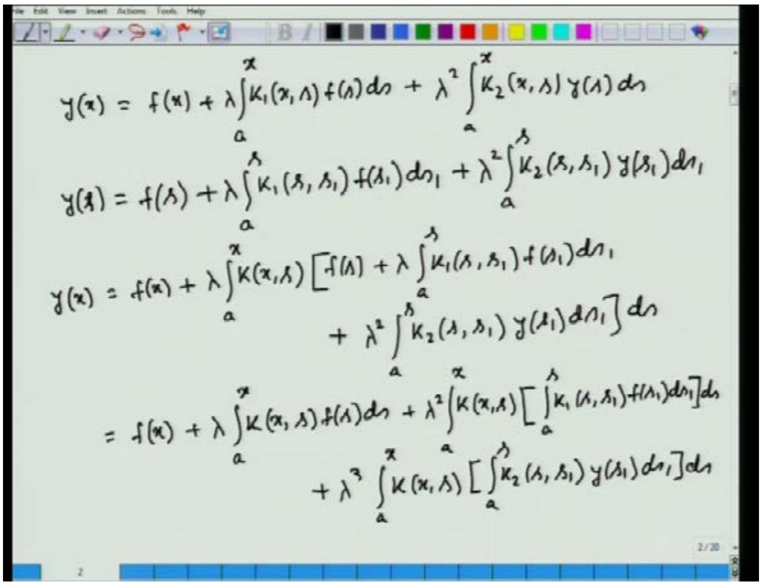
So, equation is Volterra integral equation of second kind that is f (x) plus lambda integral a to x k of x,s y(s) ds, this is the given integral equation, where, a less than equal to x,s less than equal to b; as usual f(x) is continuous over the interval a,b and the kernel k (x,s) is continuous over the squared domain that is a comma b cross a,b. Now the method is that from this given integral equation, we can calculate y (s), and then substituting into this equation, we can find out one iterates of this particular problem.

Then with the modified expression for y (x), we can calculate again y (s), and back substituting into the original equation, we can get the second modification. And proceeding in this way up to infinite number of terms that is theoretically not possible, but mathematically speaking, proceeding this step, infinite number of steps, you can see that ultimately integrals involved with that term f (x) is only; and this y (s) will disappear, and defining the resolvent kernel as infinite series of iterated kernels, we can define the solution of the given problem.

So, in order to find out y (s) from these expressions, we have to replace x by s throughout this, but already s is involved here, but it is a dummy index for this integral. So, we can replace this s s by s 1. So from here… Call it 1; from here we can write y (s) this is equal to f (s) plus lambda integral a to s k of s,s 1 y (s 1) ds 1. So, this is the expression for y (s). So, from here we can write substituting y (s) from 2 in 1 we get; so if we substitute here, we will be having y (x) is equal to f (x) plus lambda integral a to x k of x,s, then this y (s) will be replaced by the expression involved in expression 2. So, that is f (s) plus lambda integral a to s k of s,s 1 y (s 1) ds 1 ds.

From here, we can write this is equal to f (x) plus lambda integral a to x k of x,s f (s) ds plus lambda square integral a to x k of x,s then integral a to s k of s comma s 1 y s 1 ds 1 ds this is the expression. And after this, we have defined the iterated kernels; first iterated kernel k (x,s) is k (x,s) itself. And then we have defined k 2 (x,s) that is equal to integral s to x k (x,psi) k 1 (psi,s) d psi. So, with this head that is iterated kernel k 2 (x,s) defined by this one, you can recall we have converted this particular repeated integral into a single integral.

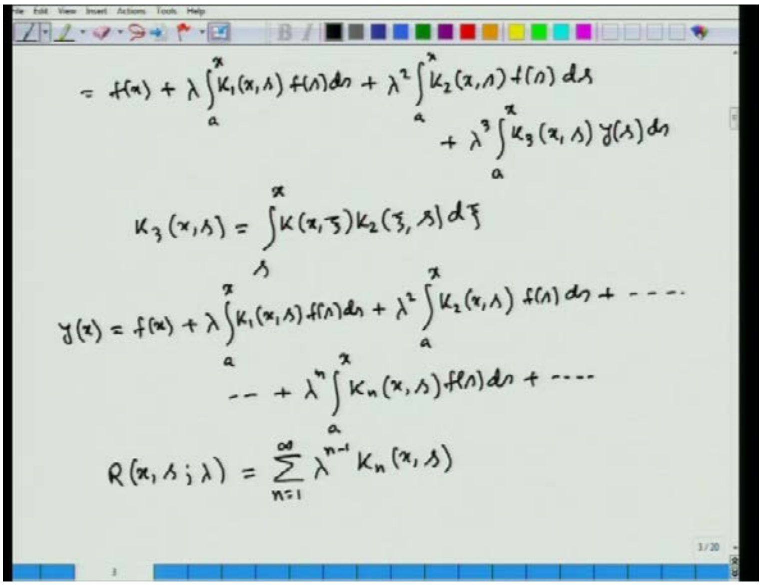
(Refer Slide Time: 06:23)



And we have obtained the result that y (x) is equal to f (x) plus lambda times integral a to x k 1 (x,s) f (s) ds, then plus lambda square integral a to x k 2 (x,s) y(s) ds. So, this is the modified expression for y (x). And you can observe that here f (s) appears into the picture into the expression for y (x). So, from this expression again we can write y (s) this is equal to f (s) plus lambda integral a to s k 1 (s,s 1) f of s 1 ds 1 plus lambda square integral a to s k 2 (s,s 1) y (s 1) ds1. So, this is the expression for y (s). And this expression for y (s) we can substitute into the first equation that is what is numbered as 1 y (x) equal to this one.

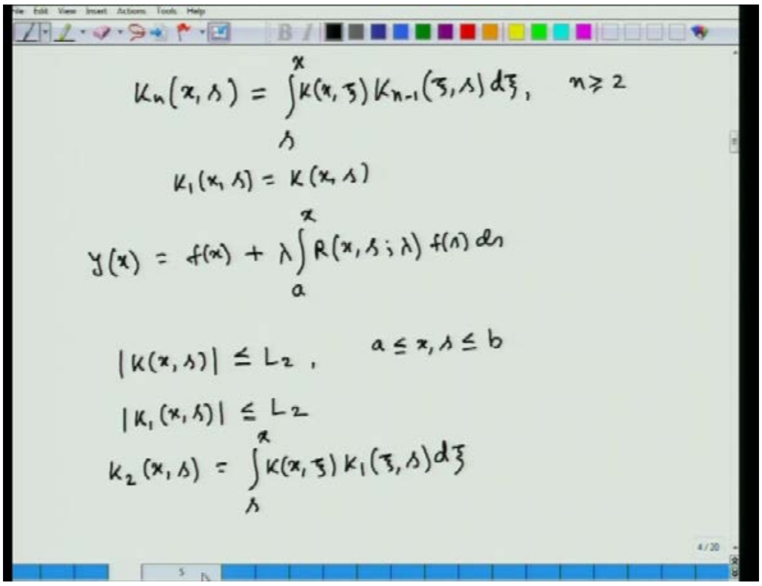
So after substituting, we can find y (x) equal to f (x) plus lambda times integral a to x k of x,s then f (s) plus lambda integral a to s k 1 (s,s 1) f (s 1) ds 1 plus lambda square integral a to s k 2 (s,s 1) y (s 1) ds 1 and then ds. So, using the procedure as we have adopted earlier, and also using the definition for third iterated kernel, we can reduce this expression into the following form using the successive steps; from here, we can write f (x) plus lambda integral a to x k of x,s f (s) ds; this is the first term coming from here k (x,s) f (s) ds, this is the first term. Then second term will be plus lambda square integral a to x k of x,s, this will be combined with integral a to s k 1 (s,s 1) f of s 1 ds 1 ds plus lambda cube integral a to x k of x,s, and then integral a to s k 2 (s,s 1) y (s 1) ds 1 ds.

(Refer Slide Time: 10:24)



And then we can write this is equal to f (x) plus lambda integral a to x k 1 (x,s) f (s) ds plus lambda square integral a to x k 2 (x,s) f (s) ds, then last term will be lambda cube integral a to x k 3 (x,s), and this is a point where we have to be careful, this will be y(s) ds; where this k 3 (x,s) this is equal to integral s to x k (x,psi) k 2 (psi,s) d psi. So, this k 3 (x,s) is defined here. So, this is the third iterated kernel. And proceeding in this way up to infinite number of steps, we can find y (x) is equal to f(x) plus lambda times integral a to x k 1 (x,s) f (s) ds plus lambda square integral a to x k 2 (x,s) f (s) ds plus dot dot, and general term will be lambda to the power n integral a to x k n (x,s) f (s) ds plus dot dot, this one. And defining this resolvent kernel R (x,s;lambda), this is equal to summation n runnings from 1 to infinity lambda to the power n minus 1 k n (x,s).

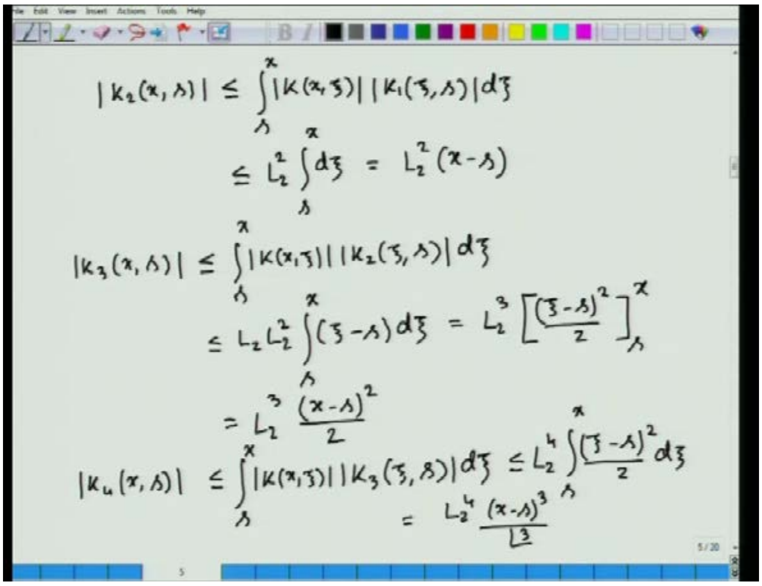
(Refer Slide Time: 12:54)



We can find the solution of the integral equation, where this general n eth iterated kernel k n (x,s) is defined by s to x k of x,psi k n minus 1 (psi,s) d xi, where n greater than equal to 2, and again k 1 (x,s) is equal to k (x,s). Now, with this particular (( )), we can find out the solution of the Volterra integral equation. Now first of all, the point is that in order to write the solution of Volterra integral equation into this form that is y (x) equal to f(x) plus lambda times integral a to x R of x,s lambda f (s) ds; we have to interchange summation and integral sign. So, the question is that this is admissible or not. In order to get answer to this question, first of all we are intended to find out the address the question of convergence of the infinite series of functions that is actually obtained for this resolvent kernel R (x,s).

So, in order to prove this, first of all we assume the bound of k (x,s). So, it is assumed that modulus of k (x,s) this is less than equal to L 2 for all a x,s lies between in this this range or we can say the component (x,s) there is a vector belongs to a,b closed interval cross the closed interval a,b. Now from here, first we can notice as k (x,s) is k 1 (x,s) itself. So, k 1 (x,s) is less than equal to L 2, then k 2 (x,s) this is equal to integral s to x k (x,psi) then k 1 (psi,s) d psi, where psi is ranging between s to x, ands and x are already bounds with in these interval a to b.

(Refer Slide Time: 15:47)

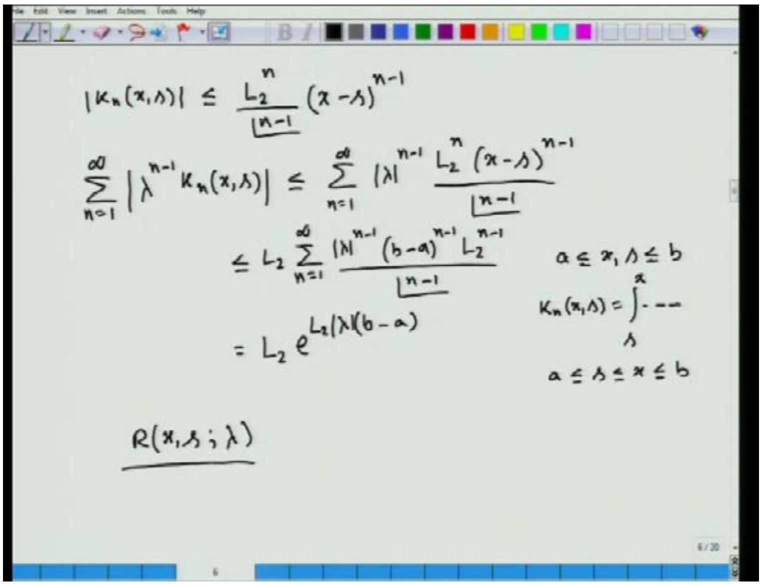


And therefore, taking modulus, and then transferring modulus sign under the integral sign, we can write modulus of k 2 (x,s) this is less than equal to integral s 2 x modulus k of x,psi, then modulus k 1 (psi,s) d psi; and this is less than equal to L 2 square integral s to x d psi. So, this is equal to L 2 square x minus s. So, this is the result for L 2 square; this is x minus s. Next for modulus k 3 (x,s) we can write as per definition, this is s to x, then taking modulus under the integral sign, this will be modulus k of x,psi then modulus k 2 (psi,s) this d psi.

Now, for modulus k (x,psi), we can use the bound for k that is L 2; and for k 2 (psi,s), we have to use the bound we have obtained at the last step. And using these results we can find this is less than equal to L 2 for this one, then L 2 square for modulus k 2 (psi,s) integral s to x psi minus s d psi. And evaluating this integral, we can find L 2 cube psi minus s whole square divided by 2 limit from s to x; at the lower limit, when substituted for psi equal to s, this is identically equal to 0. So, this is coming out to be L 2 cube x minus s whole square divided by 2.

And for the forth iterated kernel, we can calculate k 4 (x,s) this modulus is less than equal to again integral s to x modulus k (x,psi) then modulus k 3 (psi,s), this d psi. And again for k (x,psi), we can write this is less than equal to L 2; for k 3 (psi,s), we can use the result we have obtained at the last step. So, this will be less than equal to L 2 to the power 4 integral s to x psi minus s whole square by 2 d psi. And this is equal to L 2 to the power 4 times x minus s whole cube divided by factorial 3. So, this coming out to be the bound for k 4 (x,s).

(Refer Slide Time: 19:13)

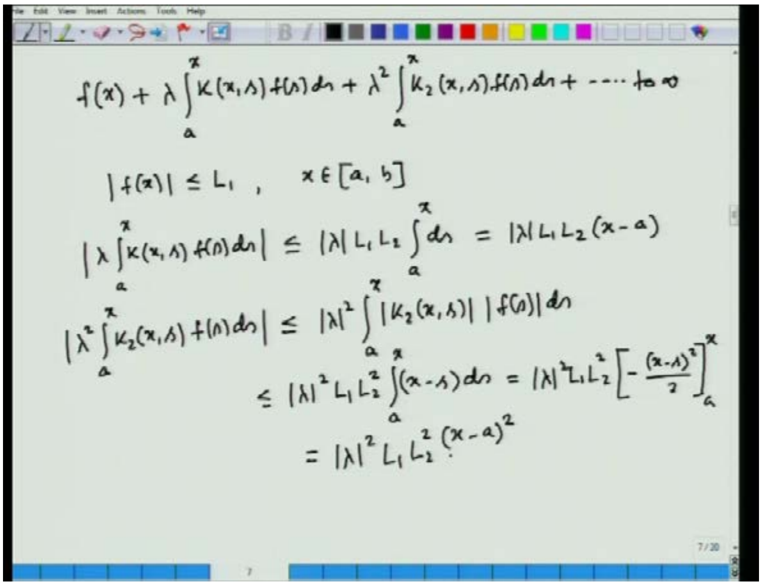


So, if we proceed in this way, in general, we can prove modulus of k n (x,s) this is less than equal to L 2 to the power n divided by factorial n minus 1, then x minus s whole to the power n minus 1; this result. And therefore, the series that is summation n runnings from 1 to infinity modulus lambda to the power n minus 1 k n (x,s), this modulus will be less than equal to sigma n runnings from 1 to infinity modulus lambda this to the power n minus 1, then L 2 to the power n x minus s to the power n minus 1 divided by factorial n minus 1.

Now, throughout this integration, we have already most of the time used the range from s to x. And initially, we have mentioned that a less than equal to x,s less than equal to b. So, combining this result that is a less than equal to x,s less than equal to b. And for the iterated kernel k n (x,s), we are considering the integral from s to x. So, we can use the ordering that is a less than equal to s less than equal to x less than equal to b. So, using this ordering, we can write this is less than equal to L 2 times sigma n runnings from 1 to infinity modulus lambda whole to the power n minus 1 then b minus a whole to the power n minus 1 L 2 to the power n minus 1 divided by factorial n minus 1.

We are taking one L 2 outside this summation in order to make the uniform index of of the all these terms, and this series is exactly equal to L 2 e to the power L 2 times modulus lambda times b minus a. So, this clearly shows that the series for R (x,s;lambda) that is for the resolvent kernels is uniformly convergent. And using the formula for successive iterates of the resolvent kernel, we can easily verify that every iterates k n (x,s) they are continuous; so that means, that series summation n runnings from 1 to infinity lambda to the power n minus 1 k n (x,s) it converges uniformly to a continuous function, and that is denoted by R x s lambda.

(Refer Slide Time: 22:52)

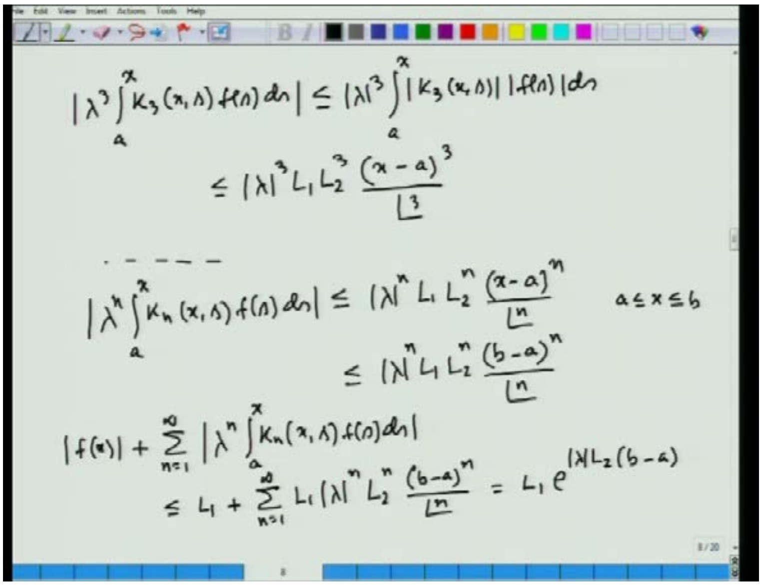


Next we are going to prove the uniform convergence of this particular series that is f(x) plus lambda times integral a to x k of x,s f (s) ds plus lambda square integral a to x k 2 (x,s) f (s) ds plus dot dot up to infinity. We are going to prove the uniform convergence of this result. Here we are assuming as f(x) is continuous over the closed interval a comma b, so therefore, we assume that modulus f(x) less than equal to L 1 whenever x belongs to this closed interval a,b. And in this case, we have to proceed in a similar fashion by which we have proved the uniform continuity of the series involved with the resolvent kernel, only one modification will comes into the picture that is involvement of f (s) here and this integral sign.

And here, first of all this modulus f(x) is less than equal to L 1, then we can write modulus lambda integral a to x k of x,s f (s) ds, this is less than equal to modulus lambda for this f (s), it will be L 1; for modulus k (x,s) it will be L 2, and then integral a to x ds. So, this is equal to modulus lambda L 1 L 2 x minus a. Using this result, we can write lambda square a to x k 2 (x,s) f (s) ds. Now this is less than equal to modulus lambda whole square integral a to x mod of k 2 (x,s) modulus of f (s) ds. And from the previous result, we have to use the bound for modulus k 2 (x,s) here.

So, using that particular result we can find this is less than equal to modulus lambda whole square, then L 1 L 2 square integral a to x x minus s ds, and after integration it will comes out to be modulus lambda whole square L 1 L 2 square, then minus x minus s whole square by 2 limit from a to x; at the upper limit, when you are substituting s equal to x, this is identically equal to 0; at the lower limit, we will be having x minus a whole square by 2, and this minus sign will be absorbed; so this is equal to modulus lambda whole square L 1 L 2 square, then x minus a whole square divided by 2.

(Refer Slide Time: 26:32)

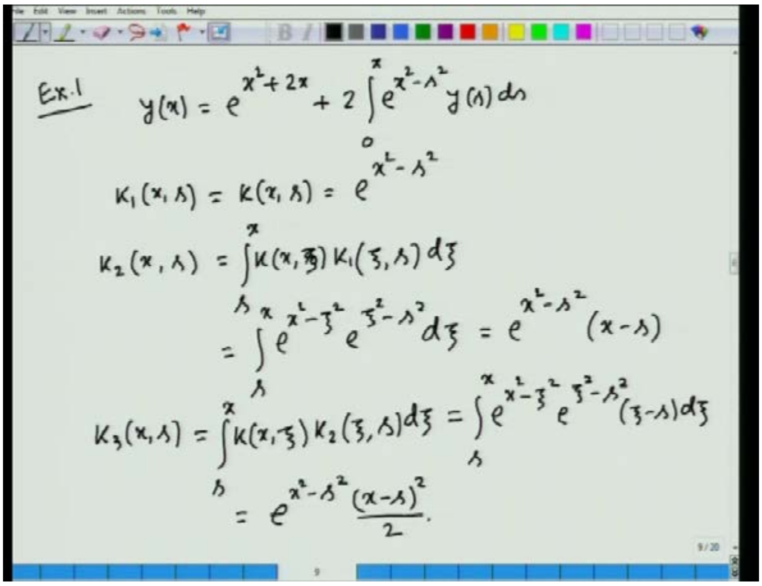


In a similar manner, we can prove modulus of lambda whole cube integral a to x k 3 (x,s) f (s) ds, this is less than equal to modulus lambda whole cube integral a to x modulus of k 3 (x,s) modulus f (s) ds, which is less than equal to modulus lambda whole cube L 1, then L 2 cube, and after integration it will come out to be x minus a whole cube divided by factorial 3. So, proceeding in this manner or by using the method of mathematical induction, you can prove the general term lambda to the power n integral a to x k n (x,s) f (s) ds, this is less than equal to modulus lambda whole to the power n L 1 L 2 to the power n x minus a whole to the power n divided by factorial n.

And here itself using the result that a less than equal to x less than equal to b, we can write this is less than equal to modulus lambda to the power n L 1 L 2 to the power n b minus a whole to the power n by factorial n, which we are going to use to prove the uniform convergence of this infinite expression that I have written here that is f(x) plus lambda integral a to x this one, but do not confuse with this part, when we are deriving the successive term of the series as there modulus is the magnitude, then we are not going to use this expression that is b minus a whole to the power n by factorial this is only in order to prove the uniform convergence.

And therefore, finally, we can write that modulus f(x) this plus sigma n runnings from 1 to infinity modulus lambda to the power n integral a to x k n (x,s) f (s) ds this is less than equal to L 1 plus summation n runnings from 1 to infinity L 1 modulus lambda whole to the power n L 2 to the power n b minus a whole to the power n by factorial n, and then taking L 1 common, we can write this is equal to L 1 times exponential of modulus lambda L 2 times b minus a. So, these particular infinite series, it converges uniformly. So, this proof is complete.

(Refer Slide Time: 29:49)

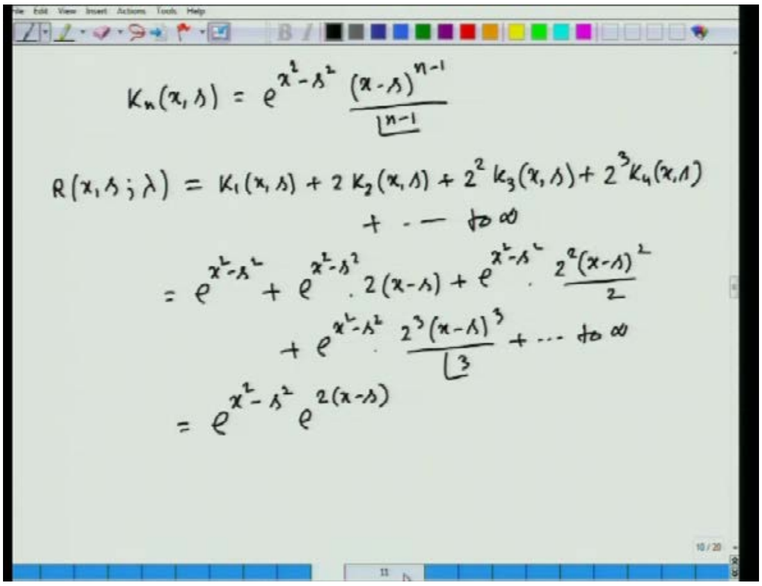


Now, before completing this discussion, I want to discuss one more example by that is the solution by method of resolvent kernel; that is y (x) is equal to e the power x square plus 2 x plus 2 integral 0 to x e to the power x square minus s square y(s) ds. And in the previous lecture, you can recall that we have considered the derivation of resolvent kernel of one function that is kernel function of this particular type e to the power x square minus s square.

Now, here k 1 (x,s) is equal to the given kernel (x,s) is e to the power x square minus s square itself. Then second iterated kernel k 2 (x,s) that is equal to integral s to x k (x,s) k 1 (psi,s) d psi; so this is equal to integral s to x e to the power x square minus psi square times e to the power psi square minus s square d psi. So, this will be equal to e to the power x square minus s square multiplied with x minus s, because this minus psi square plus psi square will cancel at the index of exponential. So, it will be e to the power x square minus s square, it will comes out from the integral sign, and integrated d psi from s to x will be having e to the power x square minus s square times x minus s.

Then k 3 (x,s), this is equal to s to x k of x,psi k 2 (psi,s) d psi. So, this is equal to integral s to x k (x,psi) is e to the power s square minus psi square; this is expression for k x comma psi. And k 2 (psi,s) we have to write from here, this is the expression for k 2 (x,s). So, k 2 (psi,s) will be e to the power psi square minus s square, then psi minus s d psi. And this will be equal to e to the power x square minus s square times x minus s whole square by 2.

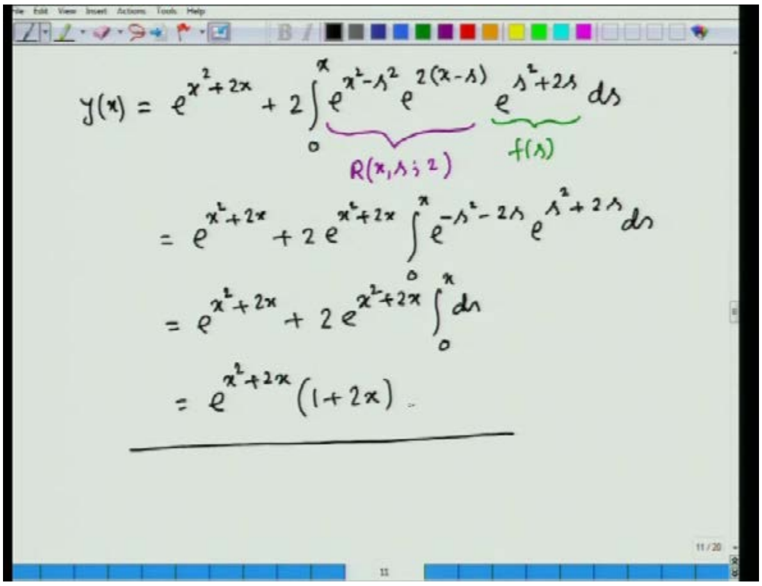
(Refer Slide Time: 32:56)



And in this way, if you calculate the general term, then you can find that k n (x,s), this will be equal to e to the power x square minus s square times x minus s whole to the power n minus 1 by factorial n minus 1; you can verify this claim by calculating the fourth iterate, and from there you can predict this will be equal to this one or you can use the standard method of mathematical induction.

So ultimately, R (x,s;lambda) that is resolvent kernel; and for this given problem, lambda is actually equal to 2. So, we will be having k 1 (x,s) plus 2 k 2 (x,s) plus 2 square k 3 (x,s) plus 2 cube k 4 (x,s) plus dot dot up to infinity; and this is equal to e to the power x square minus s square plus e to the power x square minus s square into 2 times x minus s plus e to the power x square minus s square 2 square x minus s whole square by 2 plus e to the power x square minus s square 2 cube x minus s whole cube divided by factorial 3 plus dot dot up to infinity. And clearly this is equal to e to the power x square minus s square times e to the power 2 x minus s. So, this is our resolvent kernel for the given problem.

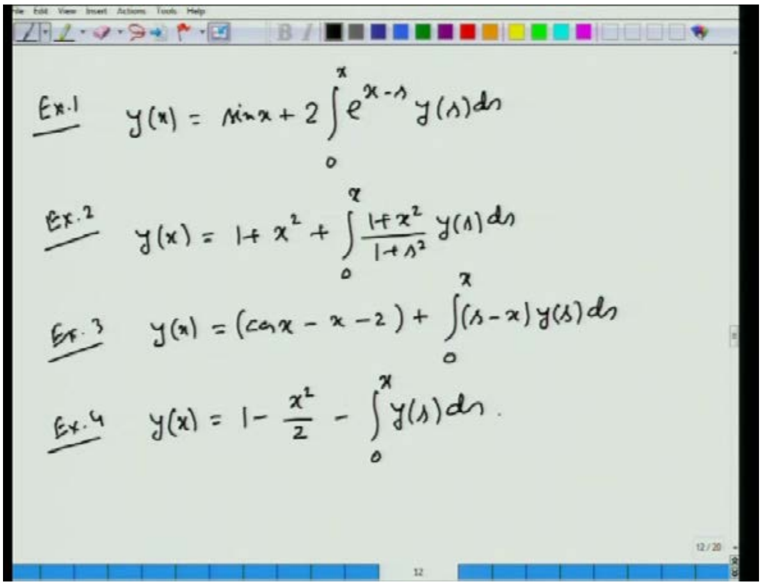
(Refer Slide Time: 34:59)



And using the formula for getting the solution of this problem, we can write y (x) this is equal to f(x). So, that is e to the power x square plus 2 x plus lambda that is 2 integral 0 to x lower limit is given as 0; then e to the power x square minus s square times e to the power 2 x minus s, then e to the power s square plus 2 s ds. Just for your convenience, I can just mention here, this is your f (s) for the given problem, and this expression is actually R (x,s;2) that is the resolvent kernel.

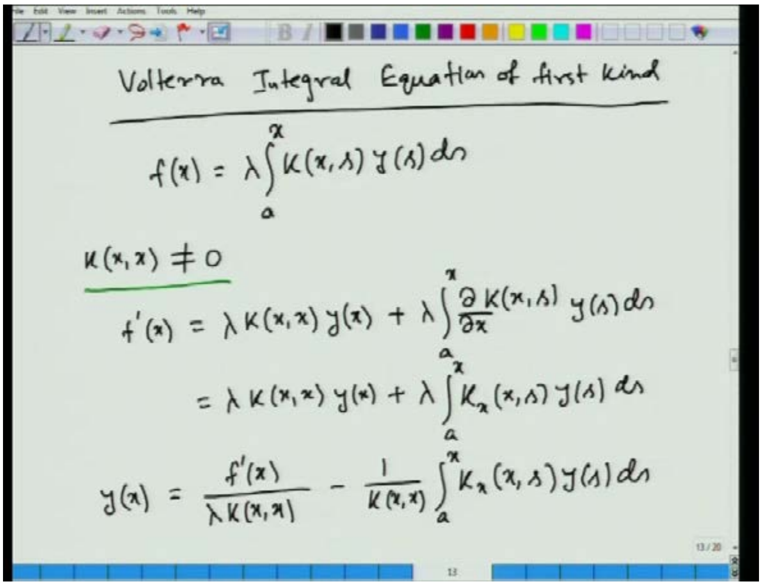
And if you evaluate this integral, then e to the power x square plus 2 x plus 2 you can take out this e to the power x square plus 2 x from here and from here, then we are left with 0 to x e to the power minus s square minus 2 s e to the power s square plus 2 s ds. So, this gives you 1. So ultimately, you will be having e to the power x square plus 2 x plus 2 e to the power x square plus 2 x integral 0 to x ds; and this is equal to e to the power x square plus 2 x this entire quantity multiplied with 1 plus 2 x. So, this is the solution to the given Volterra integral equation of the second kind.

(Refer Slide Time: 37:06)



And for your some practice problem, you can try to solve these exercises. Exercise 1 - y (x) is equal to sin x plus 2 integral 0 to x e to the power x minus s y(s) ds, this is the first problem. Then example 2 - y (x) this is equal to 1 plus x square plus integral 0 to x 1 plus x square divided by 1 plus s square y(s) ds. Third exercise - y (x) this is equal to cos x minus x minus 2 plus integral 0 to x s minus x y(s) ds. And fourth exercise y (x) this is equal to 1 minus x square by 2 minus integral 0 to x y(s) ds. These are the exercises.

(Refer Slide Time: 38:39)

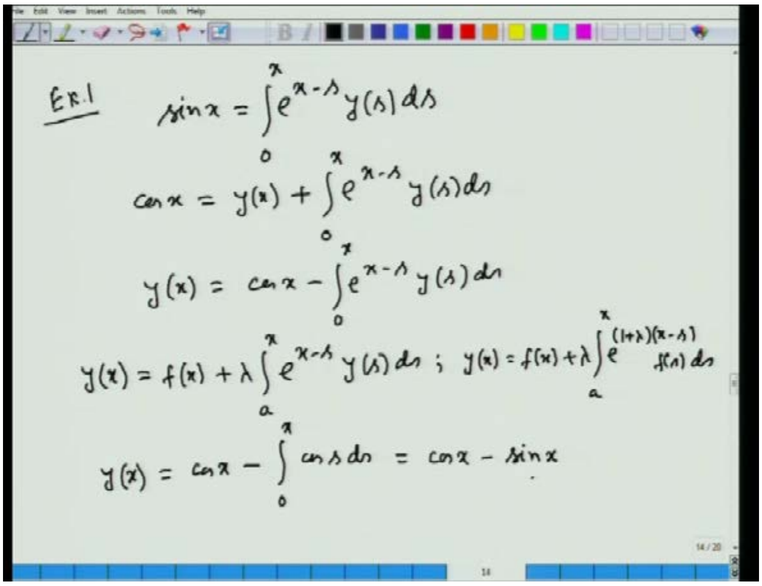


Now, we move to the Volterra integral equation of the first kind. Volterra integral equation of first kind; Volterra integral equation of first kind is given as f(x) equal to lambda times integral a to x k (x,s) y(s) ds. Now, first we are going to solve this equation by just converting it into the Volterra integral equation of the second kind, but you have to keep in mind, this conversion is possible whenever k (x,x) this is not equal to 0. So, you may ask the question or if question comes in your mind, if this is equal to 0, what can be done? After discussing this part, I will come to this particular point.

Now this is actually the equation f(x) equal to this one. You can differentiate both sides with respect to x, and in order to differentiate the right hand side, we can use the Leibniz formula. So, using Leibniz formula on to the right hand side, differentiating both sides with respect to x, we can find f dot x, this is equal to lambda k of (x,x) y (x) this 1 that means, we are substituting s equal to x into the integrant, and then plus lambda times integral a to x del del x of k (x,s) y(s) ds. And for convenience, we can write that lambda k (x,x) y (x) plus lambda integral a to x k x (x,s) y(s) ds.

So, now using this particular condition that k (x,x) not equal to 0, we can divide both the sides by k (x,x), and then transferring this integral after division by lambda k (x,x), we can obtain that transform equation that is f(x) is equal to f dot x divided by lambda k (x,x) then minus 1 by k (x,x) integral a to x k x (x,s) y(s) ds. So, this is our actually Volterra integral equation of the first kind, where f dot x by lambda times k (x,x), this is the non homogeneous part, that is the analogous expression, that was involved with equation that is f(x). And in this case, the kernel is minus partial derivative of k with respect to x divided by k (x,x).

(Refer Slide Time: 42:15)

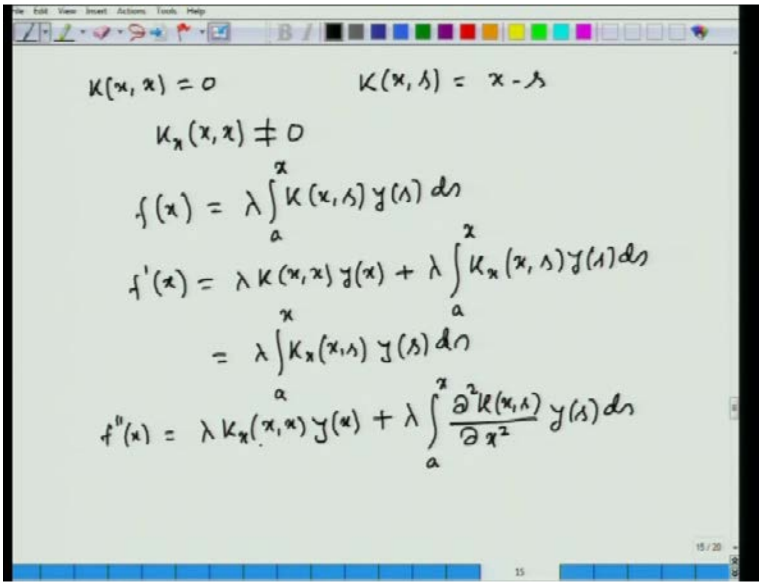


Quickly, we can have a look at a problem, how to solve or how to apply this technique. We considered the equation sin x equal to integral 0 to x e to the power x minus s y(s) ds. So, this is our f (x), this is the kernel, and this is y(s) ds. So, after differentiation, we can find cos x, this is equal to y (x) plus integral 0 to x e to the power x minus s y(s) ds. So, resulting integral equation is y (x) is equal to cos x minus cos x minus integral 0 to x e to the power x minus s y(s) ds. So, this is our integral equation, this Volterra integral equation of second kind, which we are obtained by differentiating the given Volterra integral equation of the first kind.

And here, we are write down the solution of this problem by using this result, and you can verify a similar example we have considered. If a given Volterra integral equation of first kind is given in this particular format that is y (x) equal to f(x) plus lambda a to x e to the power x minus s y(s) ds. Then its solution is given by y (x) equal to f(x) plus lambda integral a to x e to the power 1 plus lambda x minus s f (s) ds. This result you can take as an exercise and this result can be easily achieved by using the Laplace transform method. So, once you apply this formula, in order to get the solution, this of this problem.

So, you will be having y (x) this is equal to cos x minus integral 0 to x cos s ds, because if you just be careful about this one, the lambda equal to minus 1, so e to the power lambda plus 1, this is 0, so that means e to the power 1 plus lambda x minus s this is identically equal to 1. So, solution is y equal to cos x minus integral 0 to x cos s ds; and after integration, you will be having the solution this is cos x minus sin x. Now next question is if k (x,x) this is equal to 0, then what can be done?

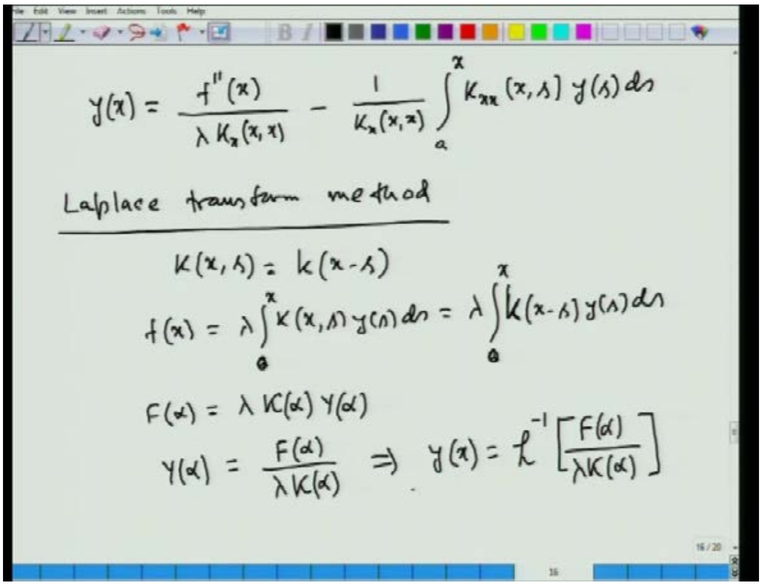
(Refer Slide Time: 45:21)



So, just for as an example, you may consider that k (x,s) is equal to x minus s itself, k (x,s) is x minus s itself. Now in this case, k (x,x) this is equal to 0, but partial derivative of k with respect to x, then after substituting s equal to x, you find that is not equal to 0. And we can exploit this idea in order to convert an Volterra integral equation of the first kind into a Volterra integral equation of the second kind. And idea is that k (x,x) equal to 0, and we are assuming k x (x,x), this is not equal to 0. So that means, if we start from this expression f(x) equal to lambda integral a to x k of x,s y(s) ds.

So, after first differentiation, you will be having f dot x equal to lambda k (x,x) y (x) this one plus lambda times integral a to x k x (x,s) y(s) ds. Now since k (x,x) equal to 0, so this reduces to again a Volterra integral equation of the first kind. So, we can apply the same tricks on this particular equation; and we can find out that if double dot x that is equal to lambda times… this will be k (x,s); k x (x,x) y (x) plus lambda times integral a to x del 2 k (x,x) del x 2 y(s) ds. And already we have assumed that k (x,x) this is not equal to 0, partial derivative of k x evaluated at s equal to x, this is not equal to 0. So; that means, this k x (x,x), this is not equal to 0.

(Refer Slide Time: 48:07)



And therefore, from here we can generate the Volterra integral equation of second kind that is given by y (x) equal to f double dot x divided by lambda times k x (x,x) minus 1 by k x (x,x) integral a to x k xx (x,s) y(s) ds. So, in case of a polynomial x minus s, if k x is a polynomial of x minus s of degree say n. So, differentiating the resulting equations in times, you can convert the equation into a Volterra integral equation of the second kind. Now, before coming to the n, we can discuss one more method to solve this kind of equation quickly; that is called Laplace transform method.

Once the kernel is a function of x minus s, so instead of differentiating this n number of times to obtain the integral equation of second kind, and then solve it; we can readily imply this particular technique that is Laplace transform method. So, whenever k (x,s), this is a function of x minus s. So, then given equation is f(x) equal to lambda times integral a to x k of x,s y(s) ds, this is equal to lambda times integral a to x k x minus s y(s) ds. So, this is the convolution integral I am sorry lower limit will be 0 0 to x.

And then taking Laplace transform of the both side, we can find F (alpha) this is equal to lambda times kappa (alpha) y (alpha) that means, Laplace transform of k (x,s), this is kappa (alpha), and Laplace transform of the unknown function y (x) is equal to y (alpha), so this one. So, if lambda and k (alpha), they not equal to 0. From here, we can write y (alpha) is equal to F (alpha) divided by lambda times kappa (alpha); and from here you can find the solution of the equation by taking the inverse Laplace transform that is L inverse F (alpha) divided by lambda times kappa (alpha). So, this is the inverse Laplace transform method.

So, or after converting the Volterra integral equation of the first kind, we have Volterra integral equation of the second kind, whatever method we have discussed any one of them can be applied to solve the problem. So, main idea is that if k (x,x) is not equal to 0, so after differentiating the given equation one times with respect to x with the help of Leibniz formula, immediately you can get a Volterra integral equation of the second kind, and you can solve the equation by using any method, what we have discussed so far. And in case, this kernel is a function of x minus s and is a polynomial such that k (x,s) is equal to 0 that means, after substituting s equal to x, this comes out to be 0, then you can take an attempt to solve this problem by the Laplace transform method. So, today I can stop at this particular point. Thank you.